Chapter 9

Green's Functions for Time-Independent Problems

9.1 Introduction

Solutions to linear partial differential equations are nonzero due to initial conditions, nonhomogeneous boundary conditions, and forcing terms. If the partial differential equation is homogeneous and there is a set of homogeneous boundary conditions, then we usually attempt to solve the problem by the method of separation of variables. In Chapter 7 we developed the method of eigenfunction expansions to obtain solutions in cases in which there were forcing terms (and/or nonhomogeneous boundary conditions).

In this chapter, we will primarily consider problems without initial conditions (ordinary differential equations and Laplace's equation with sources). We will show that there is one function for each problem called the Green's function, which can be used to describe the influence of both nonhomogeneous boundary conditions and forcing terms. We will develop properties of these Green's functions and show direct methods to obtain them. Time-dependent problems with initial conditions, such as the heat and wave equations, are more difficult. They will be used as motivation, but detailed study of their Green's functions will not be presented until Chapter 10.

9.2 One-dimensional Heat Equation

We begin by reanalyzing the one-dimensional heat equation with no sources and homogeneous boundary conditions:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}} \tag{9.2.1}$$

$$u(0,t) = 0 \tag{9.2.2}$$
$$u(L,t) = 0 \tag{9.2.3}$$
$$u(x,0) = g(x). \tag{9.2.4}$$

In Chapter 2, according to the method of separation of variables, we obtained

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-k(n\pi/L)^2 t},} \tag{9.2.5}$$

where the initial condition implied that a_n are the coefficients of the Fourier sine series of $g(x)$,

$$g(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \tag{9.2.6}$$

$$\boxed{a_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.} \tag{9.2.7}$$

We examine this solution (9.2.5) more closely in order to investigate the effect of the initial condition $g(x)$. We eliminate the Fourier sine coefficients from (9.2.7) (introducing a dummy integration variable x_0):

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L g(x_0)\sin\frac{n\pi x_0}{L}\,dx_0\right]\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}.$$

If we interchange the order of operations of the infinite summation and integration, we obtain

$$u(x,t) = \int_0^L g(x_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}\right)dx_0. \tag{9.2.8}$$

We define the quantity in parentheses as the **influence function** for the initial condition. It expresses the fact that the temperature at position x at time t is due to the initial temperature at x_0. To obtain the temperature $u(x,t)$, we sum (integrate) the influences of all possible initial positions.

Before further interpreting this result, it is helpful to do a similar analysis for a more general heat equation including sources, but still having homogeneous boundary conditions:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)} \tag{9.2.9}$$

$$u(0,t) = 0 \tag{9.2.10}$$
$$u(L,t) = 0 \tag{9.2.11}$$
$$u(x,0) = g(x). \tag{9.2.12}$$

This nonhomogeneous problem is suited for the method of eigenfunction expansions,

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}. \tag{9.2.13}$$

This Fourier sine series can be differentiated term by term since both $\sin n\pi x/L$ and $u(x,t)$ solve the same homogeneous boundary conditions. Hence, $a_n(t)$ solves the following first-order differential equation:

$$\frac{da_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 a_n = q_n(t) = \frac{2}{L}\int_0^L Q(x,t) \sin \frac{n\pi x}{L}\, dx, \tag{9.2.14}$$

where $q_n(t)$ are the coefficients of the Fourier sine series of $Q(x,t)$,

$$Q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin \frac{n\pi x}{L}. \tag{9.2.15}$$

The solution of (9.2.14) (using the integrating factor $e^{k(n\pi/L)^2 t}$) is

$$a_n(t) = a_n(0)e^{-k(n\pi/L)^2 t} + e^{-k(n\pi/L)^2 t}\int_0^L q_n(t_0) e^{k(n\pi/L)^2 t_0}\, dt_0. \tag{9.2.16}$$

$a_n(0)$ are the coefficients of the Fourier sine series of the initial condition, $u(x,0) = g(x)$:

$$g(x) = \sum_{n=1}^{\infty} a_n(0) \sin \frac{n\pi x}{L} \tag{9.2.17}$$
$$a_n(0) = \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx. \tag{9.2.18}$$

These Fourier coefficients may be eliminated, yielding

$$u(x,t) = \sum_{n=1}^{\infty}\left[\left(\frac{2}{L}\int_0^L g(x_0)\sin\frac{n\pi x_0}{L}\, dx_0\right) e^{-k(n\pi/L)^2 t} + e^{-k(n\pi/L)^2 t}\int_0^t \left(\frac{2}{L}\int_0^L Q(x_0,t_0)\sin\frac{n\pi x}{L}\, dx_0\right) e^{k(n\pi/L)^2 t_0}\, dt_0\right] \sin\frac{n\pi x}{L}.$$

After interchanging the order of performing the infinite summation and the integration (over both x_0 and t_0), we obtain

$$u(x,t) = \int_0^L g(x_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2 t}\right) dx_0 + \int_0^L\int_0^t Q(x_0,t_0)\left(\sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2(t-t_0)}\right) dt_0\, dx_0.$$

We therefore introduce the **Green's function**, $G(x,t;x_0,t_0)$,

$$G(x,t;x_0,t_0) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2(t-t_0)}. \tag{9.2.19}$$

We have shown that

$$u(x,t) = \int_0^L g(x_0)G(x,t;x_0,0)\, dx_0 + \int_0^L\int_0^t Q(x_0,t_0)G(x,t;x_0,t_0)\, dt_0\, dx_0. \tag{9.2.20}$$

The Green's function at $t_0 = 0$, $G(x,t;x_0,0)$, expresses the influence of the initial temperature at x_0 on the temperature at position x and time t. In addition, $G(x,t;x_0,t_0)$ shows the influence on the temperature at the position x and time t of the forcing term $Q(x_0,t_0)$ at position x_0 and time t_0. Instead of depending on the source time t_0 and the response time t, independently, we note that the Green's function depends only on the **elapsed time $t - t_0$**:

$$G(x,t;x_0,t_0) = G(x,t-t_0;x_0,0).$$

This occurs because the heat equation has coefficients that do not change in time; the laws of thermal physics are not changing. The Green's function exponentially decays in elapsed time $(t - t_0)$ [see (9.2.19)]. For example, this means that the influence of the source at time t_0 diminishes rapidly. It is only the most recent sources of thermal energy that are important at time t.

Equation (9.2.19) is an extremely useful representation of the Green's function if time t is large. However, for small t the series converges more slowly. In Chapter 10 we will obtain an alternative representation of the Green's function useful for small t.

In (9.2.20) we integrate over all positions x_0. The solution is the result of adding together the influences of all sources and initial temperatures. We also integrate the sources over all *past* times $0 < t_0 < t$. This is part of a **causality principle.** The temperature at time t is only due to the thermal sources that acted *before* time t. Any future sources of heat energy cannot influence the temperature now.

Among the questions we will investigate later for this and other problems are

1. Are there more direct methods to obtain the Green's function?
2. Are there any simpler expressions for it [can we simplify (9.2.19)]?
3. Can we explain the relationships between the influence of the initial condition and the influence of the forcing terms?
4. Can we account easily for nonhomogeneous boundary conditions?

EXERCISES 9.2

9.2.1. Consider

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t)\\ u(x,0) &= g(x).\end{aligned}$$

In all cases obtain formulas similar to (9.2.20) by introducing a Green's function.

(a) Use Green's formula instead of term-by-term spatial differentiation if

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0.$$

(b) Modify part (a) if

$$u(0,t) = A(t) \qquad \text{and} \qquad u(L,t) = B(t).$$

Do not reduce to a problem with homogeneous boundary conditions.

(c) Solve using any method if

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = 0.$$

*(d) Use Green's formula instead of term-by-term differentiation if

$$\frac{\partial u}{\partial x}(0,t) = A(t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = B(t).$$

9.2.2. Solve by the method of eigenfunction expansion

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q(x,t)$$

subject to $u(0,t) = 0$, $u(L,t) = 0$, and $u(x,0) = g(x)$, if $c\rho$ and K_0 are functions of x. Assume that the eigenfunctions are known. Obtain a formula similar to (9.2.20) by introducing a Green's function.

***9.2.3.** Solve by the method of eigenfunction expansion

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} + Q(x,t)\\ u(0,t) = 0 \qquad & u(x,0) = f(x)\\ u(L,t) = 0 \qquad & \frac{\partial u}{\partial t}(x,0) = g(x).\end{aligned}$$

Define functions (in the simplest possible way) such that a relationship similar to (9.2.20) exists. It must be somewhat different due to the two initial conditions. (*Hint*: See Exercise 7.5.1.)

9.2.4. Modify Exercise 9.2.3 (using Green's formula if necessary) if instead

(a) $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = 0$

(b) $u(0,t) = A(t)$ and $u(L,t) = 0$

(c) $\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = B(t)$

9.3 Green's Functions for Boundary Value Problems for Ordinary Differential Equations

9.3.1 One-dimensional Steady-State Heat Equation

Introduction. Investigating the Green's functions for the time-dependent heat equation is not an easy task. Instead, we first investigate a simpler problem. Most of the techniques discussed will be valid for more difficult problems.

We will investigate the steady-state heat equation with homogeneous boundary conditions, arising in situations in which the source term $Q(x,t) = Q(x)$ is independent of time:

$$0 = k\frac{d^2u}{dx^2} + Q(x).$$

We prefer the form

$$\boxed{\frac{d^2u}{dx^2} = f(x),} \tag{9.3.1}$$

in which case $f(x) = -Q(x)/k$. The boundary conditions we consider are

$$\boxed{u(0) = 0 \qquad \text{and} \qquad u(L) = 0.} \tag{9.3.2}$$

We will solve this problem in many different ways in order to suggest methods for other harder problems.

Limit of time-dependent problem. One way (not the most obvious or easiest) to solve (9.3.1) is to analyze our solution (9.2.20) of the time-dependent problem, obtained in the preceding section, in the special case of a steady source:

$$\begin{aligned}u(x,t) = &\int_0^L g(x_0)G(x,t;x_0,0)\,dx_0\\ &+\int_0^L -kf(x_0)\left(\int_0^t G(x,t;x_0,t_0)\,dt_0\right)dx_0,\end{aligned} \tag{9.3.3}$$

where

$$G(x,t;x_0,t_0) = \sum_{n=1}^{\infty}\frac{2}{L}\sin\frac{n\pi x_0}{L}\sin\frac{n\pi x}{L}e^{-k(n\pi/L)^2(t-t_0)}. \tag{9.3.4}$$

As $t\to\infty$, $G(x,t;x_0,0)\to 0$ such that the effect of the initial condition $u(x,0) = g(x)$ vanishes at $t\to\infty$. However, even though $G(x,t;x_0,t_0)\to 0$ as $t\to\infty$, the steady source is still important as $t\to\infty$ since

$$\int_0^t e^{-k(n\pi/L)^2(t-t_0)}\,dt_0 = \left.\frac{e^{-k(n\pi/L)^2(t-t_0)}}{k(n\pi/L)^2}\right|_{t_0=0}^{t} = \frac{1-e^{-k(n\pi/L)^2 t}}{k(n\pi/L)^2}.$$

Thus, as $t \to \infty$,

$$u(x,t) \to \boxed{u(x) = \int_0^L f(x_0) G(x, x_0)\, dx_0,} \tag{9.3.5}$$

where

$$\boxed{G(x, x_0) = -\sum_{n=1}^{\infty} \frac{2}{L} \frac{\sin n\pi x_0/L \sin n\pi x/L}{(n\pi/L)^2}.} \tag{9.3.6}$$

Here we obtained the steady-state temperature distribution $u(x)$ by taking the limit as $t \to \infty$ of the time-dependent problem with a steady source $Q(x) = -kf(x)$. $G(x, x_0)$ is the **influence** or **Green's function** for the steady-state problem. The symmetry,

$$G(x, x_0) = G(x_0, x),$$

will be discussed later.

9.3.2 The Method of Variation of Parameters

There are more direct ways to obtain the solution of (9.3.1) with (9.3.2). We consider a more general nonhomogeneous problem

$$L(u) = f(x), \tag{9.3.7}$$

defined for $a < x < b$, subject to two homogeneous boundary conditions (of the standard form discussed in Chapter 5), where L is the Sturm-Liouville operator:

$$L \equiv \frac{d}{dx}\left(p \frac{d}{dx}\right) + q. \tag{9.3.8}$$

For the simple steady-state heat equation of the preceding subsection, $p = 1$ and $q = 0$, so that $L = d^2/dx^2$.

Nonhomogeneous ordinary differential equations can always be solved by the **method of variation of parameters** if two[1] solutions of the homogeneous problem are known, $u_1(x)$ and $u_2(x)$. We briefly review this technique. In the method of variation of parameters, a particular solution of (9.3.7) is sought in the form

$$\boxed{u = v_1 \cdot u_1 + v_2 \cdot u_2,} \tag{9.3.9}$$

[1] Actually. only one homogeneous solution is necessary as the method of reduction of order is a procedure for obtaining a second homogeneous solution if one is known.

where v_1 and v_2 are functions of x to be determined. The original differential equation has one unknown function, so that the extra degree of freedom allows us to assume du/dx is the same as if v_1 and v_2 were constants:

$$\frac{du}{dx} = v_1 \frac{du_1}{dx} + v_2 \frac{du_2}{dx}.$$

Since v_1 and v_2 are not constant, this is valid only if the other terms, arising from the variation of v_1 and v_2, vanish:

$$\frac{dv_1}{dx} u_1 + \frac{dv_2}{dx} u_2 = 0$$

The differential equation $L(u) = f(x)$ is then satisfied if

$$\frac{dv_1}{dx} p \frac{du_1}{dx} + \frac{dv_2}{dx} p \frac{du_2}{dx} = f(x).$$

The method of variation of parameters at this stage yields two linear equations for the unknowns dv_1/dx and dv_2/dx. The solution is

$$\frac{dv_1}{dx} = \frac{-fu_2}{p\left(u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx}\right)} = \frac{-fu_2}{c} \tag{9.3.10}$$

$$\frac{dv_2}{dx} = \frac{fu_1}{p\left(u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx}\right)} = \frac{fu_1}{c}, \tag{9.3.11}$$

where

$$c = p\left(u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx}\right). \tag{9.3.12}$$

Using the Wronskian described shortly, we will show that c is constant. The constant c depends on the choice of homogeneous solutions u_1 and u_2. The general solution of $L(u) = f(x)$ is given by $u = u_1 v_1 + u_2 v_2$, where v_1 and v_2 are obtained by integrating (9.3.10) and (9.3.11).

Wronskian. We define the **Wronskian** W as

$$W = u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx}.$$

It satisfies an elementary differential equation:

$$\frac{dW}{dx} = u_1 \frac{d^2u_2}{dx^2} - u_2 \frac{d^2u_1}{dx^2} = -\frac{dp/dx}{p}\left(u_1 \frac{du_2}{dx} - u_2 \frac{du_1}{dx}\right) = -\frac{dp/dx}{p} W, \tag{9.3.13}$$

where the defining differential equations for the homogeneous solutions, $L(u_1) = 0$ and $L(u_2) = 0$, have been used. Solving (9.3.13) shows that

$$W = \frac{c}{p} \quad \text{or} \quad pW = c.$$

Example. Consider the problem (9.3.1) with (9.3.2):

$$\frac{d^2u}{dx^2} = f(x) \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u(L) = 0.$$

This corresponds to the general case (9.3.7) with $p = 1$ and $q = 0$. Two homogeneous solutions of (9.3.1) are 1 and x. However, the algebra is easier if we pick $u_1(x)$ to be a homogeneous solution satisfying one of the boundary conditions $u(0) = 0$ and $u_2(x)$ to be a homogeneous solution satisfying the other boundary condition:

$$\begin{aligned} u_1(x) &= x \\ u_2(x) &= L - x. \end{aligned}$$

Since $p = 1$, $c = -L$ from (9.3.12). By integrating (9.3.10) and (9.3.11) we obtain

$$\begin{aligned} v_1(x) &= \frac{1}{L}\int_0^x f(x_0)(L - x_0)\, dx_0 + c_1 \\ v_2(x) &= -\frac{1}{L}\int_0^x f(x_0)x_0\, dx_0 + c_2, \end{aligned}$$

which is needed in the method of variation of parameters ($u = u_1v_1 + u_2v_2$). The boundary condition $u(0) = 0$ yields $0 = c_2L$, whereas $u(L) = 0$ yields

$$0 = \int_0^L f(x_0)(L - x_0)\, dx_0 + c_1L,$$

so that $v_1(x) = -\frac{1}{L}\int_x^L f(x_0)(L-x_0)\, dx_0$. Thus, the solution of the nonhomogeneous boundary value problem is

$$u(x) = -\frac{x}{L}\int_x^L f(x_0)(L - x_0)\, dx_0 - \frac{L - x}{L}\int_0^x f(x_0)x_0\, dx_0. \tag{9.3.14}$$

We will transform (9.3.14) into the desired form,

$$\boxed{u(x) = \int_0^L f(x_0)G(x, x_0)\, dx_0.} \tag{9.3.15}$$

By comparing (9.3.14) to (9.3.15), we obtain

$$\boxed{G(x, x_0) = \begin{cases} \dfrac{-x(L - x_0)}{L} & x < x_0 \\ \dfrac{-x_0(L - x)}{L} & x > x_0. \end{cases}} \tag{9.3.16}$$

A sketch and interpretation of this solution will be given in Sec. 9.3.5. Although somewhat complicated, the symmetry can be seen:

$$G(x, x_0) = G(x_0, x).$$

For the steady-state heat equation, we have obtained two Green's functions, (9.3.6) and (9.3.16). They appear quite different. In Exercise 9.3.1 they are shown to be the same. In particular, (9.3.16) yields a piecewise smooth function (actually continuous), and its Fourier sine series can be shown to be given by (9.3.6).

The solution also can be derived by directly integrating (9.3.1) twice:

$$u = \int_0^x \left[\int_0^{x_0} f(\bar{x})\, d\bar{x}\right] dx_0 + c_1x + c_2. \tag{9.3.17}$$

In Exercise 9.3.2. you are asked to show that (9.3.16) can be obtained from (9.3.17). This can be done by interchanging the order of integration in (9.3.17) or by integrating (9.3.17) by parts.

9.3.3 The Method of Eigenfunction Expansion for Green's Functions

In Chapter 7, nonhomogeneous partial differential equations were solved by the eigenfunction expansion method. Here we show how to apply the same ideas to the general Sturm-Liouville nonhomogeneous ordinary differential equation:

$$L(u) = f(x) \tag{9.3.18}$$

subject to two homogeneous boundary conditions. We introduce a related eigenvalue problem,

$$L(\phi) = -\lambda\sigma\phi, \tag{9.3.19}$$

subject to the *same* homogeneous boundary conditions. The weight σ here can be chosen arbitrarily. However, there is usually at most one choice of $\sigma(x)$ such that the differential equation (9.3.19) is in fact well known.[2] We solve (9.3.18) by seeking $u(x)$ as a generalized Fourier series of the eigenfunctions

$$\boxed{u(x) = \sum_{n=1}^{\infty} a_n\phi_n(x).} \tag{9.3.20}$$

We can differentiate this twice term by term[3] since both $\phi_n(x)$ and $u(x)$ solve the same homogeneous boundary conditions:

$$\sum_{n=1}^{\infty} a_nL(\phi_n) = -\sum_{n=1}^{\infty} a_n\lambda_n\sigma\phi_n = f(x),$$

[2] For example, if $L = d^2/dx^2$, we pick $\sigma = 1$ giving trigonometric functions, but if $L = \frac{d}{dx}\left(x\frac{d}{dx}\right) - \frac{m^2}{x}$, we pick $\sigma = x$ so that Bessel functions occur.

[3] Green's formula can be used to justify this step (see Sec. 9.4).

where (9.3.19) has been used. The orthogonality of the eigenfunctions (with weight σ) implies that

$$-a_n\lambda_n = \frac{\int_a^b f(x)\phi_n \, dx}{\int_a^b \phi_n^2 \sigma \, dx}. \tag{9.3.21}$$

The solution of the boundary value problem for the nonhomogeneous ordinary differential equation is thus (after interchanging summation and integration)

$$u(x) = \int_a^b f(x_0) \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(x_0)}{-\lambda_n \int_a^b \phi_n^2 \sigma \, dx} \, dx_0. \tag{9.3.22}$$

For this problem, the Green's function has the representation in terms of the eigenfunctions:

$$\boxed{G(x, x_0) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(x_0)}{-\lambda_n \int_a^b \phi_n^2 \sigma \, dx}.} \tag{9.3.23}$$

Again the symmetry is explicitly shown. Note the appearance of the eigenvalues λ_n in the denominator. The Green's function does not exist if one of the eigenvalues is zero. This will be explained in Sec. 9.4. For now we assume that all $\lambda_n \neq 0$.

Example. For the boundary value problem,

$$\frac{d^2u}{dx^2} = f(x)$$

$$u(0) = 0 \quad \text{and} \quad u(L) = 0,$$

the related eigenvalue problem,

$$\frac{d^2\phi}{dx^2} = -\lambda\phi$$

$$\phi(0) = 0 \quad \text{and} \quad \phi(L) = 0,$$

is well known. The eigenvalues are $\lambda_n = (n\pi/L)^2$, $n = 1, 2, 3$ and the corresponding eigenfunctions are $\sin n\pi x/L$. The Fourier sine series of $u(x)$ is given by (9.3.20). In particular,

$$u(x) = \int_0^L f(x_0)G(x, x_0) \, dx_0,$$

where the Fourier sine series of the Green's function is

$$G(x, x_0) = -\frac{2}{L} \sum_{n=1}^{\infty} \frac{\sin n\pi x/L \sin n\pi x_0/L}{(n\pi/L)^2}$$

from (9.3.23), agreeing with the answer (9.3.6) obtained by the limit as $t \to \infty$ of the time-dependent problem.

9.3.4 The Dirac Delta Function and Its Relationship to Green's Functions

We have shown that

$$u(x) = \int_0^L f(x_0)G(x, x_0) \, dx_0,$$

where we have obtained different representations of the Green's function. The Green's function shows the influence of each position x_0 of the source on the solution at x. In this section, we will find a more direct way to determine the Green's function.

Dirac delta function. Our source $f(x)$ represents a forcing of our system at all points. $f(x)$ is sketched in Fig. 9.3.1. In order to isolate the effect of each individual point, we decompose $f(x)$ into a linear combination of unit pulses of duration Δx (see Fig. 9.3.2):

$$f(x) \approx \sum_i f(x_i)(\text{unit pulse starting at } x = x_i).$$

This is somewhat reminiscent of the definition of an integral. Only Δx is missing, which we introduce by multiplying and dividing by Δx:

$$f(x) = \lim_{\Delta x \to 0} \sum_i f(x_i) \frac{\text{unit pulse}}{\Delta x} \Delta x. \tag{9.3.24}$$

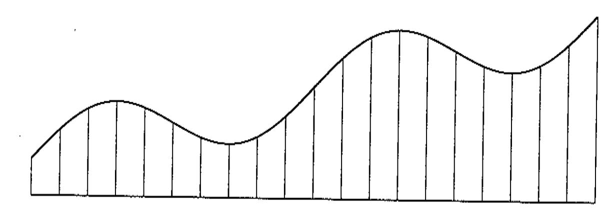

Figure 9.3.1: Piecewise constant representation of a function.

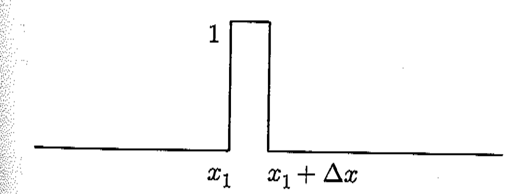

Figure 9.3.2: Pulse with unit height.

In this way we have motivated a rectangular pulse of width Δx and height $1/\Delta x$, sketched in Fig. 9.3.3. It has unit area. In the limit as $\Delta x \to 0$, this approaches an infinitely concentrated pulse (not really a function) $\delta(x - x_i)$ which would be zero everywhere except ∞ at $x = x_i$, still with unit area:

$$\delta(x - x_i) = \begin{cases} 0 & x \neq x_i \\ \infty & x = x_i, \end{cases} \tag{9.3.25}$$

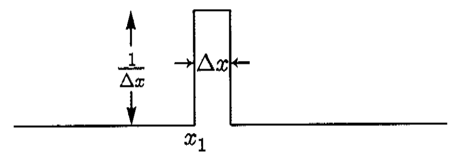

Figure 9.3.3: Rectangular pulse with unit area.

We can think of $\delta(x-x_i)$ as a **concentrated source** or **impulsive force** at $x = x_i$. According to (9.3.24) we have

$$f(x) = \int f(x_i)\delta(x - x_i)\ dx_i. \tag{9.3.26}$$

Since $\delta(x - x_i)$ is not a function, we define it as an operator with the property that for any continuous $f(x)$:

$$\boxed{f(x) = \int_{-\infty}^{\infty} f(x_i)\delta(x - x_i)\ dx_i,} \tag{9.3.27}$$

as is suggested by (9.3.26). We call $\delta(x - x_i)$, the **Dirac delta function.**[4] It is so concentrated that in integrating it with any continuous function $f(x_i)$, it "sifts" out the value at $x_i = x$. The Dirac delta function may be motivated by the "limiting function" of any sequence of concentrated pulses (the shape need not be rectangular).

Other important properties of the Dirac delta function are that it has unit area:

$$\boxed{1 = \int_{-\infty}^{\infty} \delta(x - x_i)\ dx_i;} \tag{9.3.28}$$

it is an even function

$$\delta(x - x_i) = \delta(x_i - x). \tag{9.3.29}$$

This means that the definition (9.3.27) may be used without worrying about whether $\delta(x - x_i)$ or $\delta(x_i - x)$ appears. The Dirac delta function is also the derivative of the Heaviside unit step function $H(x - x_i)$

$$H(x - x_i) \equiv \begin{cases} 0 & x < x_i \\ 1 & x > x_i; \end{cases} \tag{9.3.30}$$

$$\delta(x - x_i) = \frac{d}{dx}H(x - x_i); \tag{9.3.31}$$

$$H(x - x_i) = \int_{-\infty}^{x} \delta(x_0 - x_i)\ dx_0; \tag{9.3.32}$$

[4]Named after Paul Dirac, a twentieth-century mathematical physicist (1902–1984).

it has the following scaling property:

$$\delta[c(x - x_i)] = \frac{1}{|c|}\delta(x - x_i). \tag{9.3.33}$$

These properties are proved in the exercises.

Green's function. The solution of the nonhomogeneous problem

$$\boxed{L(u) = f(x),} \tag{9.3.34}$$

subject to two homogeneous boundary conditions is

$$\boxed{u(x) = \int_a^b f(x_0)G(x, x_0)\ dx_0.} \tag{9.3.35}$$

Here, the Green's function is the influence function for the source $f(x)$. As an example, suppose that $f(x)$ is a concentrated source at $x = x_s$, $f(x) = \delta(x - x_s)$. Then the response at $x, u(x)$, satisfies

$$u(x) = \int_a^b \delta(x_0 - x_s)G(x, x_0)\ dx_0 = G(x, x_s)$$

due to (9.3.27). This yields the fundamental interpretation of the **Green's function $G(x, x_s)$** — *it is the* **response at x due to a concentrated source at x_s**:

$$\boxed{L\left[G(x, x_s)\right] = \delta(x - x_s),} \tag{9.3.36}$$

where $G(x, x_s)$ will also satisfy the same homogeneous boundary conditions at $x = a$ and $x = b$.

As a check, let us verify that (9.3.35) satisfies (9.3.34). To satisfy (9.3.34), we must use the operator L (in the simple case, $L = d^2/dx^2$):

$$L(u) = \int_a^b f(x_0)L\left[G(x, x_0)\right]\ dx_0 = \int_a^b f(x_0)\delta(x - x_0)\ dx_0 = f(x),$$

where the fundamental property of both the Green's function (9.3.36) and the Dirac delta function (9.3.27) has been used.

Often (9.3.36) with two homogeneous boundary conditions is thought of as an *independent definition of the Green's function.* In this case we might want to *derive* (9.3.35), the representation of the solution of the nonhomogeneous problem in terms of the Green's function satisfying (9.3.36). The usual method to derive (9.3.35) involves Green's formula:

$$\int_a^b [uL(v) - vL(u)]\ dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b . \tag{9.3.37}$$

If we let $v = G(x, x_0)$, then the right-hand side vanishes since both $u(x)$ and $G(x, x_0)$ satisfy the same homogeneous boundary conditions. Furthermore, from the respective differential equations (9.3.34) and (9.3.36), it follows that

$$\int_a^b [u(x)\delta(x - x_0) - G(x, x_0)f(x)]\ dx = 0.$$

Thus, from the definition of the Dirac delta function

$$u(x_0) = \int_a^b f(x)G(x, x_0)\ dx.$$

If we interchange the variables x and x_0, we obtain (9.3.35):

$$u(x) = \int_a^b f(x_0)G(x_0, x)\ dx_0 = \int_a^b f(x_0)G(x, x_0)\ dx_0, \tag{9.3.38}$$

since the Green's function is known to be symmetric (9.3.16), $G(x_0, x) = G(x, x_0)$.

Maxwell's reciprocity. The symmetry of the Green's function is very important. We will prove it without using the eigenfunction expansion. Instead, we will directly use the defining differential equation (9.3.36). We again use Green's formula (9.3.37). Here we let $u = G(x, x_1)$ and $v = G(x, x_2)$. Since both satisfy the same homogeneous boundary conditions, it follows that the right-hand side is zero. In addition, $L(u) = \delta(x - x_1)$ while $L(v) = \delta(x - x_2)$, and thus

$$\int_a^b [G(x, x_1)\delta(x - x_2) - G(x, x_2)\delta(x - x_1)]\ dx = 0.$$

From the fundamental property of the Dirac delta function, it follows that

$$\boxed{G(x_1, x_2) = G(x_2, x_1),} \tag{9.3.39}$$

proving the symmetry from the differential equation defining the Green's function. This symmetry is remarkable; we call it **Maxwell's reciprocity. The response at x due to a concentrated source at x_0 is the same as the response at x_0 due to a concentrated source at x.** This is *not* physically obvious.

Jump conditions. The Green's function $G(x, x_s)$ may be determined from (9.3.36). For $x < x_s$, $G(x, x_s)$ must be a homogeneous solution satisfying the

homogeneous boundary condition at $x = a$. A similar procedure is valid for $x > x_s$. Jump conditions across $x = x_s$ are determined from the singularity in (9.3.36). If $G(x, x_s)$ has a jump discontinuity at $x = x_s$, then dG/dx has a delta function singularity at $x = x_s$, and d^2G/dx^2 would be more singular than the right-hand side of (9.3.36). Thus, **the Green's function $G(x, x_s)$ is continuous at $x = x_s$.** However, dG/dx is not continuous at $x = x_s$; it has a jump discontinuity obtained by integrating (9.3.36) across $x = x_s$ We illustrate this method in the next example and leave further discussion to the exercises.

Example. Consider the solution of the steady-state heat flow problem

$$\boxed{\begin{array}{c} \dfrac{d^2u}{dx^2} = f(x) \\ u(0) = 0 \quad \text{and} \quad u(L) = 0. \end{array}} \tag{9.3.40}$$

We have shown that the solution can be represented in terms of the Green's function:

$$\boxed{u(x) = \int_0^L f(x_0)G(x, x_0)\ dx_0,} \tag{9.3.41}$$

where the Green's function satisfies the following problem:

$$\boxed{\begin{array}{c} \dfrac{d^2G(x, x_0)}{dx^2} = \delta(x - x_0) \\ G(0, x_0) = 0 \quad \text{and} \quad G(L, x_0) = 0. \end{array}} \tag{9.3.42}$$

One reason for defining the Green's function by the differential equation is that it gives an alternative (and often easier) way to calculate the Green's function. Here x_0 is a parameter, representing the position of a concentrated source. For $x \neq x_0$ there are no sources and hence the steady-state heat distribution $G(x, x_0)$ must be linear ($d^2G/dx^2 = 0$):

$$G(x, x_0) = \begin{cases} a + bx & x < x_0 \\ c + dx & x > x_0, \end{cases}$$

but the constants may be different. The boundary condition at $x = 0$ applies for $x < x_0$. $G(0, x_0) = 0$ implies that $a = 0$. Similarly, $G(L, x_0) = 0$ implies that $c + dL = 0$:

$$G(x, x_0) = \begin{cases} bx & x < x_0 \\ d(x - L) & x > x_0. \end{cases}$$

This preliminary result is sketched in Fig. 9.3.4.

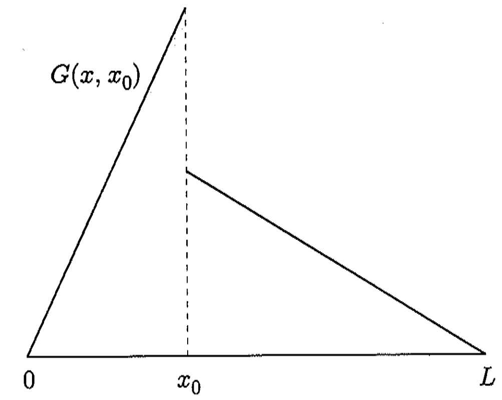

Figure 9.3.4: Green's function before application of jump conditions at $x = x_0$.

The two remaining constants are determined by two conditions at $x = x_0$. The temperature $G(x, x_0)$ must be continuous at $x = x_0$,

$$\boxed{G(x_0-, x_0) = G(x_0+, x_0),} \tag{9.3.43}$$

and there is a jump in the derivative of $G(x, x_0)$, most easily derived by integrating the defining differential equation (9.3.42) from $x = x_0-$ to $x = x_0+$:

$$\boxed{-\left.\frac{dG}{dx}\right|_{x=x_0+} + \left.\frac{dG}{dx}\right|_{x=x_0-} = -1.} \tag{9.3.44}$$

Equation (9.3.43) implies that

$$bx_0 = d(x_0 - L),$$

while (9.3.44) yields

$$b - d = -1.$$

By solving these simultaneously, we obtain

$$d = \frac{x_0}{L} \quad \text{and} \quad b = \frac{x_0 - L}{L},$$

and thus

$$\boxed{G(x, x_0) = \begin{cases} -\dfrac{x}{L}(L - x_0) & x < x_0 \\ -\dfrac{x_0}{L}(L - x) & x > x_0, \end{cases}} \tag{9.3.45}$$

agreeing with (9.3.16). We sketch the Green's function in Fig. 9.3.5. The negative nature of this Green's function is due to the negative concentrated source of thermal energy, $-\delta(x - x_0)$, since $0 = d^2G/dx^2(x, 0) - \delta(x - x_0)$.

The symmetry of the Green's function (proved earlier) is apparent in all representations we have obtained. For example, letting $L = 1$,

$$G(x, x_0) = \begin{cases} -x(1 - x_0) & x < x_0 \\ -x_0(1 - x) & x > x_0 \end{cases} \quad \text{and } G\left(\frac{1}{2}, \frac{1}{5}\right) = G\left(\frac{1}{5}, \frac{1}{2}\right) = -\frac{1}{10}.$$

We sketch $G(x, \frac{1}{5})$ and $G(x, \frac{1}{2})$ in Fig. 9.3.6. Their equality cannot be explained by simple physical symmetries.

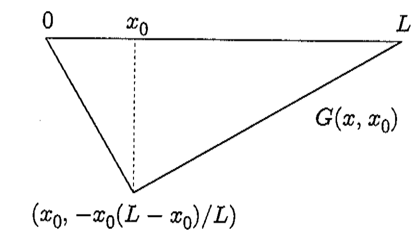

Figure 9.3.5: Green's function.

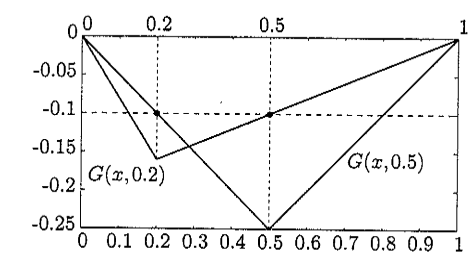

Figure 9.3.6: Illustration of Maxwell's reciprocity.

9.3.5 Nonhomogeneous Boundary Conditions

We have shown how to use Green's functions to solve nonhomogeneous differential equations with homogeneous boundary conditions. In this subsection we extend these ideas to include problems with nonhomogeneous boundary conditions:

$$\boxed{\frac{d^2u}{dx^2} = f(x)} \tag{9.3.46}$$

$$\boxed{u(0) = \alpha \quad \text{and} \quad u(L) = \beta.} \tag{9.3.47}$$

We will use the *same* Green's function as we did previously with problems with homogeneous boundary conditions:

$$\boxed{\frac{d^2G}{dx^2} = \delta(x - x_0)} \tag{9.3.48}$$

$$\boxed{G(0, x_0) = 0 \quad \text{and} \quad G(L, x_0) = 0;} \tag{9.3.49}$$

the Green's function always satisfies the related homogeneous boundary conditions.

To obtain the representation of the solution of (9.3.46) with (9.3.47) involving the Green's function, we again utilize Green's formula, with $v = G(x, x_0)$:

$$\int_0^L \left[u(x)\frac{d^2G(x,x_0)}{dx^2} - G(x,x_0)\frac{d^2u}{dx^2}\right] dx = u\frac{dG(x,x_0)}{dx} - G(x,x_0)\frac{du}{dx}\Big|_0^L.$$

The right-hand side now does not vanish since $u(x)$ does not satisfy homogeneous boundary conditions. Instead, using only the definitions of our problem (9.3.46)-(9.3.47) and the Green's function (9.3.48)-(9.3.49), we obtain

$$\int_0^L [u(x)\delta(x - x_0) - G(x,x_0)f(x)]\, dx = u(L)\frac{dG(x,x_0)}{dx}\Big|_{x=L} - u(0)\frac{dG(x,x_0)}{dx}\Big|_{x=0}.$$

We analyze this as before. Using the property of the Dirac delta function (and reversing the roles of x and x_0) and using the symmetry of the Green's function, we obtain

$$\boxed{u(x) = \int_0^L f(x_0)G(x,x_0)\, dx_0 + \beta\frac{dG(x,x_0)}{dx_0}\Big|_{x_0=L} - \alpha\frac{dG(x,x_0)}{dx_0}\Big|_{x_0=0}.} \tag{9.3.50}$$

This is a representation of the solution of our nonhomogeneous problem (including nonhomogeneous boundary conditions) in terms of the standard Green's function. We must be careful in evaluating the boundary terms. In our problem, we have already shown that

$$G(x,x_0) = \begin{cases} -\frac{x}{L}(L - x_0) & x < x_0 \\ -\frac{x_0}{L}(L - x) & x > x_0. \end{cases}$$

The derivative with respect to the source position of the Green's function is thus

$$\frac{dG(x,x_0)}{dx_0} = \begin{cases} \frac{x}{L} & x < x_0 \\ -\left(1 - \frac{x}{L}\right) & x > x_0. \end{cases}$$

Evaluating this at the end points yields

$$\frac{dG(x,x_0)}{dx_0}\Big|_{x_0=L} = \frac{x}{L} \quad \text{and} \quad \frac{dG(x,x_0)}{dx_0}\Big|_{x_0=0} = -\left(1 - \frac{x}{L}\right).$$

Consequently,

$$u(x) = \int_0^L f(x_0)G(x,x_0)\, dx_0 + \beta\frac{x}{L} + \alpha\left(1 - \frac{x}{L}\right). \tag{9.3.51}$$

The solution is the sum of a particular solution of (9.3.46) satisfying homogeneous boundary conditions obtained earlier, $\int_0^L f(x_0)G(x,x_0)\, dx_0$, and a homogeneous solution satisfying the two required nonhomogeneous boundary conditions, $\beta(x/L) + \alpha(1 - x/L)$.

9.3.6 Summary

We have described three fundamental methods to obtain Green's functions:

1. Variation of parameters
2. Method of eigenfunction expansion
3. Using the defining differential equation for the Green's function

In addition, steady-state Green's functions can be obtained as the limit as $t \to \infty$ of the solution with steady sources. To obtain Green's functions for partial differential equations we will discuss one important additional method. It will be described in Sec. 9.5.

EXERCISES 9.3

9.3.1. The Green's function for (9.3.1) is given explicitly by (9.3.16). The method of eigenfunction expansion yields (9.3.6). Show that the Fourier sine series of (9.3.16) yields (9.3.6).

9.3.2. (a) Derive (9.3.17).

(b) Integrate (9.3.17) by parts to derive (9.3.16).

(c) Instead of part (b), simplify the double integral in (9.3.17) by interchanging the orders of integration. Derive (9.3.16) this way.

9.3.3. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t)$$

subject to $u(0,t) = 0$, $\frac{\partial u}{\partial x}(L,t) = 0$, and $u(x,0) = g(x)$.

(a) Solve by the method of eigenfunction expansion.

(b) Determine the Green's function for this time-dependent problem.

(c) If $Q(x,t) = Q(x)$, take the limit as $t \to \infty$ of part (b) in order to determine the Green's function for

$$\frac{d^2u}{dx^2} = f(x) \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0.$$

9.3.4. (a) Derive (9.3.28) from (9.3.27) [*Hint:* Let $f(x) = 1$.]

(b) Show that (9.3.32) satisfies (9.3.30).

(c) Derive (9.3.29) [*Hint:* Show for any continuous $f(x)$ that

$$\int_{-\infty}^{\infty} f(x_0)\delta(x - x_0)\, dx_0 = \int_{-\infty}^{\infty} f(x_0)\delta(x_0 - x)\, dx_0$$

by letting $x_0 - x = s$ in the integral on the right.]

(d) Derive (9.3.33) [*Hint:* Evaluate $\int_{-\infty}^{\infty} f(x)\delta[c(x - x_0)]\, dx$ by making the change of variables $y = c(x - x_0)$.]

9.3.5. Consider

$$\frac{d^2u}{dx^2} = f(x) \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0.$$

*(a) Solve by direct integration.

*(b) Solve by the method of variation of parameters.

*(c) Determine $G(x, x_0)$ so that (9.3.15) is valid.

(d) Solve by the method of eigenfunction expansion. Show that $G(x, x_0)$ is given by (9.3.23).

9.3.6. Consider

$$\frac{d^2G}{dx^2} = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0 \quad \text{and} \quad \frac{dG}{dx}(L, x_0) = 0.$$

*(a) Solve directly.

*(b) Graphically illustrate $G(x, x_0) = G(x_0, x)$.

(c) Compare to Exercise 9.3.5.

9.3.7. Redo Exercise 9.3.5 with the following change: $\frac{du}{dx}(L) + hu(L) = 0,\ h > 0.$

9.3.8. Redo Exercise 9.3.6 with the following change: $\frac{du}{dx}(L) + hu(L) = 0,\ h > 0.$

9.3.9. Consider

$$\frac{d^2u}{dx^2} + u = f(x) \quad \text{with} \quad u(0) = 0 \quad \text{and} \quad u(L) = 0.$$

Assume that $(n\pi/L)^2 \neq 1$ (i.e., $L \neq n\pi$ for any n).

(a) Solve by the method of variation of parameters.

*(b) Determine the Green's function so that $u(x)$ may be represented in terms of it [see (9.3.15)].

9.3.10. Solve the problem of Exercise 9.3.9 using the method of eigenfunction expansion.

9.3.11. Consider

$$\frac{d^2G}{dx^2} + G = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0 \quad \text{and} \quad G(L, x_0) = 0.$$

*(a) Solve for this Green's function directly. Why is it necessary to assume that $L \neq n\pi$?

(b) Show that $G(x, x_0) = G(x_0, x)$.

9.3.12. For the following problems, determine a representation of the solution in terms of the Green's function. Show that the nonhomogeneous boundary conditions can also be understood using homogeneous solutions of the differential equation:

(a) $\frac{d^2u}{dx^2} = f(x),\ u(0) = A,\ \frac{du}{dx}(L) = B.$ (See Exercise 9.3.6.)

(b) $\frac{d^2u}{dx^2} + u = f(x),\ u(0) = A,\ u(L) = B.$ Assume $L \neq n\pi$. (See Exercise 9.3.11.)

(c) $\frac{d^2u}{dx^2} = f(x),\ u(0) = A,\ \frac{du}{dx}(L) + hu(L) = 0.$ (See Exercise 9.3.8.)

9.3.13. Consider the one-dimensional infinite space wave equation with a periodic source of frequency ω:

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2} + g(x)e^{-i\omega t}. \tag{9.3.52}$$

(a) Show that a particular solution $\phi = u(x)e^{-i\omega t}$ of (9.3.52) is obtained if u satisfies a nonhomogeneous Helmholtz equation

$$\frac{d^2u}{dx^2} + k^2u = f(x).$$

*(b) The Green's function $G(x, x_0)$ satisfies

$$\frac{d^2G}{dx^2} + k^2G = \delta(x - x_0).$$

Determine this infinite space Green's function so that the corresponding $\phi(x,t)$ is an outward propagating wave.

(c) Determine a particular solution of (9.3.52) above.

9.3.14. Consider $L(u) = f(x)$ with $L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$. Assume that the appropriate Green's function exists. Determine the representation of $u(x)$ in terms of the Green's function if the boundary conditions are nonhomogeneous:

(a) $u(0) = \alpha$ and $u(L) = \beta$

(b) $\frac{du}{dx}(0) = \alpha$ and $\frac{du}{dx}(L) = \beta$

(c) $u(0) = \alpha$ and $\frac{du}{dx}(L) = \beta$

*(d) $u(0) = \alpha$ and $\frac{du}{dx}(L) + hu(L) = \beta$

9.3.15. Consider $L(G) = \delta(x - x_0)$ with $L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$ subject to the boundary conditions $G(0, x_0) = 0$ and $G(L, x_0) = 0$. Introduce for all x two homogeneous solutions, y_1 and y_2, such that each solves one of the homogeneous boundary conditions:

$$\begin{array}{ll} L(y_1) = 0 & L(y_2) = 0 \\ y_1(0) = 0 & y_2(L) = 0 \\ \frac{dy_1}{dx}(0) = 1 & \frac{dy_2}{dx}(L) = 1. \end{array}$$

Even if y_1 and y_2 cannot be explicitly obtained, they can be easily calculated numerically on a computer as two *initial value problems.* Any homogeneous solution must be a linear combination of the two.

*(a) Solve for $G(x, x_0)$ in terms of $y_1(x)$ and $y_2(x)$. You may assume that $y_1(x) \neq cy_2(x)$.

(b) What goes wrong if $y_1(x) = cy_2(x)$ for all x and why?

9.3.16. Reconsider (9.3.40), whose solution we have obtained, (9.3.45). For (9.3.40) what is y_1 and y_2 in Exercise 9.3.15? Show that $G(x, x_0)$ obtained in Exercise 9.3.15 reduces to (9.3.45) for (9.3.40).

9.3.17. Consider

$$L(u) = f(x) \quad \text{with} \quad L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

$$u(0) = 0 \quad \text{and} \quad u(L) = 0.$$

Introduce two homogeneous solutions y_1 and y_2, as in Exercise 9.3.15.

(a) Determine $u(x)$ using the method of variation of parameters.

(b) Determine the Green's function from part (a).

(c) Compare to Exercise 9.3.15.

9.3.18. Reconsider Exercise 9.3.17. Determine $u(x)$ by the method of eigenfunction expansion. Show that the Green's function satisfies (9.3.23).

9.3.19. (a) If a concentrated source is placed at a node of some mode (eigenfunction), show that the amplitude of the response of that mode is zero. [*Hint:* Use the result of the method of eigenfunction expansion and recall that a node x^* of an eigenfunction means anyplace where $\phi_n(x^*) = 0$.]

(b) If the eigenfunctions are $\sin n\pi x/L$ and the source is located in the middle, $x_0 = L/2$, show that the response will have no even harmonics.

9.3.20. Derive the eigenfunction expansion of the Green's function (9.3.23) directly from the defining differential equation (9.3.40) by letting

$$G(x, x_0) = \sum_{n=1}^{\infty} a_n \phi_n(x).$$

Assume that term-by-term differentiation is justified.

***9.3.21.** Solve

$$\frac{dG}{dx} = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0.$$

Show that $G(x, x_0)$ is not symmetric even though $\delta(x - x_0)$ is.

9.3.22. Solve

$$\frac{dG}{dx} + G = \delta(x - x_0) \quad \text{with} \quad G(0, x_0) = 0.$$

Show that $G(x, x_0)$ is not symmetric even though $\delta(x - x_0)$ is.

9.3.23. Solve

$$\frac{d^4G}{dx^4} = \delta(x - x_0)$$

$$G(0, x_0) = 0 \qquad G(L, x_0) = 0$$

$$\frac{dG}{dx}(0, x_0) = 0 \qquad \frac{d^2G}{dx^2}(L, x_0) = 0.$$

9.3.24. Use Exercise 9.3.23 to solve

$$\frac{d^4u}{dx^4} = f(x)$$

$$u(0) = 0 \qquad u(L) = 0$$

$$\frac{du}{dx}(0) = 0 \qquad \frac{d^2u}{dx^2}(L) = 0$$

(*Hint:* Exercise 5.5.8 is helpful.)

9.3.25. Use the convolution theorem for Laplace transforms to obtain particular solutions of

(a) $\frac{d^2u}{dx^2} = f(x)$. (See Exercise 9.3.5.)

*(b) $\frac{d^4u}{dx^4} = f(x)$. (See Exercise 9.3.24.)

Appendix to 9.3: Establishing Green's Formula with Dirac Delta Functions

Green's formula is very important when analyzing Green's functions. However, our derivation of Green's formula requires integration by parts. Here we will show that Green's formula,

$$\int_a^b [uL(v) - vL(u)]dx = p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b, \text{ where } L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q \quad (9.3.53)$$

is valid even if v is a Green's function,

$$L(v) = \delta(x - x_0). \quad (9.3.54)$$

We will derive (9.3.53). We calculate the left-hand side of(9.3.53). Since there is a singularity at $x = x_0$, we are not guaranteed that (9.3.53) is valid. Instead, we divide the region into three parts:

$$\int_a^b = \int_a^{x_0-} + \int_{x_0-}^{x_0+} + \int_{x_0+}^b .$$

In the regions that exclude the singularity, $a \le x \le x_{0-}$ and $x_{0+} \le x \le b$, Green's formula can be used. In addition, due to the property of the Dirac delta function

$$\int_{x_0-}^{x_0+} [uL(v) - vL(u)]\ dx = \int_{x_0-}^{x_0+} [u\delta(x-x_0) - vL(u)]\ dx = u(x_0),$$

since $\int_{x_0-}^{x_0+} vL(u)\ dx = 0$. Thus, we obtain

$$\begin{aligned}\int_a^b [uL(v) - vL(u)]\ dx &= p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^{x_0-} + p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x_0+}^b + u(x_0)\\ &= p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b + p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x_0+}^{x_0-} + u(x_0).\end{aligned} \quad (9.3.55)$$

Since u, du/dx, and v are continuous at $x = x_0$; it follows that

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_{x_0+}^{x_0-} = p(x_0)u(x_0)\frac{dv}{dx}\Big|_{x_0+}^{x_0-}.$$

However, by integrating (9.3.54), we know that $p\ dv/dx|_{x_0-}^{x_0+} = 1$. Thus, (9.3.53) follows from (9.3.55). Green's formula may be utilized even if Green's functions are present.

9.4 Fredholm Alternative and Modified Green's Functions

9.4.1 Introduction

If $\lambda = 0$ is an eigenvalue, then the Green's function does not exist. In order to understand the difficulty, we reexamine the nonhomogeneous problem:

$$L(u) = f(x), \quad (9.4.1)$$

subject to homogeneous boundary conditions. By the method of eigenfunction expansion, in the preceding section we obtained

$$u = \sum_{n=1}^{\infty} a_n\phi_n(x), \quad (9.4.2)$$

where by substitution

$$-a_n\lambda_n = \frac{\int_a^b f(x)\phi_n(x)\ dx}{\int_a^b \phi_n^2\sigma\ dx}. \quad (9.4.3)$$

If $\lambda_n = 0$ (for some n, often the lowest eigenvalue), there may not be any solutions to the nonhomogeneous boundary value problem. In particular, if $\int_a^b f(x)\phi_n(x)\ dx \ne 0$, for the eigenfunction corresponding to $\lambda_n = 0$ then (9.4.3) cannot be satisfied. This warrants further explanation.

Example. Let us consider the following simple nonhomogeneous boundary value problem:

$$\frac{d^2u}{dx^2} = e^x \quad \text{with} \quad \frac{du}{dx}(0) = 0 \quad \text{and} \quad \frac{du}{dx}(L) = 0. \quad (9.4.4)$$

We attempt to solve (9.4.4) by integrating:

$$\frac{du}{dx} = e^x + c.$$

The two boundary conditions cannot be satisfied as they are contradictory:

$$\begin{aligned}0 &= 1 + c\\ 0 &= e^L + c.\end{aligned}$$

There is no guarantee that there are *any* solutions to a nonhomogeneous boundary value problem when $\lambda = 0$ is an eigenvalue for the related eigenvalue problem $[d^2\phi_n/dx^2 = -\lambda_n\phi_n$ with $d\phi_n/dx(0) = 0$ and $d\phi_n/dx(L) = 0]$.

In this example from one physical point of view, we are searching for an equilibrium temperature distribution. Since there are sources and the boundary conditions