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GREEN'S FUNCTIONS

CONSTRUCTION AND APPLICATIONS

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To our family, with love

Preface

The Green's function is one of the classic concepts in the area of differential equations. It has been intensively explored and widely implemented for almost two centuries since its introduction by George Green, a brilliant British mathematician and physicist of the nineteenth century. His elegant elaborations revealed impressive properties of this function that have been transformed into a powerful instrument for use in the qualitative theory of ordinary as well as partial differential equations.

Yet, it would be a mistake to assert that the usability of Green's functions is limited to purely theoretical aspects. As has been corroborated by numerous works in applied mathematics, published in recent decades, Green's functions also possess remarkable computational potential. They have turned out to be extremely helpful in the development of numerical algorithms for solving a vast number of problems involving differential equations, that arise in engineering and in the natural sciences.

An example of the attributes of Green's functions, that illustrate their computational potential, is the solution of a boundary value problem for a partial differential equation. By using Green's functions, the equation can be reduced to an integral equation. Applying numerical techniques to solve the reduced problem will be significantly more economical than applying them directly to the original differential equation.

We consider a number of arguments supporting this assertion: first, integral equations lend themselves to computational methods better, compared to differential equations, because the approximation to an integral is, in most cases, a well-posed problem, which is straightforward to solve. In contrast, the approximation to a differential is not, generally speaking, a quite well-posed problem. Second, procedures based on Green's functions reduce the dimensionality of the boundary-value problem under consideration, by converting it into a boundary integral equation. Thus, all the required numerical work is shifted to, the boundary of the considered region. Finally, purely numerical methods presume an approximate treatment of the governing differential equation as well as the boundary conditions, whereas in Green's function-based procedures, most of these are satisfied exactly, prior to the numerical work.

Taking all of this into account, it is reasonable to expect that Green's functions must be included as an important component in undergraduate courses on differential equations and numerical analysis. However, the importance of this topic is, in our opinion, somewhat underestimated in contemporary textbooks: not all texts on differential equations cover the topic, and even among those that do, it is hard to find textbooks, that are primarily devoted to the construction of Green's functions. With standard works on numerical analysis, things are even worse; none of those texts touches upon the development of Green's function-based numerical methods in dealing with boundary-value problems for either ordinary or partial differential equations. Hence, the situation requires special attention and the status quo in the field has to be revised.

Given the computational potential of Green's functions on the one hand, and the omission of numerical methods based on these functions in standard textbooks on the other hand, a question arises: what is it, that holds back widespread numerical use of Green's functions and causes the range of their efficient applicability to be narrowed? We believe that there is an objective explanation for this phenomenon, namely the unfortunate fact that only a limited number of computer-friendly representations of Green's functions are available in the literature. This contradiction has inspired our strong desire to create the present work. Because it was conceived as a textbook intended to at least partially resolve the conundrum, our book devotes itself to the development of efficient procedures for the construction of computer-friendly Green's functions applicable to a wide array of partial differential equations. This makes our work unique, original and unlike most others in the field.

As to the presentation of the material in the present book, we note that, although the emphasis is not on theoretical aspects, we strived to make our presentation as rigorous as possible. Whilst concentrating on formal issues that are methodologically critical for the construction of Green's functions, we tried not to do this at the expense of rigor. Have we succeeded in this? Authors are not, of course, in a position to judge the success or failure of their endeavor, which is why we look forward to hearing from the reader.

The form in which a Green's function is obtained is very important from the standpoint of applications. We don't doubt that potential users would welcome this function if and only if the form in which it is presented is suitable for immediate computer implementation. However, it appears that only a few classical Green's functions for partial differential equations are obtained, in a closed and ready to use analytical form. Most of the representations of these functions available in the literature are not suited to direct computer use. This severely limits the sphere of their effective application. Our objective in writing this book was to provide the user with as many compact and computer-friendly representations of Green's functions as the format of this textbook permits.

The book covers only a limited number of applied partial differential equations. These include the two-dimensional Laplace, the static Klein–Gordon, the biharmonic-, the diffusion- (heat), and the Black–Scholes equations. Areas of science and engineering where these equations are encountered, range from fluid and solid mechanics to financial engineering. The intention is to not only provide the reader with a review of the classical approaches, that are traditionally used for the construction of Green's functions, but also to introduce our readers to several nontrivial construction techniques and invite them to a challenging research in this productive area of applied mathematics. The authors believe that the present volume can be use as a principal text for elective advanced undergraduate and graduate courses, within the field of applied mathematics and related disciplines.

It is with great pleasure that we express our sincere gratefulness to a number of our colleagues who, made either direct or indirect contributions to this project. Among those, the first author is especially thankful to two brilliant individuals under whose irresistible influence and professional guidance he gained his own experience and expertise in the area of analytical and numerical exploration of Green's functions and matrices. Professors V. D. Kupradze and S. P. Gavelya have been widely recognized advocates for the use of the Green's function method in the theory of elasticity as well as in plate and shell theory. Their encouragement and parting words stressed our hard but very fascinating work on this manual.

Murfreesboro/Lebanon, September 2011

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Chapter 0
Introduction

The scope of this book is, to a large extent, limited to applications of elliptic and parabolic partial differential equations in two variables, although a few three-dimensional settings are also covered in Chapters 2, 3, and 7. These chapters provide the reader with an idea of how the described methods extend to three dimensional problems. The elliptic PDE include the Laplace equation, the static Klein–Gordon equation, and the biharmonic equation. Parabolic PDE are represented by the classical diffusion (heat) equation and the Black–Scholes equation. The latter has emerged as a mathematical model in financial mathematics just a few decades ago [8, 52, 55, 65, 75]. In this work, Green's functions are constructed, for a variety of problems, involving the above equations.

The Green's function formalism is traditionally referred to as one of the most efficient instruments in the field of differential equations [3, 5, 12, 13, 68, 18, 22, 23, 25, 28, 29, 39, 47, 53, 54, 57, 61, 66, 67]. But until recently, it was mostly employed for exploring qualitative aspects of differential equations like the existence and uniqueness of a solution, dependence of the latter on initial data and so on. In a series of publications from recent decades [6, 14, 17, 21, 26, 30, 31, 32, 40, 42, 47, 48, 50, 69, 70], one can find convincing evidence for the efficiency of Green's function-based numerical methods when solving boundary-value problems for ordinary as well as partial differential equations, arising in various areas of engineering and science.

Practicality of a Green's-function-based numerical procedure depends, to a great extent, on the form in which the required Green's function is presented: identical Green's functions, that are constructed with different methods, may appear in different forms. Some of those forms are compact, ready for immediate computer implementation, and simple to use. Others might not necessarily be that computer-friendly. This book reviews various differing methods providing a variety of different forms of Green's functions to be applied to partial differential equations.

Regarding the two-dimensional Laplace equation,

$$\nabla^2 u = 0$$

which finds numerous applications in a variety of areas of engineering and science, we find that some of its Green's functions, available in the current literature, can be stated in closed form (only containing elementary functions). To name a few such forms, we recall the classical [18, 22, 37, 39, 43, 45, 53, 57, 66, 67]

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{(x-\xi)^2 + (y+\eta)^2}{(x-\xi)^2 + (y-\eta)^2} \tag{1}$$

expression of the Green's function used in the Dirichlet problem for the upper half-plane $\{-\infty < x < \infty, 0 < y < \infty\}$.

As another classical examples of a closed compact form of Green's function, one might recall

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{\cosh \omega(x-\xi) - \cos \omega(y+\eta)}{\cosh \omega(x-\xi) - \cos \omega(y-\eta)}, \quad \omega = \frac{\pi}{b}, \tag{2}$$

and

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{\cosh \omega(x+\xi) - \cos \omega(y-\eta)}{\cosh \omega(x-\xi) - \cos \omega(y-\eta)} + \frac{1}{4\pi} \ln \frac{\cosh \omega(x-\xi) - \cos \omega(y+\eta)}{\cosh \omega(x+\xi) - \cos \omega(y+\eta)}, \quad \omega = \frac{\pi}{b}, \tag{3}$$

representing the Green's functions of the Dirichlet problem for the Laplace equation, in an infinite strip $\{-\infty < x < \infty, 0 < y < b\}$ and in the infinite semi-strip $\{0 < x < \infty, 0 < y < b\}$, respectively.

Recall also the following (see, for example, [17, 18, 22, 37, 39, 43, 45, 54, 57, 61, 66]) closed form

$$G(r, \varphi; \varrho, \psi) = \frac{1}{4\pi} \ln \frac{a^4 - 2a^2 r\varrho \cos(\varphi - \psi) + r^2\varrho^2}{a^2\left(r^2 - 2r\varrho\cos(\varphi - \psi) + \varrho^2\right)} \tag{4}$$

that represents the Green's function for the Dirichlet problem for the Laplace equation on a disk $\{0 < r < a, 0 < \varphi \leq 2\pi\}$ of radius a.

From this brief review on the availability of Green's functions for the Laplace equation, one might assume that the situation for the two-dimensional static Klein–Gordon equation

$$\nabla^2 u - k^2 u = 0$$

must be similar. Indeed, it sounds reasonable to expect that a similarity of the two equations might support this assumption. But it appears in reality that this is not quite true. The point is that none of Green's functions for the static Klein–Gordon equation can be expressed in a closed form, in the sense that was explained earlier. This is predetermined by the form of the fundamental solution of this equation, which is not, unlike in the case of the Laplace equation, an elementary function.

In contrast with other existing texts on partial differential equations, the current book provides a unique and extensive list of Green's functions, also for the static Klein–Gordon equation. And – important especially from the point of applicability – all of them are computer-friendly.

Another elliptic partial differential equation that finds numerous applications in engineering and science, and whose Green's functions are under continuous consideration in this book, is the two-dimensional biharmonic equation

$$\nabla^2\nabla^2 w = 0$$

It is traditionally used to simulate phenomena and processes in mechanics of solids (in plate and shell theory, in particular). The number of available in the literature Green's functions for this equation is severely limited. The only known [39, 59] closed form is

$$G(r,\varphi;\varrho,\psi) = \frac{1}{16\pi}\left[\frac{1}{a^2}(a^2-\varrho^2)(a^2-r^2)\right.$$
$$\left. - \left(r^2 - 2r\varrho\cos(\varphi-\psi)+\varrho^2\right)\ln\frac{a^4 - 2a^2 r\varrho\cos(\varphi-\psi)+r^2\varrho^2}{a^2\left(r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2\right)}\right] \quad (5)$$

representing the Green's function for the Dirichlet problem

$$w(a,\varphi) = \frac{\partial w(a,\varphi)}{\partial r} = 0$$

for the biharmonic equation on a disk $\{0 < r < a, 0 < \varphi \leq 2\pi\}$ of radius a.

We will make a strong effort to provide the reader with an exhaustive list of computer-friendly representations of Green's functions for a number of boundary-value problems for the biharmonic equation.

Note that from all the classical applied partial differential equations, the Green's function-based methods appeared to be the most widely applicable to the parabolic equations [13, 17, 29, 37, 46, 57, 66]. A field where these methods are especially popular is continuum mechanics (mass and heat transfer, in particular), with the classical diffusion (heat) equation

$$\frac{\partial u(x,t)}{\partial t} = \kappa\frac{\partial^2 u(x,t)}{\partial x^2} \quad (6)$$

representing the key mathematical instrument.

It is worth noting that the number of closed analytical forms of Green's functions for the heat equation is very limited too. One might recall, for example,

$$G(x,t;\xi,\tau) = \frac{1}{2\sqrt{\kappa\pi(t-\tau)}}\left[\exp\left(-\frac{(x-\xi)^2}{4\kappa(t-\tau)}\right) - \exp\left(-\frac{(x+\xi)^2}{4\kappa(t-\tau)}\right)\right] \quad (7)$$

and

$$G(x,t;\xi,\tau) = \frac{1}{2\sqrt{\kappa\pi(t-\tau)}}\left[\exp\left(-\frac{(x-\xi)^2}{4\kappa(t-\tau)}\right) + \exp\left(-\frac{(x+\xi)^2}{4\kappa(t-\tau)}\right)\right] \quad (8)$$

representing [13, 66] the Green's functions for the first $u(0,t) = 0$ and the second $\partial u(0,t)/\partial x = 0$ initial-boundary value problem, respectively, for equation (6) on the region $\{0 < x < \infty, t > 0\}$.

Another function

$$G(s,t;\xi) = \frac{\exp(-r(T-t))}{\sigma\xi\sqrt{2\pi(T-t)}} \exp\left(-\frac{[\ln(s/\xi) + (r-\sigma^2/2)(T-t)]^2}{2\sigma^2(T-t)}\right) \quad (9)$$

that is frequently [8, 36, 49, 52, 55, 58, 65, 75] used and cited in the field, is the so-called *Green's function of the Black–Scholes equation*

$$\frac{\partial u(s,t)}{\partial t} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 u(s,t)}{\partial s^2} + rs\frac{\partial u(s,t)}{\partial s} - ru(s,t) = 0 \quad (10)$$

which represents an equation with variable coefficients, where s and t are the independent variables, while σ and r represent constant parameters.

Note that (10) represents another parabolic equation, which is widely used in financial mathematics. We would like comment the name, commonly accepted in the field, for the function in (9). Naming it a Green's function is not justified, and we can readily challenge it: a Green's function, by definition, must be associated with a certain problem formulated for a given differential equation. In other words, it must satisfy some imposed uniqueness conditions (initial, boundary, or other). This is of course not the case for the function in (9), since it represents just a special (parameter ξ containing) solution of the Black–Scholes equation, and is not explicitly associated with any uniqueness conditions.

Taking the above arguments into account, one must refer to (9) as the *fundamental solution* to (10). We might, nevertheless, offer a reasonable compromise calling (9) the *Green's function of the Black–Scholes equation for the entire feasible space* $\{0 < s < \infty, -\infty < t < T\}$ of the independent variables s and t.

Later in this text, the reader will find a number of actual Green's functions constructed for a few particular terminal-boundary value problems posed for the Black–Scholes equation. Many of those have recently been reported and published in [49], but have not yet been in a book format.

As to the representations in (1) through (9), it is hard to argue that they are compact enough and convenient to work with in numerical applications. It is worth noting however, that unfortunately there exist only a few such closed analytical forms of Green's functions in the standard textbooks on applied partial differential equations.

Some other Green's functions available in the literature for application to partial differential equations are expressed in a form containing elementary functions and series components. As an example of such a form, we note the following function

$$G(r,\varphi;\varrho,\psi) = \frac{1}{4\pi}\left[\frac{2}{a\beta} - 4a\beta\sum_{n=1}^{\infty}\frac{1}{n(n+a\beta)}\left(\frac{r\varrho}{a^2}\right)^n \cos n(\varphi-\psi) \right. \quad (11)$$

$$\left. - \ln\left(\frac{1}{a^3}\left(r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2\right)\left(a^4 - 2a^2 r\varrho\cos(\varphi-\psi) + r^2\varrho^2\right)\right)\right].$$

It represents the Green's function for the mixed boundary-value problem

$$\left(\frac{\partial}{\partial r}+\beta\right)u(a,\varphi)=0, \quad \beta>0,$$

for the Laplace equation on a disk $\{0<r<a, 0<\varphi\le 2\pi\}$ of radius a. This Green's function can be found in [43] and [45].

Another example of a series-containing form is

$$\begin{aligned}G(r,\varphi;\varrho,\psi)=&\frac{1}{16\pi}\Bigg\{\frac{1}{a^2}(a^2-\varrho^2)(a^2-r^2)\\ &\times\Bigg[\frac{3+\sigma}{1+\sigma}-\ln\left(1-2\frac{r\varrho}{a^2}\cos(\varphi-\psi)+\left(\frac{r\varrho}{a^2}\right)^2\right)\\ &-2h\sum_{n=1}^{\infty}\frac{1}{n(n+h)}\left(\frac{r\varrho}{a^2}\right)^n\cos n(\varphi-\psi)\Bigg]\\ &-\left(r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2\right)\ln\frac{a^4-2a^2r\varrho\cos(\varphi-\psi)+r^2\varrho^2}{a^2\left(r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2\right)}\Bigg\}\end{aligned} \tag{12}$$

representing the Green's function (see [45, 47]) for the boundary-value problem

$$w(a,\varphi)=0, \text{and } \left(\frac{\partial^2}{\partial r^2}+\frac{\sigma}{a}\left(\frac{\partial}{\partial r}+\frac{1}{a}\frac{\partial^2}{\partial\varphi^2}\right)\right)w(a,\varphi)=0$$

for the biharmonic equation on a disk $\{0<r<a, 0<\varphi\le 2\pi\}$ of radius a. Here σ is a constant parameter, while h is expressed in terms of σ as $h=(1+\sigma)/2$.

In some other cases, Green's functions for applied partial differential equations are expressed series form, for example the classical [17, 22, 29, 41, 45, 48, 57] representation

$$G(x,y;\xi,\eta)=\frac{4}{ab}\sum_{m,n=1}^{\infty}\frac{\sin\mu x\sin\nu y\sin\mu\xi\sin\nu\eta}{\mu^2+\nu^2} \tag{13}$$

of the Green's function for the Dirichlet problem posed for the Laplace equation on the rectangle $\{0<x<a, 0<y<b\}$.

Another series-only form [45, 47, 72]

$$G(x,y;\xi,\eta)=-\frac{4}{ab}\sum_{m,n=1}^{\infty}\frac{\sin\mu x\sin\nu y\sin\mu\xi\sin\nu\eta}{\left(\mu^2+\nu^2\right)^2} \tag{14}$$

represents the Green's function for the boundary-problem

$$w(0,y)=\frac{\partial^2 w(0,y)}{\partial x^2}=w(a,y)=\frac{\partial^2 w(a,y)}{\partial x^2}=0$$

and

$$w(x,0) = \frac{\partial^2 w(x,0)}{\partial y^2} = w(x,b) = \frac{\partial^2 w(x,b)}{\partial y^2} = 0$$

posed for the biharmonic equation on the rectangle $\{0 < x < a, 0 < y < b\}$. Note that the parameters μ and ν in (13) and (14) read as

$$\mu = \frac{m\pi}{a} \quad \text{and} \quad \nu = \frac{n\pi}{b}$$

in terms of the dimensions a and b of the rectangle, and the summation indices m and n of the series.

The series form

$$G(x,t;\xi,\tau) = \frac{1}{2\sqrt{\kappa\pi(t-\tau)}} \times \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x-\xi+2na)^2}{4\kappa(t-\tau)}\right) - \exp\left(-\frac{(x+\xi+2na)^2}{4\kappa(t-\tau)}\right) \right] \tag{15}$$

represents the Green's function of the problem

$$u(0,t) = u(a,t) = 0$$

for the diffusion equation in the region $\{0 < x < a, t > 0\}$.

The representations in (11) through (15) are certainly not as computer-ready as those in (1) through (9). Indeed, the convergence of their series components is a critical issue, that must be considered before practical use. In most cases a certain regularization is required in order to adjust a series-containing Green's function to a form appropriate for an immediate computer implementation.

This book is primarily intended as a graduate/undergraduate text. However, it also aims at researchers working in fields of engineering and science involving a practical solution of partial differential equations. To make this textbook attractive for practical needs, it must contain easily computable or computer-friendly representations of Green's functions. With this in mind, our objective in this book is threefold. First, we want to include herein as many Green's functions, that are already available in the standard textbooks and handbooks, as possible. Second, our goal is to maximally extend that list of available Green's functions with as many nontrivial ones as possible. Third, we aim at providing the reader with some traditional and innovative approaches, that might help in constructing new Green's functions that are not yet available.

We believe that with all the objectives attained, our book will be of use to the indicated groups of audience. As authors, we are committed to providing the reader with the most informative and complete source of reference material required for the construction of Green's functions.

To guide the reader through the book, we briefly will review its organization. Chapter 1 is preparatory in nature and deals with classical procedures traditionally employed in the construction of Green's functions for boundary-value problems for higher order linear ordinary differential equations. The chapter lays out a methodological background for a significant part of our further work on partial differential equations. A construction procedure based on the defining properties of Green's functions is analyzed in detail in Section 1.1. A special symmetry feature of Green's functions and its role in this volume is discussed in Section 1.2. An alternative procedure for the construction of Green's functions is described in Section 1.3. It is based on a corollary of the second Green's formula and the method of variation of parameters.

Chapter 2 is devoted to the Laplace equation. The first two sections review trivial methods, that are traditionally used for the construction of Green's functions for this equation in two dimensions. These are the methods of images and conformal mapping. A number of classical Green's functions are derived with the aid of these methods. In Section 2.3, our focus is switched to another method that is traditional in the field, namely the eigenfunction expansion, producing a variety of computer-friendly representations of Green's functions for boundary-value problems formulated in Cartesian and polar coordinates, and in geographical coordinates on surfaces of revolution. It is noted that the first two methods covered in this chapter are of no use in most of the problems considered in Section 2.3. Section 2.4 deals with a number of three-dimensional problems for the Laplace equation.

The static Klein–Gordon equation is a topic that is dealt with in Chapter 3. Note that this equation is rarely included in standard textbooks. Initially, we define the Green's function for this equation and focus the reader's attention on the type of singularity that this function possesses. Section 3.2 then, analyzes those boundary-value problems for the Klein–Gordon equation, for which the method of images appears efficient, while in Section 3.3 the focus is on the method of eigenfunction expansion. Many of the Green's functions obtained in this chapter have so far never been exposed in book format. In Section 3.4, a few three-dimensional problems are considered, with their Green's functions obtained in a computer-friendly form.

Higher order two-dimensional elliptic partial differential equations, which find their application in structural mechanics, are the topic of Chapter 4. After introducing the notion of Green's functions for such equations, our focus is, in Sections 4.2 and 4.3, on problem settings for the biharmonic equation in Cartesian and polar coordinates. Section 4.4 is devoted to the biharmonic type fourth-order equation

$$\nabla^2\nabla^2 w + \lambda^4 w = 0$$

which is commonly used in structural mechanics [47, 72] to simulate the stress-strain state of thin plates resting on an elastic foundation. In Section 4.5, an algorithm is sketched for the construction of Green's matrices for several elliptic systems, that simulate different problem settings in the thin shells theory [72].

In Chapter 5 we return to linear ordinary differential equations. However, we turn our focus from the classical material covered in Chapter 1 on the extension of the Green's function formalism to special sets of ordinary differential equations with discontinuous coefficients. This will lead us to the notion of *matrices of Green's type* in Section 5.1, as a consequence of that extension, and we will construct a number of such matrices. In Section 5.2, we will illustrate the practical application of the extended formalism, by applying it to the analysis of the stress-strain state of multi-span beams. In Section 5.3, the Green's matrix formalism is further extended to sets of ordinary differential equations on graphs.

The matrix of Green's type notion, introduced in Chapter 5, is further extended in Chapter 6, where it is employed to treat special sets of two-dimensional elliptic partial differential equations. After a brief introduction in Section 6.1, we describe the construction of Green's matrices for such sets in detail in Section 6.2. Section 6.3 is devoted to a specific application, where fields of potential generated by point sources are analyzed on surface structures consisting of cylindrical, spherical and toroidal fragments.

In Chapter 7, a classical parabolic partial differential equation is brought forward. Green's functions and matrices of Green's type are constructed for a variety of initial-boundary-value problems for the diffusion (heat) equation. The construction procedure is based on a combination of the integral Laplace transform with the techniques described in detail in Chapters 2 through 6 for elliptic partial differential equations and their systems. To aid ease-of reading and comprehension of the material in Chapter 7 we present a brief review of the basics of the Laplace transform, in Section 7.1. In Section 7.2, the required Green's functions are defined and constructed. Construction of matrices of Green's type for compound regions is described in Section 7.3.

Chapter 8 deals with another parabolic partial differential equation, namely the Black–Scholes equation which was mentioned before in (10) of this introductory section. In Section 8.1 we focus on the fundamental solution of this equation, which is generally referred to in literature as the *Green's function for the Black–Scholes equation*, and offer some terminological comments in this regard. In Section 8.2, we construct a number of Green's functions for a variety of terminal-boundary-value problems.We discuss a specific methodological issue in Section 8.3, whilst Section 8.4 illustrates some computational features of Green's functions of the Black–Scholes equation.

Since one of the most important features of this book is its practical value, a great deal of effort was made to enhance it. To increase both its readability and its comprehensibility for a broader audience, each book's section is accompanied by relevant examples, illustrating the text. In addition, a carefully designed set of review exercises accompanies every chapter. Answers and helpful hints to most of the review exercises can be found in the Appendix.

Chapter 1
Green's Functions for ODE

The primary interest of this textbook are applications of linear partial differential equations. We will consider a variety of elliptic and parabolic equations. However, the present chapter steers clear of, the book's main topic, and deals with ordinary differential equations. The focus is on Green's functions of boundary-value-problems for higher-order linear equations. We opted for this, for various reasons. First of all, these functions represent an important instrument in the qualitative as well as quantitative analysis of ODE. Second, standard textbooks within the field generally do not cover the topic in sufficient detail. This is especially true for the construction Green's functions. Finally, Green's functions for ODE play a significant role in some of the procedures that we later offer for PDE.

We discuss two alternative techniques, traditionally applied to the construction of Green's functions for linear ODE. One of these techniques, which is our focus in Section 1.1, follows from a proof of the existence and uniqueness theorem for the Green's function. The proof is constructive in nature and uses the defining properties of the Green's function. We discuss the second technique in Sections 1.2 and 1.3. This technique based on the fact that the solution of an inhomogeneous equation can be written in terms of the Green's function for the corresponding homogeneous equation. The intricacies of both techniques, when applied to various types of problem settings, are illustrated by numerous examples.

1.1 Standard Procedure for Construction

In this section, we introduce the notion of Green's function by means of a boundary-value-problem for an ordinary nth order linear differential equation. We then provide a detailed description of the classical method for the construction of Green's functions, based on their defining properties. A number of examples in this section illustrate different aspects of the construction procedure.

Our discussion is concerned with the linear homogeneous boundary-value problem

$$L[y(x)] \equiv p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n(x)y = 0, \tag{1.1}$$

$$M_i(y(a), y(b)) \equiv \sum_{k=0}^{n-1}\left(\alpha_k^i \frac{d^k y(a)}{dx^k} + \beta_k^i \frac{d^k y(b)}{dx^k}\right) = 0, \quad i = \overline{1, n}, \tag{1.2}$$

which is assumed well-posed on the interval (a, b). The coefficients of the equation $p_i(x)$, $i = \overline{0, n}$, are continuous functions on (a, b), where the leading coefficient $p_0(x)$ must be non-zero in all points in (a, b).

A special explanation is appropriate with respect to the abbreviated form of the boundary conditions in (1.2). M_i, $i = \overline{1, n}$, with constant coefficients α_k^i and β_k^i, are presumed to be linearly independent. This restricts the total number of boundary conditions to n, ensuring that $y \equiv 0$ is the only solution for the problem in (1.1) and (1.2). Note that the superscript i on the coefficients α_k^i and β_k^i is the boundary condition number and not an exponent.

The relations in (1.2) are written in a *two-point form.* This means that some of them may include both the endpoints a and b of the interval. However, if a certain boundary condition in the setting is given in a *single-point* form – i.e. it is imposed for say, a only – then all the coefficients β_k^i in (1.2) are zero, whereas at least one of the coefficients α_k^i is not. Similarly, if a certain condition is imposed for b only, then all the coefficients α_k^i are zero, while at least one of the coefficients β_k^i is not.

At this point, we introduce the classical [20, 22, 29, 37, 57, 66] definition of the *Green's function* for the homogeneous boundary-value-problem that appears in (1.1) and (1.2). Theorems about its existence and uniqueness will be formulated and proven later, providing a method for its construction.

Definition. The function $g(x, s)$ is said to be the *Green's function* for the boundary-value problem in (1.1) and (1.2), if, as a function of its first variable x, it meets the following defining criteria, for any $s \in (a, b)$:

1. On both intervals $[a, s)$ and $(s, b]$, $g(x, s)$ is a continuous function having continuous derivatives up to nth order, and satisfies the governing equation in (1.1) on (a, s) and (s, b), i.e.:

$$L[g(x, s)] = 0, x \in (a, s); \quad L[g(x, s)] = 0, \ x \in (s, b).$$

2. For $x = s$, $g(x, s)$ and all its derivatives up to $(n-2)$ are continuous

$$\lim_{x \to s+} \frac{\partial^k g(x, s)}{\partial x^k} - \lim_{x \to s-} \frac{\partial^k g(x, s)}{\partial x^k} = 0, \quad k = \overline{0, n-2}.$$

3. The $(n-1)$th derivative of $g(x, s)$ is discontinuous when $x = s$, providing

$$\lim_{x \to s+} \frac{\partial^{n-1} g(x, s)}{\partial x^{n-1}} - \lim_{x \to s-} \frac{\partial^{n-1} g(x, s)}{\partial x^{n-1}} = -\frac{1}{p_0(s)}$$

with $p_0(s)$ the leading coefficient in (1.1).

4. $g(x, s)$ satisfies the boundary conditions in (1.2), i.e.:

$$M_i(g(a, s), g(b, s)) = 0, \quad i = \overline{1, n}.$$

The arguments x and s in the Green's function are conventionally referred to as the *observation* (*field*) *point* and the *source point*, respectively. The following theorem specifies the conditions for existence and uniqueness of the Green's function.

Theorem 1.1 (existence and uniqueness). *If the homogeneous boundary-value problem in* (1.1) *and* (1.2) *has only a trivial solution, then there exists an unique Green's function* $g(x,s)$ *associated with the problem.*

We suggest the reader to read through the proof of this theorem carefully. The important point is that this is a constructive proof, which implies that it delivers a straightforward algorithm for the actual construction of Green's functions. Throughout the present textbook, we will frequently apply this algorithm to a variety of problems.

Proof. Let the functions $y_j(x)$, $j = \overline{1,n}$, be a fundamental set of solutions for (1.1). That is, $y_j(x)$ are particular solutions of (1.1), linearly independent on (a,b)

In numerous practical situations, one can find the fundamental set of solutions for (1.1) analytically. This can, be done for equations with constant coefficients in particular. If, however, the governing differential equation does not allow an analytical solution, appropriate numerical procedures may be employed for obtaining approximate solutions. Later in this book we will discuss this point in more detail.

In compliance with property 1 of the definition, for any arbitrarily fixed value of $s \in (a,b)$, the Green's function $g(x,s)$ has to be a solution of (1.1) in (a,s) (to the left of s), as well as in (s,b) (to the right of s). As soon as $y_j(x)$, $j = \overline{1,n}$, constitutes a fundamental set of solutions for (1.1), any one of its solutions can be expressed as a linear combination of the components $y_j(x)$. Consequently, one may write $g(x,s)$ in the following form

$$g(x,s) = \sum_{j=1}^{n} \begin{cases} y_j(x)A_j(s), & \text{for } a \le x \le s, \\ y_j(x)B_j(s), & \text{for } s \le x \le b, \end{cases} \tag{1.3}$$

where $A_j(s)$ and $B_j(s)$ represent functions yet to be determined. Clearly, the number of functions is $2n$ and the number of linear relations, which can be derived for $g(x,s)$ from properties 2, 3, and 4 of the definition, is also $2n$. Thus, we are going to derive a system of $2n$ linear equations with $2n$ unknowns $A_j(s)$ and $B_j(s)$. It can be shown that $(n-1)$ of those equations can be obtained from property 2, one equation from property 3, and n equations from property 4.

Hence, the key issue to be resolved in the remaining part of this proof is whether the $2n \times 2n$ system in $A_j(s)$ and $B_j(s)$ is well-posed. This implies that the system is consistent, and the solution is unique.

By virtue of property 2, which states the continuity of $g(x,s)$ itself and its partial derivatives with respect to x of up to $(n-2)$ order, for $x = s$, one derives the following system of $(n-1)$ linear algebraic equations

$$\sum_{j=1}^{n} C_j(s)\frac{d^k y_j(s)}{dx^k} = 0, \quad k = \overline{0, n-2}, \tag{1.4}$$

in n unknown functions

$$C_j(s) = B_j(s) - A_j(s), \quad j = \overline{1, n}. \tag{1.5}$$

The system in (1.4) is underdetermined, because the number of equations in it $(n - 1)$ is less than the number of unknowns (n). This can be circumvent, however, by applying property 3 to the expression in (1.3). This yields a single linear algebraic equation

$$\sum_{j=1}^{n} C_j(s)\frac{d^{n-1} y_j(s)}{dx^{n-1}} = -\frac{1}{p_0(s)} \tag{1.6}$$

with the same set $\{C_j(s) | j = \overline{1, n}\}$ of unknowns. Hence, the relations (1.4) along with that of (1.6) constitute a system of n simultaneous linear algebraic equations in n unknowns. The determinant of the coefficient matrix in this system is non-zero, because it represents the Wronskian for the fundamental set of solutions $\{y_j(x) | j = \overline{1, n}\}$. Thus, the system has a unique solution. In other words, one can readily obtain explicit expressions for $C_j(s)$.

Once the functions $C_j(s)$ are found, the relations in (1.5) form another underdetermined system of n linear algebraic equations in the $2n$ functions $A_j(s)$ and $B_j(s)$. To eliminate the underdeterminedness of the system, we take advantage of the defining property 4. In doing so, we first break down the forms $M_i(y(a), y(b))$ in (1.2) into two additive parts as

$$M_i(y(a), y(b)) = P_i(y(a)) + Q_i(y(b)), \quad i = \overline{1, n},$$

with $P_i(a)$ and $Q_i(b)$ being defined as

$$P_i(y(a)) = \sum_{k=0}^{n-1} \alpha_k^i y^{(k)}(a), \quad Q_i(y(b)) = \sum_{k=0}^{n-1} \beta_k^i y^{(k)}(b).$$

Using property 4, we now substitute the expression for $g(x, s)$ from (1.3) into (1.2)

$$M_i(g(a, s), g(b, s)) \equiv P_i(g(a, s)) + Q_i(g(b, s)) = 0, \quad i = \overline{1, n}. \tag{1.7}$$

Since the values of $g(a, s)$ and its derivatives at the left endpoint $x = a$ of the interval $[a, b]$ are determined by the operator P_i in (1.7), while Q_i governs those at the right endpoint $x = b$, the branch of $g(x, s)$ for $a \le x \le s$ from (1.3) goes to

$P_i(g(a,s))$, while the branch for $s \le x \le b$ must be substituted into $Q_i(g(b,s))$. This yields

$$M_i(g(a,s), g(b,s)) \equiv \sum_{j=1}^{n} [P_i(g(a,s))A_j(s) + Q_i(g(b,s))B_j(s)] = 0, \quad i = \overline{1,n}.$$

Replacing the values of $A_j(s)$ in accordance with (1.5), we can rewrite the above equation in the form

$$\sum_{j=1}^{n} [P_i(g(a,s))(B_j(s) - C_j(s)) + Q_j(g(b,s))B_j(s)] = 0, \quad i = \overline{1,n}.$$

Combining the terms with $B_j(s)$ and taking the term with $C_j(s)$ on the right-hand side, one obtains

$$\sum_{j=1}^{n} [P_i(g(a,s)) + Q_i(g(b,s))]B_j(s) = \sum_{j=1}^{n} P_i(g(a,s))C_j(s), \quad i = \overline{1,n}.$$

Recalling the relations from (1.7), the above equations can finally be rewritten in the form

$$\sum_{j=1}^{n} M_i(g(a,s), g(b,s))B_j(s) = \sum_{j=1}^{n} P_i(g(a,s))C_j(s), \quad i = \overline{1,n}. \tag{1.8}$$

It is evident that the above relations form a system of n linear algebraic equations in n unknown functions $B_j(s)$. The coefficient matrix of this system is not singular, since the forms M_i are assumed to be linearly independent. The right-hand side vector in (1.8) is defined in terms of the functions $C_j(s)$ which have already been determined. Thus, in compliance with the fundamental theorem of linear algebra [37, 76], the system has a unique solution. Once the functions $B_j(s)$ are obtained, the functions $A_j(s)$ can be readily found from (1.5) in terms of $B_j(s)$ and $C_j(s)$.

Hence, we have provided a constructive proof for the theorem, implying that the proof is an algorithm for obtaining an explicit expression for the Green's function $g(x,s)$. □

As we mentioned before, the proof suggests a consistent and practical way to construct the Green's function. Below, we will support this statement with a series of particular examples, in each of which we present and analyze the various intricacies of statements of boundary-value problems, which may occur while considering practical situations in applied sciences.

Example 1.1. We begin with probably the most trivial case by considering the following differential equation

$$\frac{d^2y(x)}{dx^2} = 0, \quad x \in (0, a), \tag{1.9}$$

subject to the boundary conditions

$$y(0) = y(a) = 0. \tag{1.10}$$

To make sure that a Green's function exists for the above problem, let us find out if the setting in (1.9) and (1.10) has only the trivial solution. The most elementary set of functions constituting a fundamental set of solutions for (1.9) is represented by

$$y_1(x) \equiv 1, \quad y_2(x) \equiv x.$$

This yields the general solution $y_g(x)$ for equation (1.9) written as

$$y_g(x) = D_1 + D_2 x,$$

where D_1 and D_2 represent arbitrary constants.

A substitution of this function into the boundary conditions in (1.10) yields the homogeneous system of linear algebraic equations in D_1 and D_2, with a well-posed coefficient matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & a \end{pmatrix}.$$

Hence, the problem in (1.9) and (1.10) has only the trivial solution. Thus, there exists a unique Green's function for this problem. According to the procedure described in the proof of Theorem 1.1, it can be found to be of the form

$$g(x,s) = \begin{cases} A_1(s) + xA_2(s), & \text{for } 0 \le x \le s, \\ B_1(s) + xB_2(s), & \text{for } s \le x \le a. \end{cases} \tag{1.11}$$

Introducing then the functions $C_1(s)$ and $C_2(s)$, as suggested in (1.5), we form a system of linear algebraic equations in these functions (see the system in (1.4) and (1.6)) written as

$$\begin{cases} C_1(s) + sC_2(s) = 0, \\ C_2(s) = -1. \end{cases} \tag{1.12}$$

The obvious solution is $C_1(s) = s$ and $C_2(s) = -1$.

$A_1(s) = 0$ follows from the first boundary condition $y(0) = 0$ in (1.10), satisfied by the upper branch of $g(x,s)$. The upper branch is chosen because $x = 0$ belongs to the interval $0 \le x \le s$. Since $B_1(s) = C_1(s) + A_1(s)$, we conclude that $B_1(s) = s$.

The second boundary condition $y(a) = 0$ in (1.10), being satisfied by the lower branch of $g(x, s)$, yields $B_1(s) + aB_2(s) = 0$. Hence, $B_2(s) = -s/a$. Finally, since $A_2(s) = B_2(s) - C_2(s)$, we find that $A_2(s) = 1 - s/a$. Substituting these values into (1.11), we ultimately obtain the sought-after Green's function of the form

$$g(x,s) = \begin{cases} a^{-1}x(a-s), & \text{for } 0 \le x \le s, \\ a^{-1}s(a-x), & \text{for } s \le x \le a. \end{cases}$$

Example 1.2. We formulate another boundary-value problem

$$\frac{dy(0)}{dx} = 0, \quad \frac{dy(a)}{dx} = 0 \tag{1.13}$$

for (1.9) on the same interval $(0, a)$.

It is evident that this problem does not have a unique solution. Indeed, one can clearly see that any constant function $y(x) \equiv$const satisfies the both the equation and the boundary conditions in (1.13). Hence, the condition of existence and uniqueness for Green's function does not hold. Therefore, a Green's function does not exist.

Example 1.3. Consider another boundary-value problem

$$\frac{dy(0)}{dx} = 0, \quad \frac{dy(a)}{dx} + hy(a) = 0 \tag{1.14}$$

for equation in (1.9) on $(0, a)$, with h a nonzero constant.

It can readily be shown that the problem in (1.9) and (1.14) only has the trivial solution. Consequently, there exists a unique Green's function for this problem.

The first part of the construction process mirrors that of the problem in Example 1.1. The Green's function is again expressed by (1.11), and the coefficients $C_1(s)$ and $C_2(s)$ satisfy the system in (1.12), resulting in $C_1(s) = s$ and $C_2(s) = -1$.

The first boundary condition in (1.14), being satisfied by the upper branch in (1.11), yields $A_2(s) = 0$. This immediately results in $B_2(s) = -1$. The second condition in (1.14), being satisfied by the lower branch of (1.11), yields the following equation

$$B_2(s) + h[B_1(s) + aB_2(s)] = 0$$

in $B_1(s)$ and $B_2(s)$. Based on the known $B_2(s)$, one obtains the function $B_1(s) = (1 + ha)/h$. This in turn yields $A_1(s) = [1 + h(a - s)]/h$.

Substituting the values we obtained for $A_j(s)$ and $B_j(s)$ into (1.11), we get the Green's function for the boundary-value problem in (1.9) and (1.14) of the form

$$g(x,s) = \begin{cases} (a-s) + h^{-1}, & \text{for } 0 \le x \le s, \\ (a-x) + h^{-1}, & \text{for } s \le x \le a. \end{cases} \tag{1.15}$$

Notice that if h goes to infinity, the second term h^{-1} in (1.15) vanishes yielding the Green's function

$$g(x,s) = \begin{cases} a-s, & \text{for } 0 \le x \le s, \\ a-x, & \text{for } s \le x \le a, \end{cases}$$

for equation (1.9) subject to the following boundary conditions

$$\frac{dy(0)}{dx} = 0, \quad y(a) = 0.$$

In the applied sciences, it is frequently required to formulate research projects to study phenomena occurring in infinite media. The Green's function formalism can successfully be extended to the associated boundary-value problems formulated over infinite intervals. In our next example, we construct the Green's function for such a problem.

Example 1.4. Consider the following differential equation

$$\frac{d^2y(x)}{dx^2} - k^2y(x) = 0, \quad x \in (0,\infty), \tag{1.16}$$

subject to boundary conditions

$$y(0) = 0, \quad \lim_{x\to\infty} |y(x)| < \infty. \tag{1.17}$$

It can be shown that the conditions for existence and uniqueness for Green's function are met, assuring a unique Green's function for this problem.

Since the roots of the characteristic (auxiliary) equation for (1.16) are k and $-k$, the following two exponential functions

$$y_1(x) \equiv \exp(kx), \quad y_2(x) \equiv \exp(-kx)$$

represent a fundamental set of solutions for (1.16). Hence, the Green's function for the boundary-value problem in (1.16) and (1.17) is of the form

$$g(x,s) = \begin{cases} A_1(s)\exp(kx) + A_2(s)\exp(-kx), & \text{for } x \le s, \\ B_1(s)\exp(kx) + B_2(s)\exp(-kx), & \text{for } s \le x. \end{cases} \tag{1.18}$$

Denoting $C_i(s) = B_i(s) - A_i(s)$, $i = 1, 2$, we obtain the following system of linear algebraic equations

$$\begin{cases} \exp(ks)C_1(s) + \exp(-ks)C_2(s) = 0, \\ k\exp(ks)C_1(s) - k\exp(-ks)C_2(s) = -1, \end{cases}$$

in $C_1(s)$ and $C_2(s)$. The solution of this system is

$$C_1(s) = -\frac{1}{2k}\exp(-ks), \quad C_2(s) = \frac{1}{2k}\exp(ks). \tag{1.19}$$

The first condition in (1.17) implies

$$A_1(s) + A_2(s) = 0 \tag{1.20}$$

while the second condition leads to $B_1(s) = 0$, because the exponential function $\exp(kx)$ is unbounded as x approaches infinity, and the only way to satisfy the second condition in (1.17) is to set $B_1(s)$ to zero. This immediately yields

$$A_1(s) = \frac{1}{2k}\exp(-ks).$$

The relation in (1.20) consequently provides

$$A_2(s) = -\frac{1}{2k}\exp(-ks).$$

Hence, based on the known functions $C_2(s)$ and $A_2(s)$, we obtain

$$B_2(s) = \frac{1}{2k}[\exp(ks) - \exp(-ks)].$$

Upon substitution of the coefficients $A_j(s)$ and $B_j(s)$ into (1.18), we finally obtains the Green's function for the problem (1.16) and (1.17) in the form

$$g(x,s) = \frac{1}{2k}\begin{cases}\exp(k(x-s)) - \exp(-k(x+s)), & \text{for } x \le s,\\ \exp(k(s-x)) - \exp(-k(s+x)), & \text{for } s \le x,\end{cases}$$

which can be rewritten in a more compact form as

$$g(x,s) = \frac{1}{2k}\left(\exp(-k|x-s|) - \exp(-k(x+s))\right) \tag{1.21}$$

in terms of the absolute value function.

Example 1.5. Consider a boundary-value problem for the same equation as in Example 1.4, now formulated on a different domain

$$\frac{d^2y(x)}{dx^2} - k^2y(x) = 0, \quad x \in (0,a), \tag{1.22}$$

and subject to a specific set of boundary conditions

$$y(0) = y(a), \quad \frac{dy(0)}{dx} = \frac{dy(a)}{dx}. \tag{1.23}$$

This boundary-value problem represents an important type of problem in applied sciences. The relations (1.23) specify conditions of the a-periodicity of the solution.

We leave it as an exercise to the reader, to show that this boundary-value problem has only the trivial solution, proving the existence of a unique Green's function associated with it.

Since (1.22) and (1.23) are the same differential equation as in Example 1.4, the beginning stage of the construction procedure for the Green's function resembles that from the previous problem. We again express the Green's function by (1.18), and the coefficients $C_1(s)$ and $C_2(s)$ are found as in (1.19).

It is important to notice that to satisfy the boundary conditions in this case, we need both branches of (1.18). That is, satisfying the first condition in (1.23), we utilize the upper branch in (1.18) in order to calculate the value of $y(0)$, while its lower branch is used for computing the value of $y(a)$. This results in

$$A_1(s) + A_2(s) = B_1(s)\exp(ka) + B_2(s)\exp(-ka). \tag{1.24}$$

Using the second condition in (1.23), we calculate the derivative of $y(x)$ at $x = 0$ by using the upper branch in (1.18), while the value of the derivative of $y(x)$ at $x = a$ is calculated by using the lower branch of (1.18). This yields

$$A_1(s) - A_2(s) = B_1(s)\exp(ka) - B_2(s)\exp(-ka) \tag{1.25}$$

implying that the relations in (1.24) and (1.25) along with those in (1.19) form a well-posed system of four linear algebraic equations in $A_1(s)$, $A_2(s)$, $B_1(s)$, and $B_2(s)$. To find $A_1(s)$ and $B_1(s)$, we add up equations (1.24) and (1.25). This leads to

$$A_1(s) - B_1(s)\exp(ka) = 0. \tag{1.26}$$

Now, the first relation in (1.19) can be rewritten in the form

$$-A_1(s) + B_1(s) = -\frac{1}{2k}\exp(-ks). \tag{1.27}$$

Solving (1.26) and (1.27) simultaneously, we obtain

$$A_1(s) = \frac{\exp(k(a-s))}{2k[\exp(ka)-1]}, \quad B_1(s) = \frac{\exp(-ks)}{2k[\exp(ka)-1]}.$$

To find the functions $A_2(s)$ and $B_2(s)$, we subtract (1.25) from (1.24), leading to

$$A_2(s) - B_2(s)\exp(-ka) = 0. \tag{1.28}$$

Rewriting then the second relation in (1.19) as

$$-A_2(s) + B_2(s) = \frac{1}{2k}\exp(ks) \tag{1.29}$$

we have to solve (1.28) and (1.29) simultaneously. This yields

$$A_2(s) = \frac{\exp(ks)}{2k[\exp(ka) - 1]}, \quad B_2(s) = \frac{\exp(k(s+a))}{2k[\exp(ka) - 1]}.$$

Substituting $A_1(s)$, $A_2(s)$, $B_1(s)$, and $B_2(s)$ into (1.18), we finally obtain the Green's function for the boundary-value problem in (1.22) and (1.23):

$$g(x,s) = K_0 \begin{cases} \exp(k(x-s+a)) + \exp(k(s-x)), & \text{for } x \le s, \\ \exp(k(s-x+a)) + \exp(k(x-s)), & \text{for } s \le x, \end{cases} \tag{1.30}$$

where the constant factor $K_0 = \{2k[\exp(ka) - 1]\}^{-1}$.

In all the particular problems that we have dealt with so far, the governing differential equations had constant coefficients. In this case finding a fundamental set of solution is a routine procedure. It is worth noting that variable coefficients do not decisively restrict the described algorithm, if a fundamental set of solutions is obtained in terms of either elementary or well-tabulated special functions. In other words, if the governing equation allows for an exact solution, one can readily construct a Green's function by means of this algorithm. The following example illustrates this point.

Example 1.6. Consider the equation with variable coefficients

$$\frac{d}{dx}\left((mx+b)\frac{dy}{dx}\right) = 0, \quad x \in (0,a). \tag{1.31}$$

To ensure the fact that the above equation is not degenerate in any single point in $(0,a)$, it is important to impose certain limitations on the constant parameters m and b. With this in mind, we assume $m > 0$ and $b > 0$, which clearly implies that the function $mx + b$ does not become zero anywhere on $[0,a]$.

It can be readily shown that, if boundary conditions are imposed for (1.31) as

$$\frac{dy(0)}{dx} = 0, \quad y(a) = 0, \tag{1.32}$$

then the problem in (1.31) and (1.32) is well-posed, which allows only the trivial solution.

The fundamental set of solutions for (1.31)

$$y_1(x) \equiv 1, \quad y_2(x) \equiv \ln(mx+b)$$

can be obtained by two successive integrations. Indeed, the first integration yields

$$(mx+b)\frac{dy}{dx} = C_1$$

while, dividing the above equation by $mx+b$ and multiplying by dx, we can separate the variables

$$dy = C_1 \frac{dx}{mx+b}$$

and, after the second integration, we finally obtain

$$y(x) = \frac{C_1}{m} \ln(mx+b) + C_2.$$

Since the boundary-value problem in (1.31) and (1.32) has only the trivial solution, there exists a unique Green's function associated with it, which can be written as

$$g(x,s) = \begin{cases} A_1(s) + \ln(mx+b)A_2(s), & \text{for } 0 \le x \le s, \\ B_1(s) + \ln(mx+b)B_2(s), & \text{for } s \le x \le a. \end{cases} \tag{1.33}$$

Following our now customary procedure, we obtain the system of linear algebraic equations

$$\begin{cases} C_1(s) + \ln(ms+b)C_2(s) = 0, \\ m(ms+b)^{-1}C_2(s) = -(ms+b)^{-1} \end{cases}$$

in $C_j(s) = B_j(s) - A_j(s), (j = 1, 2)$. Its solution is

$$C_1(s) = \frac{1}{m}\ln(ms+b), \quad C_2(s) = -\frac{1}{m}.$$

The first boundary condition in (1.32) yields $A_2(s) = 0$. Consequently, $B_2(s) = -1/m$. The second condition leads to

$$B_1(s) + \ln(ma+p)B_2(s) = 0$$

resulting in $B_1(s) = [\ln(ma+b)]/m$, which gives us

$$A_1(s) = \frac{1}{m}\ln\frac{ma+b}{ms+b}.$$

Substituting the values of $A_j(s)$ and $B_j(s)$ into (1.33), we obtains the sought-after Green's function of the form

$$g(x,s) = \frac{1}{m}\begin{cases} \ln[(ma+b)(ms+b)^{-1}], & \text{for } 0 \le x \le s, \\ \ln[(ma+b)(mx+b)^{-1}], & \text{for } s \le x \le a. \end{cases} \tag{1.34}$$

Sometimes in applied sciences, boundary-value problems on finite intervals arise, where one of the endpoints is a singularity of the governing differential equation. The algorithm in this section can also be used to construct Green's functions in these cases. To illustrate this point, we provide the following example.

Example 1.7. Consider a boundary value problem for the following differential equation

$$\frac{d}{dx}\left(x\frac{dy(x)}{dx}\right) = 0, \quad x \in (0, a), \tag{1.35}$$

subject to boundary conditions

$$\lim_{x\to 0} |y(x)| < \infty, \quad \frac{dy(a)}{dx} + hy(a) = 0. \tag{1.36}$$

Clearly, the left endpoint $x = 0$ of the domain is a singular point of the equation (1.35). Therefore, instead of a traditional boundary condition at this point, we require in equation (1.36) for $y(x)$ to be bounded as x approaches zero.

Integrating the equation (1.35) successively two times, we obtains, similarly to the case in Example 1.6, its fundamental set of solutions

$$y_1(x) \equiv 1, \quad y_2(x) \equiv \ln x.$$

The problem in (1.35) and (1.36) has only the trivial solution, allowing us to look for its unique Green's function of the form

$$g(x, s) = \begin{cases} A_1(s) + \ln x A_2(s), & \text{for } 0 \le x \le s, \\ B_1(s) + \ln x B_2(s), & \text{for } s \le x \le a. \end{cases}$$

Following our customary procedure, we now form the system of linear algebraic equations

$$\begin{cases} C_1(s) + \ln s C_2(s) = 0, \\ s^{-1} C_2(s) = -s^{-1} \end{cases}$$

in $C_1(s)$ and $C_2(s)$, with the solution $C_1(s) = \ln s$ and $C_2(s) = -1$.

The boundedness of the Green's function at $x = 0$ implies $A_2(s) = 0$. Consequently, we find $B_2(s) = -1$. The second condition in (1.36) yields

$$B_2(s)/a + h[B_1(s) + \ln a B_2(s)] = 0$$

resulting in $B_1(s) = 1/ah + \ln a$, and ultimately, $A_1(s) = 1/ah - \ln s/a$. Hence, we finally obtain

$$g(x, s) = \begin{cases} (ah)^{-1} - \ln[(a)^{-1}s], & \text{for } 0 \le x \le s, \\ (ah)^{-1} - \ln[(a)^{-1}x], & \text{for } s \le x \le a. \end{cases} \tag{1.37}$$

Notice that as the value of h goes to infinity, the first term $(ah)^{-1}$ in (1.37) vanishes, yielding the Green's function

$$g(x,s) = \begin{cases} -\ln[(a)^{-1}s], & \text{for } 0 \le x \le s, \\ -\ln[(a)^{-1}x], & \text{for } s \le x \le a, \end{cases} \tag{1.38}$$

for equation (1.35), subject to the boundary conditions

$$\lim_{x\to 0} |y(x)| < \infty, \quad y(a) = 0.$$

Example 1.8. For the following example, we present a boundary-value problem for the simplest fourth order linear equation

$$\frac{d^4y(x)}{dx^4} = 0, \quad x \in (0,1), \tag{1.39}$$

with boundary conditions written

$$y(0) = \frac{dy(0)}{dx} = 0, \quad y(1) = \frac{d^2y(1)}{dx^2} = 0. \tag{1.40}$$

As is known from applied mechanics [7, 19, 28, 45, 56, 71], this setting simulates bending of an elastic beam of unit length, with the left edge clamped and the right edge resting on a support.

Clearly, the following set of functions

$$y_1(x) \equiv 1, \quad y_2(x) \equiv x, \quad y_3(x) \equiv x^2, \quad y_4(x) \equiv x^3 \tag{1.41}$$

is the simplest fundamental set of solutions for (1.39). Hence, the general solution is

$$y_g(x) = D_1 + D_2x + D_3x^2 + D_4x^3.$$

Applying the first two boundary conditions in (1.40), we obtain $D_1 = D_2 = 0$. At $x = 1$ this leads to $D_3 = D_4 = 0$. Hence, the boundary-value problem in (1.39) and (1.40) has only the trivial solution. There exists, consequently, a unique Green's function.

Based on the fundamental set of solutions in (1.41), the Green's function can be written as

$$g(x,s) = \begin{cases} A_1(s) + A_2(s)x + A_3(s)x^2 + A_4(s)x^3, & \text{for } x \le s, \\ B_1(s) + B_2(s)x + B_3(s)x^2 + B_4(s)x^3, & \text{for } s \le x. \end{cases} \tag{1.42}$$

From properties 2 and 3 of the definition of the Green's function, we derive the following system of linear equations in $C_i(s) = B_i(s) - A_i(s)$, written in matrix form as

$$\begin{pmatrix} 1 & s & s^2 & s^3 \\ 0 & 1 & 2s & 3s^2 \\ 0 & 0 & 2 & 6s \\ 0 & 0 & 0 & 6 \end{pmatrix} \times \begin{pmatrix} C_1(s) \\ C_2(s) \\ C_3(s) \\ C_4(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{1.43}$$

the solution of which

$$C_1 = \frac{1}{6}s^3, \quad C_2 = -\frac{1}{2}s^2, \quad C_3 = \frac{1}{2}s, \quad C_4 = -\frac{1}{6} \tag{1.44}$$

is obtained immediately, because of the upper triangular form of the coefficient matrix.

By virtue of property 4 of the definition, the boundary conditions in (1.40) provide

$$A_1 = 0, \quad A_2 = 0, \quad B_1 + B_2 + B_3 + B_4 = 0, \quad 2B_3 + 6B_4 = 0$$

while the rest of the coefficients for $g(x, s)$

$$A_3 = -\frac{1}{4}s^3 + \frac{3}{4}s^2 - \frac{1}{2}s, \quad A_4 = \frac{1}{12}s^3 - \frac{1}{4}s^2 + \frac{1}{6},$$

$$B_1 = \frac{1}{6}s^3, \qquad B_2 = -\frac{1}{2}s^2 \quad \text{and}$$

$$B_3 = -\frac{1}{4}s^3 + \frac{3}{4}s^2, \qquad B_4 = \frac{1}{12}s^3 - \frac{1}{4}s^2$$

are calculated with the help of functions $C_j(s)$ in (1.44).

Substituting the coefficients $A_j(s)$ and $B_j(s)$ into (1.42), we obtain the Green's function $g(x, s)$ for the boundary-value problem posed by (1.39) and (1.40). For $x \le s$, it is written as

$$g(x, s) = \left(\frac{1}{12}s^3 - \frac{1}{4}s^2 + \frac{1}{6}\right)x^3 - \left(\frac{1}{4}s^3 - \frac{3}{4}s^2 + \frac{1}{2}s\right)x^2 \tag{1.45}$$

while for $x \ge s$, we get

$$g(x, s) = \left(\frac{1}{12}x^3 - \frac{1}{4}x^2 + \frac{1}{6}\right)s^3 - \left(\frac{1}{4}x^3 - \frac{3}{4}x^2 + \frac{1}{2}x\right)s^2. \tag{1.46}$$

This example shows that even for higher-order equations, the procedure for the construction of Green's functions utilized in this section is compact enough and requires only limited amount of work.

If we examine the form of the Green's functions we constructed so far in this section, we may notice a common property: they are symmetric in a certain sense. That is, the interchange of x with s in a Green's function valid for $x \le s$ yields one valid for $x \ge s$ and vice versa. In the next section, we will discuss this point in more detail. Certain conditions will be found, under which the symmetry occurs. In the mean time, however, we will consider a problem whose Green's function appears, in contrast to all the previous settings, to be in a asymmetric form.

Example 1.9. We present a boundary-value problem for the following second order linear equation

$$\frac{d^2y(x)}{dx^2}+\frac{dy(x)}{dx}-2y(x)=0, \quad x\in(0,\infty), \tag{1.47}$$

subject to boundary conditions

$$y(0)=0, \quad \lim_{x\to\infty}|y(x)|<\infty. \tag{1.48}$$

Direct analysis shows that the above problem has only the trivial solution, allowing, subsequently, a unique Green's function. Since $y_1(x) = \exp(x)$ and $y_2(x) = \exp(-2x)$ constitute a fundamental set of solutions to (1.47), we write the Green's function to this problem

$$g(x,s)=\begin{cases} A_1(s)\exp(x)+A_2(s)\exp(-2x), & \text{for } x\le s,\\ B_1(s)\exp(x)+B_2(s)\exp(-2x), & \text{for } s\le x.\end{cases} \tag{1.49}$$

This leads to the system of linear equations in $C_j(s) = B_j(s) - A_j(s)$

$$\begin{pmatrix} \exp(s) & \exp(-2s) \\ \exp(s) & -2\exp(-2s)\end{pmatrix}\times\begin{pmatrix} C_1(s)\\ C_2(s)\end{pmatrix}=\begin{pmatrix}0\\-1\end{pmatrix}$$

the solution of which is

$$C_1(s)=-\frac{1}{3}\exp(-s), \quad C_2(s)=\frac{1}{3}\exp(2s).$$

The first condition in (1.48) gives $A_1(s)+A_2(s) = 0$. The second condition implies $B_1(s) = 0$. Therefore $A_1(s) = 1/3\exp(-s)$, resulting in $A_2(s) = -1/3\exp(-s)$, and, finally, $B_2(s) = 1/3[\exp(2s) - \exp(-s)]$. Substituting these into (1.49), we obtain the Green's function for (1.47) and (1.48) as

$$g(x,s)=\frac{1}{3}\begin{cases} \exp(-s)[\exp(x)-\exp(-2x)], & \text{for } x\le s,\\ \exp(-2x)[\exp(2s)-\exp(-s)], & \text{for } s\le x.\end{cases} \tag{1.50}$$

It is evident that the Green's function in (1.50) is not symmetric. The question that arises with regard to this fact as to why this is so? What makes the statement of the boundary-value problem in (1.47) and (1.48) different from all the others considered earlier in this section? The reader will find the reasoning behind this fact in the next section.

1.2 Symmetry of Green's Functions

In order to address the issue of symmetry for Green's functions with respect to the observation and the source point, we have to carry out some preparatory work, which we will do in this section.

Let us write down the linear nth order homogeneous differential equation

$$L[y(x)] \equiv p_0 \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0$$

in $y = y(x)$, with variable coefficients $p_i = p_i(x), i = \overline{0, n}$.

From the qualitative theory of linear equations (see, for example, [18, 20, 29, 37, 54, 66]), we learn that the equation

$$L_a[y(x)] \equiv (-1)^n \frac{d^n(p_0 y)}{dx^n} + (-1)^{n-1} \frac{d^{n-1}(p_1 y)}{dx^{n-1}} + \cdots + p_n y = 0$$

is called the *adjoint* of $L[y(x)] = 0$. If L_a is the *adjoint* of L, and $L \equiv L_a$, then L is said to be a *self-adjoint* operator and the equation $L[y(x)] = 0$ is called *self-adjoint*.

For the sake of simplicity, the discussion in this section is limited to the second order equation

$$L[y(x)] \equiv p_0 \frac{d^2 y}{dx^2} + p_1 \frac{dy}{dx} + p_2 y = 0, \quad x \in (a, b). \tag{1.51}$$

We can make this limitation without loss of generality, but it markedly condenses it and makes it easier to comprehend.

The leading coefficient $p_0(x)$ in (1.51) must be non-zero in all single points of (a, b) with the possible exception of one of the endpoints. Additionally, we require the coefficient $p_0(x)$ to be twice differentiable and $p_1(x)$ just differentiable on (a, b).

According to the terminology introduced above, the equation

$$L_a[y(x)] \equiv \frac{d^2(p_0 y)}{dx^2} - \frac{d(p_1 y)}{dx} + p_2 y = 0 \tag{1.52}$$

is the adjoint of (1.51).

We will briefly review here the self-adjointness of differential equations and other relevant issues important to the analysis of symmetry of Green's functions. A more complete discussion on self-adjointness can be found in standard textbooks on ODE.

Using the product rule, the operator L_a in (1.52) can be rewritten as

$$L_a[y(x)] \equiv \frac{d}{dx}\left(y \frac{dp_0}{dx} + p_0 \frac{dy}{dx}\right) - \left(y \frac{dp_1}{dx} + p_1 \frac{dy}{dx}\right) + p_2 y.$$

Differentiating further and combining like terms, we obtain

$$L_a[y(x)] \equiv p_0 \frac{d^2 y}{dx^2} + \left(2 \frac{dp_0}{dx} - p_1\right) \frac{dy}{dx} + \left(\frac{d^2 p_0}{dx^2} - \frac{dp_1}{dx} + p_2\right) y. \tag{1.53}$$

Suppose (1.51) is self-adjoint, that is $L[y(x)] \equiv L_a[y(x)]$. Then, from comparing the coefficients of dy/dx in $L[y(x)]$ and $L_a[y(x)]$ in (1.51) and (1.53), we obtain the following relation for the coefficients $p_0(x)$ and $p_1(x)$

$$2\frac{dp_0(x)}{dx} - p_1(x) = p_1(x)$$

which must hold for (1.51) to be self-adjoint. This implies

$$p_1(x) = \frac{dp_0(x)}{dx}. \tag{1.54}$$

From differentiating the above relation, we find that the sum of the first two terms in the coefficient

$$\frac{d^2p_0(x)}{dx^2} - \frac{dp_1(x)}{dx} + p_2(x)$$

of $y(x)$ in (1.53) equals zero. This means that self-adjointness of (1.51) implies the relation between the coefficients $p_0(x)$ and $p_1(x)$ and does not constrain the coefficient $p_2(x)$. In other words, if (1.51) is self-adjoint, it can be written as

$$p_0(x)\frac{d^2y(x)}{dx^2} + \frac{dp_0(x)}{dx}\frac{dy(x)}{dx} + p_2(x)y(x) = 0$$

which gives, shortened,

$$\frac{d}{dx}\left(p_0(x)\frac{dy(x)}{dx}\right) + p_2(x)y(x) = 0. \tag{1.55}$$

The above is usually referred to as the *standard form of a second order self-adjoint equation*.

Thus, if the coefficients $p_0(x)$ and $p_1(x)$ in (1.51) satisfy the relation in (1.54), then (1.51) is in self-adjoint form. The fact that (1.54) does not involve the coefficient $p_2(x)$ prompts a simple method for reducing a linear second order differential equation to self-adjoint form.

Indeed, multiplying (1.51) by a certain nonzero function (we call it the *integrating factor*) and applying the relation (1.54) to the coefficients of d^2y/dx^2 and dy/dx of the resultant equation, we can readily formulate a relation from which the integrating factor can be determined. The procedure of finding the integrating factor is quite straightforward. In the series of examples that follow, we consider particular equations and discuss their self-adjointness in detail.

Example 1.10. Find out if the equation

$$e^x\frac{d^2y(x)}{dx^2} + (2 - \cos x)y(x) = 0 \tag{1.56}$$

is in self-adjoint form and if not, reduce it to such.

It is clear that this equation is not in self-adjoint form, since $p_0(x)$ is e^x, whilst $p_1(x)$ equals zero and the condition in (1.54) is not met. The integrating factor e^{-x} is also evident, in this case, because if the equation in (1.56) is multiplied by e^{-x}, then it reduces to the self-adjoint equation

$$\frac{d^2y(x)}{dx^2} + e^{-x}(2 - \cos x)y(x) = 0$$

which meets the condition in (1.54).

Example 1.11. Consider the equation

$$x^3\frac{d^2y(x)}{dx^2} + 3x^2\frac{dy(x)}{dx} - y(x) = 0 \tag{1.57}$$

which is in self-adjoint form; it is evident that the condition in (1.54) is met, because the derivative of x^3 is $3x^2$. Let us change the statement in (1.57) slightly, and consider the equation

$$x^3\frac{d^2y(x)}{dx^2} + x^2\frac{dy(x)}{dx} - y(x) = 0 \tag{1.58}$$

instead, which is not self-adjoint. However, finding an integrating factor for (1.58) is unproblematic; after multiplying (1.58) with x^{-2} we convert the equation to

$$x\frac{d^2y(x)}{dx^2} + \frac{dy(x)}{dx} - x^{-2}y(x) = 0$$

which meets condition (1.54).

It is worth noting that in applied sciences, in most cases, when the self-adjointness is an issue and an equation under consideration is not in such a form, we have to follow a routine procedure to finding an integrating factor. The following example illustrates this point.

Example 1.12. The condition in (1.54) is evidently not met for the equation

$$\frac{d^2y(x)}{dx^2} + 4x\frac{dy(x)}{dx} - 2y(x) = 0, \tag{1.59}$$

so it is not in self-adjoint form; also, there is no obvious guess for the integrating factor. However, following the procedure outlined before, we multiply this equation by an integrating factor $\mu(x)$

$$\mu(x)\frac{d^2y(x)}{dx^2} + 4x\mu(x)\frac{dy(x)}{dx} - 2\mu(x)y(x) = 0. \tag{1.60}$$

The leading coefficient $p_0(x)$ of this equation is $\mu(x)$, while the coefficient $p_1(x)$ equals $4x\mu(x)$. Thus, equation (1.60) would be self-adjoint if (according to condition (1.54))

$$\frac{d\mu(x)}{dx} = 4x\mu(x)$$

which is a separable first order differential equation in $\mu(x)$. Multiplying it by dx and dividing by $\mu(x)$, we separate variables

$$\frac{d\mu(x)}{\mu(x)} = 4xdx$$

and then integrate both sides

$$\ln|\mu(x)| = 2x^2 + C.$$

Solving this equation for $\mu(x)$, we obtain

$$\mu(x) = e^{2x^2+C}$$

Any function in this family can be considered an integrating factor for equation (1.59). In other words, the constant C can be arbitrarily fixed and upon assumption of, say, $C = 0$, we obtain

$$\mu(x) = e^{2x^2}. \tag{1.61}$$

Finally, on substituting (1.61) into (1.60), we can reduce (1.59) to the self-adjoint form

$$e^{2x^2}\frac{d^2y(x)}{dx^2} + 4xe^{2x^2}\frac{dy(x)}{dx} - 2e^{2x^2}y(x) = 0.$$

At this point in our presentation we assume L to be a self-adjoint operator of the second order. That is:

$$L \equiv \frac{d}{dx}\left(p_0(x)\frac{d}{dx}\right) + p_2(x).$$

Consider two functions $u(x)$ and $v(x)$, and assume that each of them is twice continuously differentiable on (a, b). We form the bilinear combination

$$u(x)L[v(x)] - v(x)L[u(x)]$$

which can be rewritten explicitly as

$$u\left(\frac{d}{dx}\left(p_0(x)\frac{dv}{dx}\right) + p_2(x)v\right) - v\left(\frac{d}{dx}\left(p_0(x)\frac{du}{dx}\right) + p_2(x)u\right). \tag{1.62}$$

Removing the outer parentheses in both the components, the above transforms to

$$uL(v) - vL(u) = u\frac{d}{dx}\left(p_0(x)\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p_0(x)\frac{du}{dx}\right).$$

After applying the product rule and rearranging terms, the above expression simplifies to

$$\begin{aligned}
&u\frac{d}{dx}\left(p_0(x)\frac{dv}{dx}\right) - v\frac{d}{dx}\left(p_0(x)\frac{du}{dx}\right)\\
&\quad = u\left(\frac{dp_0(x)}{dx}\frac{dv}{dx} + p_0(x)\frac{d^2v}{dx^2}\right) - v\left(\frac{dp_0(x)}{dx}\frac{du}{dx} + p_0(x)\frac{d^2u}{dx^2}\right)\\
&\quad = u\frac{dp_0(x)}{dx}\frac{dv}{dx} - v\frac{dp_0(x)}{dx}\frac{du}{dx} + p_0(x)u\frac{d^2v}{dx^2} - p_0(x)v\frac{d^2u}{dx^2}\\
&\quad = \frac{dp_0(x)}{dx}\left(u\frac{dv}{dx} - v\frac{du}{dx}\right) + p_0(x)\frac{d}{dx}\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\\
&\quad = \frac{d}{dx}\left(p_0(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right).
\end{aligned}$$

Hence, the bilinear combination in (1.62) reduces to

$$uL(v) - vL(u) = \frac{d}{dx}\left(p_0(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\right). \tag{1.63}$$

Integrating both sides of (1.63) from a to b, we obtains the following relation

$$\int_a^b [uL(v) - vL(u)]dx = p_0(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b \tag{1.64}$$

which is usually referred to as *Green's formula* for a self-adjoint operator. This all leads us to conclude that Green's formula holds for a self-adjoint operator L and two functions $u(x)$ and $v(x)$, continuously differentiable on (a, b).

If, in addition to being twice continuously differentiable on (a, b), $u(x)$ and $v(x)$ are functions for which the right-hand side in (1.64) vanishes, we can reduce Green's formula to a compact form. That is, if

$$p_0(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Bigg|_a^b = 0 \tag{1.65}$$

we get

$$\int_a^b [uL(v) - vL(u)]dx = 0. \tag{1.66}$$

So, the Green's formula in (1.66) is valid for a self-adjoint operator L, with $u(x)$ and $v(x)$ twice continuously differentiable on (a, b) and satisfying the relation in (1.65). This relation is, however, implicit in nature, which makes it too cumbersome to deal with in real-life calculations. Therefore, it is important to find several of its explicit equivalents which are more convenient for practical use.

In doing so, we rewrite the relation in (1.65) in the extended form

$$p_0(b)\left(u(b)\frac{dv(b)}{dx} - v(b)\frac{du(b)}{dx}\right) - p_0(a)\left(u(a)\frac{dv(a)}{dx} - v(a)\frac{du(a)}{dx}\right) = 0 \tag{1.67}$$

Since this equation contains $u(x)$ and $v(x)$, as well as their derivatives at the endpoints of the interval $[a, b]$, it is immediately clear that the relation in (1.67) holds if both $u(x)$ and $v(x)$ satisfy one of the following types of boundary condition in $x = a$ and $x = b$:

(1) $y(a) = 0, y(b) = 0$;

(2) $y(a) = 0, y'(b) = 0$;

(3) $y'(a) = 0, y'(b) = 0$.

We can also see immediately that condition (1.67) is satisfied in the so-called *singular* case, when the leading coefficient $p_0(x)$ in (1.55) equals zero at one of the endpoints of $[a, b]$. In such a case we usually require $y(x)$ to be bounded at that endpoint, with a value of either $y(x)$ or $y'(x)$ being zero at the other endpoint, that is:

(4) $\lim_{x\to a} |y(x)| < \infty, y(b) = 0$;

(5) $\lim_{x\to a} |y(x)| < \infty, y'(b) = 0$.

In addition, on a close analysis it follows that condition (1.67) also holds for both $u(x)$ and $v(x)$, satisfying one of the following sets of boundary conditions:

(6) $y(a) = 0, y'(b) + hy(b) = 0$;

(7) $y'(a) = 0, y'(b) + hy(b) = 0$;

(8) $y'(a) + h_1 y(a) = 0, y'(b) + h_2 y(b) = 0$;

(9) $y(a) = y(b), p_0(a)y'(a) = p_0(b)y'(b)$;

(10) $\lim_{x\to a} |y(x)| < \infty, y'(b) + hy(b) = 0$.

This last set of conditions presupposes (similar to (4) and (5)) that the leading coefficient $p_0(x)$ of (1.55) equals zero at $x = a$.

Note that the endpoints a and b, in all boundary conditions (1)–(10), are interchangeable. That is, the set of conditions

$$y(b) = 0, \quad y'(a) + hy(a) = 0$$

substitutes (6). This is also true for the boundary conditions (4), (5), (7) and (10).

We can now introduce another important terminological issue. A boundary-value problem for (1.55), subject to either one of the types of boundary conditions listed above, belongs to the class of so-called *self-adjoint boundary-value problems*.

We are now ready to turn our attention to one of the basic questions posed in this section: what makes a Green's function symmetric in the sense mentioned in Section 1.1? Theorem 1.2 below specifies the conditions which a boundary-value problem should satisfy for its Green's function to be symmetric.

Theorem 1.2. *If a well-posed boundary-value problem*

$$\frac{d}{dx}\left(p_0(x)\frac{dy(x)}{dx}\right) + p_2(x)y(x) = 0, \tag{1.68}$$

$$M_1[y(a), y(b)] = 0, \quad M_2[y(a), y(b)] = 0 \tag{1.69}$$

is self-adjoint, then its Green's function $g(x,s)$ is symmetric, provided that we can obtain its value for $x \geq s$ from that valid for $x \leq s$ by interchanging x and s.

Proof. This proof is based on a slight modification of the procedure we used in the proof of Theorem 1.1. Again, we choose two linearly independent particular solutions $y_1(x)$ and $y_2(x)$ of the governing equation (1.68). But contrary to Theorem 1.1, additional constraints are supposed to be put on $y_1(x)$ and $y_2(x)$, for which we will make specific choices:

let $y_1(x)$ and $y_2(x)$ be two nonzero linearly independent particular solutions of (1.68). Additionally, let $y_1(x)$ satisfy the first boundary condition in (1.69) and $y_2(x)$ satisfy the second condition in (1.69). Clearly, neither $y_1(x)$ nor $y_2(x)$ can satisfy both boundary conditions in (1.69). In fact, such an assumption conflicts with the well-posedness of (1.68) and (1.69), which implies that the trivial solution is the only solution of the problem in (1.68) and (1.69).

We now form the bilinear combination from $y_1(x)$ and $y_2(x)$

$$y_1(x)L[y_2(x)] - y_2(x)L[y_1(x)],$$

which is exactly zero in all points of (a,b), since $L[y_1(x)] \equiv 0$ and $L[y_2(x)] \equiv 0$ for $x \in (a,b)$.

Recalling (1.63) and rewriting it in terms of $y_1(x)$ and $y_2(x)$ yields

$$y_1L(y_2) - y_2L(y_1) = \frac{d}{dx}\left(p_0(x)\left(y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right)\right).$$

Since the left-hand side of the relation is zero, the right-hand side is as well. That is

$$\frac{d}{dx}\left(p_0(x)\left(y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right)\right) = 0,$$

which implies

$$p_0(x)\left(y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right) = C \tag{1.70}$$

with C a constant.

Note that $y_1(x)$ and $y_2(x)$ are determined up to a constant factor. If $y_1(x)$, for example, satisfies both the governing equation (1.68) and the first boundary condition in (1.69), then, for any nonzero constant α, the product $\alpha y_1(x)$ also satisfies both these relations. This is equally true for $y_2(x)$, which allows us to arbitrarily fix the constant C in (1.70). Hence, without loss of generality, we can read (1.70) as

$$p_0(x)\left(y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right) = -1. \tag{1.71}$$

Hence, we can assume that, if the particular solutions to (1.68), $y_1(x)$ and $y_2(x)$, are chosen in the manner specified, these solutions meet condition (1.71) on all of (a,b). This implies that for any point $s \in (a,b)$ we can write Green's function $g(x,s)$ for the problem in (1.68) and (1.69) in the form

$$g(x,s) = \begin{cases} c_1(s)y_1(x), & \text{for } a \le x \le s, \\ c_2(s)y_2(x), & \text{for } s \le x \le b. \end{cases} \tag{1.72}$$

This function satisfies the boundary conditions in (1.69) regardless of $c_1(s)$ and $c_2(s)$, because $y_1(x)$ and $y_2(x)$ satisfy the first and the second of those boundary conditions, respectively. Hence, $g(x,s)$ as in (1.72) already meets properties 1 and 4 of the definition of Green's function.

Making use of properties 2 and 3 of the definition, we obtain the following system of linear algebraic equations

$$\begin{pmatrix} y_2(s) & -y_1(s) \\ y_2'(s) & -y_1'(s) \end{pmatrix} \times \begin{pmatrix} c_2(s) \\ c_1(s) \end{pmatrix} = \begin{pmatrix} 0 \\ -p_0^{-1}(s) \end{pmatrix} \tag{1.73}$$

in $c_1(s)$ and $c_2(s)$. Clearly, the coefficient matrix of this system is not singular, because its determinant

$$W(s) \equiv y_1(s)y_2'(s) - y_2(s)y_1'(s)$$

is the Wronskian of the two linearly independent functions $y_1(s)$ and $y_2(s)$. Hence, system (1.73) has a unique solution of the form

$$c_1(s) = -\frac{y_2(s)}{p_0(s)W(s)}, \quad c_2(s) = -\frac{y_1(s)}{p_0(s)W(s)}.$$

Substituting these expressions for $c_1(s)$ and $c_2(s)$ into (1.72), we obtain, for the branch of the Green's function with $x \le s$ as

$$g(x,s) = -\frac{y_1(x)y_2(s)}{p_0(s)W(s)}, \quad x \le s, \tag{1.74}$$

while for the other branch, we have

$$g(x,s) = -\frac{y_2(x)y_1(s)}{p_0(s)W(s)}, \quad s \le x. \tag{1.75}$$

According relation (1.71), the denominator in (1.73) and (1.74) satisfies the condition

$$p_0(s)W(s) \equiv p_0(s)\left(y_1(s)y_2'(s) - y_2(s)y_1'(s)\right) \equiv -1.$$

This allows us to write the Green's function $g(x,s)$ for the boundary-value problem in (1.68) and (1.69) in the following "symmetric" form

$$g(x,s) = \begin{cases} y_2(s)y_1(x), & \text{for } a \le x \le s, \\ y_1(s)y_2(x), & \text{for } s \le x \le b. \end{cases} \tag{1.76}$$

Hence, the theorem has been proven. Indeed, from the above representation, it follows that the Green's function $g(x,s)$ of a self-adjoint boundary-value problem is invariant under exchange of the observation point x and the source point s. In other words, the Green's function is symmetric in the sense that whenever x and s are exchanged on one of the branches of $g(x,s)$, we obtain the other branch. □

The symmetry of Green's functions, the analysis of which we completed, has important applications in various applied sciences. It is related to the so called *Maxwell's reciprocity* [29, 37, 66] asserting that the response of a field at an observation point x due to a source at s is the same as the response at s due to a point source at x.

In the next section, we will revisit the basic issue of this chapter, which is the construction of Green's functions for ODE, and we will present another standard construction procedure in detail.

1.3 Alternative Construction Procedure

As can be found in classical textbooks on differential equations, the notion of a Green's function is introduced for a boundary-value problem where both the governing differential equation and the boundary conditions are homogeneous. Such settings are referred to as *homogeneous boundary-value problems*. In this section, we will turn our attention to inhomogeneous linear differential equations subject to homogeneous boundary conditions.

In this section, we will state an prove an important theorem. It builds up a theoretical background for the utilization of Green's functions in solving boundary-value problems for inhomogeneous linear equations. After that, we will review the classical [3, 45, 53, 66] procedure for the construction of Green's functions, which is based on the aforementioned theorem and the Lagrange method of variation of parameters, which is traditionally used in ODE to solve inhomogeneous linear differential equations analytically, if the fundamental set of solutions is available for the corresponding homogeneous equation.

Consider the linear inhomogeneous equation

$$L[y(x)] \equiv p_0(x)\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n(x)y = -f(x) \tag{1.77}$$

subject to the homogeneous boundary conditions

$$M_i(y(a), y(b)) \equiv \sum_{k=0}^{n-1}[\alpha_k^i \frac{d^k y(a)}{dx^k} + \beta_k^i \frac{d^k y(b)}{dx^k}] = 0, \quad i = \overline{1, n}, \tag{1.78}$$

where the coefficients $p_j(x)$ and the right-hand side term $f(x)$ in the governing equation are continuous functions, with $p_0(x) \neq 0$ on (a, b), and M_i represent linearly independent forms with constant coefficients.

We will establish the link between the uniqueness of the solution of (1.77) and (1.78) and the corresponding homogeneous problem. For this reason, we focus on a theorem that makes things ready for the use of a Green's function, constructed for a homogeneous problem, in solving corresponding inhomogeneous equations.

Theorem 1.3. *If the homogeneous boundary-value problem corresponding to* (1.77) *and* (1.78) *has only the trivial solution, then the problem in* (1.77) *and* (1.78) *has a unique solution.*

Proof. The theorem follows from the linearity of the setting: let $Y_1(x)$ and $Y_2(x)$ represent two distinct solutions of (1.77) and (1.78). This means that each of these solutions is supposed to make equation (1.77) hold. That is,

$$p_0(x)\frac{d^n Y_1}{dx^n} + p_1(x)\frac{d^{n-1} Y_1}{dx^{n-1}} + \cdots + p_n(x)Y_1 = -f(x)$$

and

$$p_0(x)\frac{d^n Y_2}{dx^n} + p_1(x)\frac{d^{n-1} Y_2}{dx^{n-1}} + \cdots + p_n(x)Y_2 = -f(x).$$

Subtracting these term-by-term, we obtain

$$p_0(x)\frac{d^n (Y_1 - Y_2)}{dx^n} + p_1(x)\frac{d^{n-1} (Y_1 - Y_2)}{dx^{n-1}} + \cdots + p_n(x)(Y_1 - Y_2) = 0.$$

Thus, if $Y_1(x)$ and $Y_2(x)$ represent two distinct solutions of (1.77), then their difference $Y_{12}(x) = Y_1(x) - Y_2(x)$ is a solution of the corresponding homogeneous equation. In the same fashion, taking advantage of the linearity of the forms M_i, we can show that $Y_{12}(x)$ should satisfy the homogeneous boundary conditions in (1.78). In other words, $Y_{12}(x)$ is a solution of the homogeneous boundary-value problem corresponding to (1.77) and (1.78). But, according to the statement in this theorem, the corresponding homogeneous problem has only the trivial solution, which implies that the difference $Y_1(x) - Y_2(x)$ should be identical to zero for all x.

So, our assumption about the existence of two distinct solutions of the original equations (1.77) and (1.78) is wrong. Therefore, there exists a unique solution, if the corresponding homogeneous problem only has the trivial solution. □

The following theorem establishes a direct way for expressing the solution of (1.77) and (1.78) in terms of the Green's function, constructed for the corresponding homogeneous boundary value problem.

Theorem 1.4. *If the boundary-value problem stated by the inhomogeneous equation in* (1.77) *subject to the homogeneous conditions in* (1.78) *is well-posed, then the unique solution for* (1.77) *and* (1.78) *can be expressed by the integral*

$$y(x) = \int_a^b g(x,s) f(s) ds \tag{1.79}$$

whose kernel $g(x,s)$ is the Green's function of the corresponding homogeneous problem.

Proof. The theorem requires that we prove two independent statements. First, that the integral in (1.79) satisfies the equation in (1.77), and second, that it satisfies the boundary conditions of (1.78).

Since the Green's function $g(x,s)$ consists of two segments, we break down the integral in (1.79) into two integrals as shown

$$y(x) = \int_a^x g^-(x,s) f(s) ds + \int_x^b g^+(x,s) f(s) ds, \tag{1.80}$$

where by $g^+(x,s)$ and $g^-(x,s)$ we denote the branches of $g(x,s)$ where $x \le s$ and $x \ge s$, respectively.

As complying with the governing equation (1.77), we take into account a specific instance of $y(x)$ in (1.80); it is defined in terms of two definite integrals (with s representing the integration variable), which contain a parameter x and have variable limits depending on x. Therefore, one has to recall from the fundamental theorem of integral calculus [37, 66] that, if a function $\Phi(x)$ is defined in integral form

$$\Phi(x) = \int_{\alpha(x)}^{\beta(x)} F(x,s) ds,$$

then its derivative (with respect to x) is written as

$$\frac{d\Phi(x)}{dx} = \int_{\alpha(x)}^{\beta(x)} \frac{\partial F(x,s)}{\partial x} ds$$
$$= F(x, \beta(x))\beta'(x) - F(x, \alpha(x))\alpha'(x). \tag{1.81}$$

This implies that, since both integrals in (1.80) contain x as a parameter and their limits depend upon x, the derivative of $y(x)$ can formally be written as

$$\frac{dy(x)}{dx} = \int_a^x \frac{\partial g^-(x,s)}{\partial x} f(s)ds + g^-(x,x)f(x)$$
$$+ \int_x^b \frac{\partial g^+(x,s)}{\partial x} f(s)ds - g^+(x,x)f(x).$$

Combining these integrals and realizing that non-integral terms are eliminated due to the continuity of the Green's function for $x = s$, we obtain

$$\frac{dy(x)}{dx} = \int_a^b \frac{\partial g(x,s)}{\partial x} f(s)ds.$$

Thus, the derivative of (1.79) can be taken by direct differentiation of its kernel. Taking into account the continuity of the derivatives of the Green's function, up to $(n-2)$nd order, as $x = s$ (see property 2 of the definition), the higher-order derivatives of the integrals in (1.80), up to $(n-1)$st order, can be calculated analogously to the first derivative:

$$\frac{d^k y(x)}{dx^k} = \int_a^b \frac{\partial^k g(x,s)}{\partial x^k} f(s)ds, \quad k = \overline{1, n-1}. \tag{1.82}$$

Since the boundary conditions in (1.78) involve only the derivatives of $y(x)$ up to $(n-1)$st order, all the derivatives in $M_i(y(a), y(b))$ can be calculated formally within the integral. This yields

$$M_i(y(a), y(b)) \equiv \sum_{k=0}^{n-1} \left[\alpha_k^i \int_a^b \frac{\partial^k g(a,s)}{\partial x^k} f(s)ds + \beta_k^i \int_a^b \frac{\partial^k g(b,s)}{\partial x^k} f(s)ds \right]$$
$$= \int_a^b \sum_{k=0}^{n-1} \left[\alpha_k^i \frac{\partial^k g(a,s)}{\partial x^k} + \beta_k^i \frac{\partial^k g(b,s)}{\partial x^k} \right] f(s)ds = 0, \quad i = \overline{1, n},$$

because the expressions in the brackets equals zero due to property 4 in the definition of the Green's function. Hence, the boundary conditions in (1.78) are indeed satisfied by the formula in (1.79).

To complete the part of the theorem related to complying with the governing equation in (1.77), we calculate the nth derivative of $y(x)$ by differentiating (1.82), with $k = n - 1$ fixed. This yields

$$\frac{d^n y(x)}{dx^n} = \int_a^b \frac{\partial^n g(x,s)}{\partial x^n} f(s)ds + \left[\frac{\partial^{n-1} g^-(x,x)}{\partial x^{n-1}} - \frac{\partial^{n-1} g^+(x,x)}{\partial x^{n-1}}\right] f(x)$$

which, in compliance with property 3 of the definition, transforms to

$$\frac{d^n y(x)}{dx^n} = \int_a^b \frac{\partial^n g(x,s)}{\partial x^n} f(s)ds - f(x)p_0^{-1}(x).$$

After substituting $y(x)$ and its derivatives found earlier into (1.77) and combining all the integral terms into a single term, we finally obtain

$$\int_a^b L[g(x,s)]f(s)ds - f(x) = -f(x).$$

The above equality is an identity, since $L[g(x,s)] = 0$ on (a,b). With this, we have proven the theorem. □

In Section 1.1, we obtained Green's functions for a number of boundary-value problems, which were constructed using a technique based on the defining properties of the Green's function. In the following, we will present an alternative approach, utilizing the theorem just proven, which can also be used to construct Green's functions. The idea behind this approach is to employ Lagrange's method of variation of parameters, which is traditionally used in solving inhomogeneous linear differential equations.

For the sake of simplicity, we limit ourselves to the second order equation

$$p_0(x)\frac{d^2 y(x)}{dx^2} + p_1(x)\frac{dy(x)}{dx} + p_2(x)y(x) = -f(x) \tag{1.83}$$

on the interval (a,b) and subject to the simplest set of boundary conditions

$$y(a) = 0, \quad y(b) = 0, \tag{1.84}$$

Assuming that the above boundary-value problem has a unique solution, we know that the corresponding homogeneous problem has only the trivial solution. Let $y_1(x)$ and $y_2(x)$ be linearly independent particular solutions of the associated homogeneous equation (1.83) forming, in other words, its fundamental set of solutions. Following this, express the general solution of (1.83), following the Lagrangian method of variation of parameters [66], as

$$y(x) = C_1(x)y_1(x) + C_2(x)y_2(x) \tag{1.85}$$

with $C_1(x)$ and $C_2(x)$ differentiable functions on (a, b), to be found in what follows.

Our first impression is that the idea of a solution of (1.83) of the form of (1.85) might not be useful for determining the two functions $C_1(x)$ and $C_2(x)$. This is correct, unless a second relation, in addition to (1.83), is established to find a unique $C_1(x)$ and $C_2(x)$. Lagrange's method provides an effective and elegant choice for such a relation.

It is obvious that direct substitution of $y(x)$ obtained from (1.85) into (1.83) would result in a single second order differential equation in two unknown functions $C_1(x)$ and $C_2(x)$, which is cumbersome. In order to avoid such a complication, Lagrangian method suggests the following: First, we differentiate $y(x)$ in (1.85)

$$y'(x) = C_1'(x)y_1(x) + C_1(x)y_1'(x) + C_2'(x)y_2(x) + C_2(x)y_2'(x)$$

and then, to simplify matters, we assume that

$$C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0. \tag{1.86}$$

This yields

$$y'(x) = C_1(x)y_1'(x) + C_2(x)y_2'(x) \tag{1.87}$$

resulting in the second derivative of $y(x)$ expressed as follows:

$$y''(x) = C_1'(x)y_1'(x) + C_1(x)y_1''(x) + C_2'(x)y_2'(x) + C_2(x)y_2''(x). \tag{1.88}$$

Now, we substitute $y(x)$, $y'(x)$, and $y''(x)$ from (1.85), (1.87), and (1.88) into (1.83)

$$p_0(C_1'y_1' + C_1y_1'' + C_2'y_2' + C_2y_2'') + p_1(C_1y_1' + C_2y_2') \\ + p_2(C_1y_1 + C_2y_2) = -f(x).$$

Rearranging the order of terms, we rewrite this as

$$C_1(p_0y_1'' + p_1y_1' + p_2y_1) + C_2(p_0y_2'' + p_1y_2' + p_2y_2) \\ + p_0(C_1'y_1' + C_2'y_2') = -f(x).$$

Since $y_1(x)$ and $y_2(x)$ represent particular solutions of the homogeneous equation in (1.83), the coefficients of $C_1(x)$ and $C_2(x)$ are zero. This yields

$$C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = -f(x)p_0^{-1}(x). \tag{1.89}$$

The relations (1.86) and (1.89) constitute a well-posed linear system in $C_1'(x)$ and $C_2'(x)$. This assertion is based on the fact that the determinant of the coefficient matrix of the system is the Wronskian

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

for the two linearly independent functions $y_1(x)$ and $y_2(x)$, and therefore must be nonzero. Upon solving the system, we obtain

$$C_1'(x) = -\frac{y_2(x)f(x)}{p_0(x)W(x)}, \quad C_2'(x) = \frac{y_1(x)f(x)}{p_0(x)W(x)}.$$

Straightforward integration of the derivatives $C_1'(x)$ and $C_2'(x)$ yields

$$C_1(x) = -\int_a^x \frac{y_2(s)f(s)}{p_0(s)W(s)}ds + H_1$$

and

$$C_2(x) = \int_a^x \frac{y_1(s)f(x)}{p_0(s)W(s)}ds + H_2.$$

Substitution in (1.85) gives

$$y(x) = H_1 y_1(x) + H_2 y_2(x) \\ + y_2(x)\int_a^x \frac{y_1(s)f(x)}{p_0(s)W(s)}ds - y_1(x)\int_a^x \frac{y_2(s)f(x)}{p_0(s)W(s)}ds.$$

Taking the factors $y_1(x)$ and $y_2(x)$ inside the integral (which is a formal operation since the variable of integration is s but not x) and combining the two integrals in one, we find

$$y(x) = \int_a^x \frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{p_0(s)W(s)}ds + H_1 y_1(x) + H_2 y_2(x). \tag{1.90}$$

In order to satisfy the boundary conditions in (1.84), with $y(x)$ as expressed above, we obtain the following linear system

$$\begin{pmatrix} y_1(a) & y_2(a) \\ y_1(b) & y_2(b) \end{pmatrix} \times \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} = \begin{pmatrix} 0 \\ P(a,b) \end{pmatrix} \tag{1.91}$$

in H_1 and H_2, with $P(a,b)$ defined as

$$P(a,b) = \int_a^b \frac{R(b,s)}{p_0(s)W(s)} f(s)ds,$$

where

$$R(b,s) = y_1(b)y_2(s) - y_1(s)y_2(b).$$

We now arrive at the solution to the system (1.91)

$$H_1 = -\int_a^b \frac{y_2(a)R(b,s)f(s)}{p_0(s)R(a,b)W(s)}ds$$

and

$$H_2 = \int_a^b \frac{y_1(a)R(b,s)f(s)}{p_0(s)R(a,b)W(s)}ds.$$

Upon substituting these expressions into (1.90), we obtain the solution of the boundary-value problem (1.83) and (1.84):

$$y(x) = -\int_a^x \frac{R(x,s)f(s)}{p_0(s)W(s)}ds + \int_a^b \frac{R(a,x)R(b,s)f(s)}{p_0(s)R(a,b)W(s)}ds$$

which can be rewritten as a single integral

$$y(x) = \int_a^b g(x,s)f(s)ds, \tag{1.92}$$

with the kernel $g(x,s)$ consisting of two segments, one of which is

$$g(x,s) = \frac{R(a,x)R(b,s)}{p_0(s)R(a,b)W(s)}, \quad x \le s, \tag{1.93}$$

while for $x \ge s$, we obtain immediately

$$g(x,s) = \frac{R(a,x)R(b,s) - R(x,s)R(a,b)}{p_0(s)R(a,b)W(s)}, \quad x \ge s.$$

After a trivial but quite cumbersome transformation, the above expression simplifies to

$$g(x,s) = \frac{R(a,s)R(b,x)}{p_0(s)R(a,b)W(s)}, \quad x \ge s. \tag{1.94}$$

Note that, since the solution to (1.83) and (1.84) is found to be the integral (1.92), we conclude, by virtue of Theorem 1.4, that its kernel $g(x,s)$ does in fact represent the Green's function for the homogeneous boundary-value problem corresponding to (1.83) and (1.84).

From the closing part of Section 1.2, recall that, if the setting in (1.83) and (1.84) is self-adjoint, then the product $p_0(s)W(s)$ is a constant (see the relation in (1.71)). This obviously makes (1.93) and (1.94) symmetric in the sense discussed in Sections 1.1 and 1.2.

Hence, the approach based on the method of variation of parameters can be used successfully to construct Green's functions; once the solution to an inhomogeneous linear differential equation, subject to homogeneous boundary conditions, is expressed in integral form (1.92), the kernel of the latter is the Green's function to the corresponding homogeneous boundary-value problem.

This approach is an alternative to the method based on the defining properties as described in Section 1.1. In the following, we present a number of examples, illustrating several of the intricacies of its application.

Example 1.13. In order to apply the procedure based on the method of variation of parameters to the construction of the Green's function, we consider the inhomogeneous equation

$$\frac{d^2y(x)}{dx^2} + k^2y(x) = -f(x), \quad x \in (0,a), \tag{1.95}$$

subject to homogeneous boundary conditions imposed as

$$y'(0) = 0, \quad y'(a) = 0. \tag{1.96}$$

We assume that the right-hand side function $f(x)$ in (1.95) is continuous on $(0,a)$.

It can be easily shown that the corresponding homogeneous boundary-value problem has only the trivial solution. This implies that the conditions for existence and uniqueness of its Green's function are satisfied.

Since the functions $y_1(x) \equiv \sin kx$ and $y_2(x) \equiv \cos kx$ constitute a fundamental set of solutions for the homogeneous equation corresponding to (1.95), the general solution to (1.95) itself becomes

$$y(x) = C_1(x)\sin kx + C_2(x)\cos kx. \tag{1.97}$$

The system of linear algebraic equations in $C_1'(x)$ and $C_2'(x)$ (which has been derived for (1.83) and (1.84) in (1.86) and (1.89)) appears, in this case, as

$$\begin{pmatrix} \sin kx & \cos kx \\ k\cos kx & -k\sin kx \end{pmatrix} \times \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -f(x) \end{pmatrix}$$

providing us with the following solution:

$$C_1'(x) = \frac{1}{k}\cos kx f(x), \quad C_2'(x) = -\frac{1}{k}\sin kx f(x).$$

After integration, we obtain

$$C_1(x) = \int_0^x \frac{1}{k}\cos ks f(s)ds + H_1$$

and

$$C_2(x) = -\int_0^x \frac{1}{k}\sin ks f(s)ds + H_2.$$

From substitution into (1.97) and carrying out an obvious transformation, we get

$$y(x) = \int_0^x \frac{1}{k}\sin k(x-s) f(s)ds + H_1\sin kx + H_2\cos kx. \tag{1.98}$$

To determine H_1 and H_2, we differentiate $y(x)$

$$y'(x) = \int_0^x \cos k(x-s) f(s)ds + H_1 k \cos kx - H_2 k \sin kx.$$

Note that in performing this differentiation, it turns out that, due to the specific form of the integrand $\sin k(x-s) f(s)$ in (1.98) – which vanishes for $x = s$ – the non-integral terms (see (1.81)) do not show up in this case.

From the first condition in (1.96), it follows that $H_1 = 0$, while the second condition yields

$$\int_0^a \cos k(a-s) f(s)ds - H_2 k \sin ka = 0$$

from which we obtain immediately

$$H_2 = \int_0^a \frac{\cos k(a-s)}{k \sin ka} f(s)ds.$$

Upon substituting H_1 and H_2 in (1.98), we obtain the solution of (1.95) and (1.96) as

$$y(x) = \int_0^x \frac{\sin k(x-s)}{k} f(s)ds + \int_0^a \cos(kx) \frac{\cos k(a-s)}{k \sin ka} f(s)ds. \tag{1.99}$$

This formula transforms into a single integral with its kernel expressed in two pieces. To ease the reader through such a transformation, we leave the first integral in (1.99) in its current form and decompose the second integral into

$$\int_0^a \cos kx \frac{\cos k(a-s)}{k \sin ka} f(s)ds = \int_0^x \cos kx \frac{\cos k(a-s)}{k \sin ka} f(s)ds + \int_x^a \cos kx \frac{\cos k(a-s)}{k \sin ka} f(s)ds.$$

With this, equation (1.99) becomes

$$y(x) = \int_0^x \frac{\sin k(x-s)}{k} f(s)ds + \int_0^x \cos kx \frac{\cos k(a-s)}{k \sin ka} f(s)ds + \int_x^a \cos kx \frac{\cos k(a-s)}{k \sin ka} f(s)ds.$$

Combining the first two integrals, we rewrite the above as

$$\begin{aligned} y(x) &= \int_0^x \left[\frac{\sin k(x-s)}{k} + \cos kx \frac{\cos k(a-s)}{k \sin ka} \right] f(s)ds \\ &\quad + \int_x^a \cos kx \frac{\cos k(a-s)}{k \sin ka} f(s)ds \\ &= \int_0^x \cos ks \frac{\cos k(a-x)}{k \sin ka} f(s)ds + \int_x^a \cos kx \frac{\cos k(a-s)}{k \sin ka} f(s)ds. \end{aligned}$$

Note that the kernel in the first of the above two integrals is valid for $x \geq s$, because x represents its upper limit of integration, whereas the kernel in the second integral, by similar reasoning, is valid for $x \leq s$. With this in mind, $y(x)$ can indeed be viewed as a single integral

$$y(x) = \int_0^a g(x,s)f(s)ds \tag{1.100}$$

with a kernel $g(x,s)$ consisting of two segments

$$g(x,s) = \frac{1}{k \sin ka} \begin{cases} \cos kx \cos k(a-s), & \text{for } x \leq s, \\ \cos ks \cos k(a-x), & \text{for } s \leq x. \end{cases} \tag{1.101}$$

Hence, as long as the solution to the boundary-value problem in (1.95) and (1.96) has the form of a single integral similar to (1.100), its kernel $g(x,s)$ is, in compliance with Theorem 1.4, the Green's function for the corresponding homogeneous boundary-value problem.

Example 1.14. Consider the inhomogeneous equation

$$\frac{d^2y(x)}{dx^2} - k^2y(x) = -f(x) \tag{1.102}$$

subject to the homogeneous boundary conditions

$$y'(0) = 0, \quad y(a) = 0. \tag{1.103}$$

We leave it as an exercise to the reader to prove that the homogeneous boundary-value problem in (1.102) and (1.103) has only the trivial solution, hence justifying the existence and uniqueness of the Green's function.

The following set of functions

$$y_1(x) \equiv \exp(kx), \quad y_2(x) \equiv \exp(-kx)$$

constitutes a fundamental set of solutions for the homogeneous equation corresponding to that in (1.102). Hence, the general solution of the latter can be represented by

$$y(x) = C_1(x)\exp(kx) + C_2(x)\exp(-kx). \tag{1.104}$$

Following Lagrange's method, we obtain the expressions

$$C_1(x) = -\int_0^x \frac{1}{2k}\exp(-ks)f(s)ds + H_1$$

and

$$C_2(x) = \int_0^x \frac{1}{2k}\exp(ks)f(s)ds + H_2$$

for the coefficients $C_1(x)$ and $C_2(x)$ in (1.104). After substituting these into (1.102) we get

$$y(x) = -\int_0^x \frac{1}{k}\sinh k(x-s)f(s)ds + H_1\exp(kx) + H_2\exp(-kx). \qquad (1.105)$$

The first boundary condition $y'(0) = 0$ in (1.103) implies $H_1 = H_2$, while the second condition $y(a) = 0$ yields

$$H_1 = H_2 = \int_0^a \frac{\sinh k(a-s)}{2k\cosh ka} f(s)ds.$$

Substituting into (1.105), it transforms to

$$y(x) = \int_0^a \frac{\cosh kx \sinh k(a-s)}{k\cosh ka} f(s)ds - \int_0^x \frac{1}{k}\sinh k(x-s)f(s)ds.$$

Hence, again following the procedure described in Example 1.13, we convert the above expression for $y(x)$ to a single integral form whose kernel

$$g(x,s) = \frac{1}{k\cosh ka}\begin{cases}\cosh kx \sinh k(a-s), & \text{for } x \le s,\\ \cosh ks \sinh k(a-x), & \text{for } s \le x,\end{cases} \qquad (1.106)$$

represents the Green's function for the homogeneous boundary-value problem, corresponding to that in (1.102) and (1.103).

Example 1.15. Let us consider equation (1.102) again, but now impose a different set of boundary conditions, namely

$$y'(0) - hy(0) = 0, \qquad \lim_{x\to\infty}|y(x)| < \infty. \qquad (1.107)$$

This example is designed to show how we deal with imposed boundedness, as in (1.107), when applying Lagrange's method.

It can be easily shown that there exists a unique Green's function for the homogeneous boundary-value problem, corresponding to (1.102) and (1.107).

Recall that the general solution of (1.102) was already derived in (1.105). In this case, however, it is preferable to express it in terms of the exponential functions

$$y(x) = H_1\exp(kx) + H_2\exp(-kx) + \int_0^x \frac{1}{2k}[\exp(k(s-x)) - \exp(k(x-s))]f(s)ds, \qquad (1.108)$$

in contrast to the mixed hyperbolic-exponential function form of (1.105); the formula in (1.108) is more practical regarding the necessity of taking into account that we impose boundedness with (1.107): splitting off the exponential terms under the integral sign, taking out the exponential functions of x, and grouping the terms containing $\exp(kx)$ and the terms containing $\exp(-kx)$, we can rewrite (1.108) as

$$y(x) = \left(H_1 - \int_0^x \frac{\exp(-ks)}{2k} f(s)ds\right) \exp(kx) + \left(H_2 + \int_0^x \frac{\exp(ks)}{2k} f(s)ds\right) \exp(-kx). \tag{1.109}$$

It is evident that the imposing boundedness requires the coefficient of the positive exponential term $\exp(kx)$ in (1.109) to be zero as x approaches infinity. This yields

$$H_1 = \int_0^\infty \frac{1}{2k} \exp(-ks) f(s)ds,$$

whilst the first condition in (1.107) yields

$$H_2 = \frac{k-h}{k+h} H_1 = \int_0^\infty \frac{k-h}{2k(k+h)} \exp(-ks) f(s)ds.$$

After substituting H_1 and H_2 into (1.108) and rewriting its first term in a more compact hyperbolic form, we obtain

$$y(x) = -\int_0^x \frac{1}{k} \sinh k(x-s) f(s)ds + \int_0^\infty \frac{1}{2k} \exp(-ks) \left(\exp(kx) + h^* \exp(-kx)\right) f(s)ds,$$

where $h^* = (k-h)/(k+h)$. From this we obtain the Green's function $g(x,s)$ for equation (1.102) and for (1.107)

$$g(x,s) = \frac{1}{2k} \begin{cases} \exp(-ks)\left(\exp(kx) + h^* \exp(-kx)\right), & \text{for } x \le s, \\ \exp(-kx)\left(\exp(ks) + h^* \exp(-ks)\right), & \text{for } s \le x, \end{cases}$$

which can be rewritten in a single expression as

$$g(x,s) = \frac{1}{2k} \left(\exp(-k|x-s|) + h^* \exp(-k(x+s))\right). \tag{1.110}$$

Example 1.16. Consider a boundary-value problem for the equation with variable coefficients

$$\frac{d}{dx}\left((x^2+1)\frac{dy(x)}{dx}\right) = f(x), \quad x \in (0,a), \tag{1.111}$$

with boundary conditions

$$y(0) = 0, \quad y(a) = 0, \tag{1.112}$$

and briefly describe the construction procedure for the Green's function of the corresponding homogeneous problem.

The form of the homogeneous equation corresponding to that in (1.111) suggests

$$(x^2 + 1)\frac{dy(x)}{dx} = C,$$

where C is an arbitrary constant. Separating the variables in the above equation

$$dy = \frac{C dx}{x^2 + 1}$$

and integrating, we get

$$y(x) = C \arctan x + D.$$

Hence, a fundamental set of solutions for the homogeneous equation corresponding to (1.111) can be formed from the functions $y_1(x) \equiv 1$ and $y_2(x) \equiv \arctan x$, which yields the general solution to (1.111)

$$y(x) = \int_0^x (\arctan s - \arctan x) f(s) ds + D_1 + D_2 \arctan x. \tag{1.113}$$

From the boundary conditions, we find D_1 and D_2 to be

$$D_1 = 0, \quad D_2 = \int_0^a \frac{\omega - \arctan s}{\omega} f(s) ds,$$

where $\omega = \arctan a$.

Substituting these into the above expression for the general solution and rearranging the integral terms, we obtain the solution to the original boundary-value problem as the single integral

$$y(x) = \int_0^a g(x,s) f(s) ds$$

the kernel of which

$$g(x,s) = \frac{1}{\omega}\begin{cases} \arctan x(\omega - \arctan s), & \text{for } 0 \le x \le s, \\ \arctan s(\omega - \arctan x), & \text{for } x \le s \le a, \end{cases}$$

represents the sought-after Green's function.

Example 1.17. Construct Green's function for the homogeneous boundary-value problem for the equation

$$\frac{d^4y(x)}{dx^4} - 2k^2\frac{d^2y(x)}{dx^2} + k^4y(x) = -f(x) \tag{1.114}$$

defined on $x \in (0, \infty)$ and subject to the boundary conditions

$$y(0) = 0, \quad y'(0) = 0, \tag{1.115}$$

as well as the boundedness conditions

$$\lim_{x\to\infty} |y(x)| < \infty, \quad \lim_{x\to\infty} |y'(x)| < \infty. \tag{1.116}$$

In structural mechanics, this setting is associated with a semi-infinite elastic beam resting on a specific elastic foundation [7, 45]. The beam's edge $x = 0$ is assumed to be clamped, in order to satisfy (1.115).

Existence and uniqueness of the Green's function for the above problem can be easily established. In order to construct it, we first obtain the general solution to (1.114). In doing so, write down the characteristic (auxiliary) equation

$$m^4 - 2k^2m^2 + k^4 = 0$$

of the homogeneous equation corresponding to (1.114). It has two pairs of repeated real roots: $m_{1,2} = k$ and $m_{3,4} = -k$, which implies that the general solution of (1.114) can be expressed as

$$y(x) = C_1(x)e^{kx} + C_2(x)e^{-kx} + C_3(x)xe^{kx} + C_4(x)xe^{-kx}. \tag{1.117}$$

We can obtain the coefficient matrix for the system of linear algebraic equations in the derivatives of $C_i(x)$, $i = 1, 2, 3, 4$, as

$$\begin{pmatrix} e^{kx} & e^{-kx} & xe^{kx} & xe^{-kx} \\ ke^{kx} & -ke^{-kx} & (1+kx)e^{kx} & (1-kx)e^{-kx} \\ k^2e^{kx} & k^2e^{-kx} & k(2+kx)e^{kx} & -k(2-kx)e^{-kx} \\ k^3e^{kx} & -k^3e^{-kx} & k^2(3+kx)e^{kx} & k^2(3-kx)e^{-kx} \end{pmatrix},$$

where the right-hand side vector of the system is $(0, 0, 0, -f(x))^T$. Clearly, the above matrix is non-singular, because its determinant represents the Wronskian of the fundamental set of solutions in (1.117). Therefore, we immediately obtain

$$C_1'(x) = \frac{1+kx}{4k^3}e^{-kx}f(x), \quad C_2'(x) = -\frac{1-kx}{4k^3}e^{kx}f(x),$$
$$C_3'(x) = -\frac{1}{4k^2}e^{-kx}f(x), \quad C_4'(x) = -\frac{1}{4k^2}e^{kx}f(x).$$

Upon integration, we find $C_i(x)$ to be

$$C_1(x) = \int_0^x \frac{1+ks}{4k^3}\mathrm{e}^{-ks} f(s)ds + H_1, \quad C_2(x) = -\int_0^x \frac{1-ks}{4k^3}\mathrm{e}^{ks} f(s)ds + H_2,$$
$$C_3(x) = -\int_0^x \frac{1}{4k^2}\mathrm{e}^{-ks} f(s)ds + H_3, \quad C_4(x) = -\int_0^x \frac{1}{4k^2}\mathrm{e}^{ks} f(s)ds + H_4.$$

Hence, the function $y(x)$ in (1.117), together with $C_i(x)$, is the general solution to (1.114). The constants H_i have to be calculated by applying the boundary and the boundedness conditions in (1.115) and (1.116). Before going any further, to aid clarity of the following derivation, we first differentiate $y(x)$ in (1.117) by using the product rule

$$\begin{aligned} y'(x) = {} & k\mathrm{e}^{kx}C_1(x) + \underline{\mathrm{e}^{kx}\left(\frac{1+kx}{4k^3}\mathrm{e}^{-kx}f(x)\right)} \\ & - k\mathrm{e}^{-kx}C_2(x) + \underline{\mathrm{e}^{-kx}\left(-\frac{1-kx}{4k^3}\mathrm{e}^{kx}f(x)\right)} \\ & + (1+kx)\mathrm{e}^{kx}C_3(x) + \underline{x\mathrm{e}^{kx}\left(-\frac{1}{4k^2}\mathrm{e}^{-kx}f(x)\right)} \\ & + (1-kx)\mathrm{e}^{-kx}C_4(x) + \underline{x\mathrm{e}^{-kx}\left(-\frac{1}{4k^2}\mathrm{e}^{kx}f(x)\right)}. \end{aligned}$$

Note that the sum of the underlined terms is zero. Substituting the values of $C_i(x)$ into the part of $y'(x)$ that remains, we obtain

$$\begin{aligned} y'(x) = {} & k\mathrm{e}^{kx}\left(\int_0^x \frac{1+ks}{4k^3}\mathrm{e}^{-ks}f(s)ds + H_1\right) \\ & - k\mathrm{e}^{-kx}\left(-\int_0^x \frac{1-ks}{4k^3}\mathrm{e}^{ks}f(s)ds + H_2\right) \\ & + (1+kx)\mathrm{e}^{kx}\left(-\int_0^x \frac{1}{4k^2}\mathrm{e}^{-ks}f(s)ds + H_3\right) \\ & + (1-kx)\mathrm{e}^{-kx}\left(-\int_0^x \frac{1}{4k^2}\mathrm{e}^{ks}f(s)ds + H_4\right). \end{aligned}$$

Now recall the boundary conditions imposed by (1.115). The first one $y(0) = 0$ yields

$$H_1 + H_2 = 0$$

whilst the second $y'(0) = 0$ results in

$$kH_1 + H_3 - kH_2 + H_4 = 0.$$

The boundedness conditions (1.116) lead us to

$$H_1 = -\int_0^\infty \frac{1+ks}{4k^3}\mathrm{e}^{-ks} f(s)ds, \quad H_3 = \int_0^\infty \frac{1}{4k^2}\mathrm{e}^{-ks} f(s)ds$$

which subsequently leads to

$$H_2 = \int_0^\infty \frac{1+ks}{4k^3}\mathrm{e}^{-ks} f(s)ds, \quad H_4 = \int_0^\infty \frac{1+2ks}{4k^2}\mathrm{e}^{-ks} f(s)ds.$$

After substituting H_1, H_2, H_3, and H_4 into (1.117) and combining like integrals, we finally obtain

$$y(x) = \int_0^x \left(\frac{1-k(x-s)}{4k^3}\mathrm{e}^{k(x-s)} - \frac{1+k(x-s)}{4k^3}\mathrm{e}^{-k(x-s)}\right) f(s)ds$$
$$+ \int_0^\infty \left(\frac{1+k(x+s)+2k^2xs}{4k^3}\mathrm{e}^{-k(x+s)} - \frac{1-k(x-s)}{4k^3}\mathrm{e}^{k(x-s)}\right) f(s)ds.$$

Similarly to earlier examples, $y(x)$ can be rewritten as a single integral

$$y(x) = \int_0^\infty g(x,s) f(s)ds$$

the kernel of which $g(x,s)$ consists of two segments. The branch for $x \le s$ is found to be

$$g(x,s) = \frac{1}{4k^3}[(1+k(x+s)+2k^2xs)\mathrm{e}^{-k(x+s)} - (1-k(x-s))\mathrm{e}^{k(x-s)}] \tag{1.118}$$

whereas the branch for $s \le x$ can be obtained by exchanging x and s. Hence, by virtue of Theorem 1.4, $g(x,s)$ represents the Green's function for the homogeneous boundary-value problem corresponding to equations (1.114)–(1.116).

The following example shows how a Green's function can be used practically to calculate the solution for an inhomogeneous equation subject to homogeneous boundary conditions.

Example 1.18. Using a corresponding Green's function and applying Theorem 1.4, find the solution of the inhomogeneous equation

$$\frac{d^4y(x)}{dx^4} = -P_0 \sin \pi x, \quad P_0 = \text{const} \tag{1.119}$$

subject to the homogeneous boundary conditions

$$y(0) = \frac{dy(0)}{dx} = 0, \quad y(1) = \frac{d^2y(1)}{dx^2} = 0. \tag{1.120}$$

In structural mechanics, to name an example, this setting models deflection $y(x)$ of an elastic beam of a unit length with the left edge clamped and the right edge resting on a support [7, 47]. A transverse load given as $f(x) = P_0 \sin \pi x$ is applied to the beam.

The Green's function for the homogeneous problem associated with (1.119) and (1.120) was derived in Section 1.1 (see (1.45)). The branch for $x \leq s$ is

$$g^{+}(x,s) = -\left(\frac{s^3}{4} - \frac{3s^2}{4} + \frac{s}{2}\right)x^2 + \left(\frac{s^3}{12} - \frac{s^2}{4} + \frac{1}{6}\right)x^3.$$

Due to the self-adjointness of the original statement, the branch $g^{-}(x,s)$ of the Green's function for $x \geq s$, can be obtained from the above by exchanging x and s.

It follows from Theorem 1.4, that the solution of the problem in (1.119) and (1.120) can be found by straightforward integration

$$y(x) = P_0 \int_0^1 g(x,s)(\sin \pi s)ds. \tag{1.121}$$

In order to evaluate this integral, we recall that $g(x,s)$ is defined in two pieces, and decompose the integral into

$$\begin{aligned} y(x) &= P_0\left[\int_0^x g^{-}(x,s)(\sin \pi s)ds + \int_x^1 g^{+}(x,s)(\sin \pi s)ds\right] \qquad (1.122)\\ &= P_0\left[\int_0^x \left(\left(\frac{x^3}{12} - \frac{x^2}{4} + \frac{1}{6}\right)s^3 - \left(\frac{x^3}{4} - \frac{3x^2}{4} + \frac{x}{2}\right)s^2\right)(\sin \pi s)ds\right.\\ &\quad \left. + \int_x^1 \left(\left(\frac{s^3}{12} - \frac{s^2}{4} + \frac{1}{6}\right)x^3 - \left(\frac{s^3}{4} - \frac{3s^2}{4} + \frac{s}{2}\right)x^2\right)(\sin \pi s)ds\right]. \end{aligned}$$

The calculation of the integrals in (1.122) is a routine procedure, which leads to

$$y(x) = \frac{P_0}{2\pi^4}[\pi x(x-1)(x-2) - 2\sin \pi x].$$

It is important to note that the Green's function-based approach to the boundary-value problem in (1.119) and (1.120) is especially effective if we are required to calculate a number of solutions for different right-hand side functions. In such a situation, the direct use of Theorem 1.4 markedly simplifies the calculations.

To illustrate this, let us assume that we are required to find a solution to (1.119) and (1.120), where the right hand side in (1.119) is another function – say, the first-order polynomial $f(x) = mx + b$. We can then replace $P_0 \sin \pi s$ in (1.121) with $ms + b$ and, after an elementary transformation, obtain

$$y(x) = \frac{x^2(x-1)}{240}[2mx^2 + 2(5b+m)x - (15b+7m)].$$

In this chapter, we considered classical boundary-value problems for linear ordinary differential equations. The Green's function method for this class of problems is well-developed and its implementation is, as the latter example illustrates, a quite straightforward procedure.

In Chapter 5 we will further extend the Green's function formalism. We will treat a nontrivial sphere of possible applications where, until recently, this formalism has not been used [45, 47, 50, 51]. A specific class of problems will be considered, which occur in various areas of the applied sciences. The so-called *multi point-posed boundary-value problems* for special systems of linear ordinary differential equations will be treated by means of the Green's function method.

1.4 Chapter Exercises

1. Determine whether the following boundary-value problem has only the trivial solution:

   (a) $y''(x) = 0$, with $y'(0) = 0$ and $y'(a) + my(a) = 0$;

   (b) $y''(x) - k^2 y(x) = 0$, with $y(0) = 0$ and $\lim_{x\to\infty} |y(x)| < \infty$;

   (c) $y''(x) - k^2 y(x) = 0$, with $y(0) = y(a)$ and $y'(0) = y'(a)$;

   (d) $((mx + p)y'(x))' = 0$, with $y'(0) = 0$ and $y(a) = 0$;

   (e) $(xy'(x))' = 0$, with $\lim_{x\to\infty} |y(x)| < \infty$ and $y'(a) + hy(a) = 0$, $h > 0$;

   (f) $y''(x) + y'(x) - 2y(x) = 0$, with $y(0) = 0$ and $\lim_{x\to\infty} |y(x)| < \infty$;

   (g) $y''(x) + y'(x) = 0$, with $y'(0) = 0$ and $y'(a) = 0$;

   (h) $y^{\mathrm{IV}}(x) = 0$, with $y(0) = y''(0) = 0$ and $y''(a) = y'''(a)$;

   (i) $y^{\mathrm{IV}}(x) = 0$, with $y(0) = y'(0) = 0$ and $y'(a) = y'''(a) = 0$;

   (j) $y^{\mathrm{IV}}(x) = 0$, with $y(0) = y'(0) = 0$ and $y''(a) - ky'(a) = y'''(a) = 0$, $k > 0$;

   (k) $y^{\mathrm{IV}}(x) = 0$, with $y'(0) = y''(0) = 0$ and $y'(a) = y'''(a) - ky(a) = 0$, $k > 0$;

   (l) $y^{\mathrm{IV}}(x) = 0$, with $y''(0) = y'''(0) = 0$ and $y''(a) = y'''(a) = 0$.

2. Construct the Green's functions for the following boundary-value problems on the indicated intervals:

   (a) $y''(x) = 0$, with $y(0) = 0$ and $y'(a) = 0$;

   (b) $y''(x) = 0$, with $y(0) = 0$ and $y'(a) + hy(a) = 0$, $h \geq 0$. Show that if $h = 0$, the Green's function for this problem reduces to that in Exercise 2 (a);

(c) $y''(x) = 0$, with $y'(0) - h_1 y(0) = 0$ and $y'(a) + h_2 y(a) = 0$, when h_1 and h_2 are non-zero at the same time. Show that if $h_1 = 0$, the Green's function reduces to that of Example 1.3 in Section 1.1;

(d) $((mx + p)y'(x))' = 0$, with $y(0) = 0$ and $y(a) = 0$, when $m > 0$ and $p > 0$;

(e) $(\exp(\beta x)y'(x))' = 0$, with $y(0) = 0$ and $y(a) = 0$;

(f) $(\exp(\beta x)y'(x))' = 0$, with $y(0) = 0$ and $y'(a) = 0$;

(g) $y''(x) + k^2 y(x) = 0$, with $y(0) = 0$ and $y(a) = 0$;

(h) $y^{\mathrm{IV}}(x) = 0$, with $y(0) = y'(0) = 0$ and $y''(a) = y'''(a) - ky(a) = 0$, $k > 0$;

(i) $y^{\mathrm{IV}}(x) = 0$, with $y(0) = y'(0) = 0$ and $y'(a) = y'''(a) = 0$;

(j) $y^{\mathrm{IV}}(x) = 0$, with $y(0) = y'(0) = 0$ and $y''(a) + ky'(a) = y'''(a) = 0$, $k > 0$;

(k) $y^{\mathrm{IV}}(x) = 0$, with $y'(0) = y'''(0) + ky(0) = 0$ and $y'(a) = y'''(a) = 0$, $k > 0$;

(l) $y^{\mathrm{IV}}(x) = 0$, with $y(0) = y'(0) = 0$ and $y(a) = y'(a) = 0$.

3. Determine whether the following equations are self-adjoint:

   (a) $y''(x) + k^2 y(x) = 0$;

   (b) $x^2 y''(x) + 2xy'(x) - (x^2 - 1)y(x) = 0$;

   (c) $x^2 y''(x) - 2xy'(x) + y(x) = 0$;

   (d) $y''(x) + 3y'(x) + 9y(x) = 0$;

   (e) $\sin^2(x)y''(x) + \sin(2x)y'(x) - y(x) = 0$.

4. Reduce the following differential equations to self-adjoint form by introducing integrating factors:

   (a) $y''(x) - 2y'(x) + 4y(x) = 0$;

   (b) $y''(x) + xy'(x) - x^2 y(x) = 0$;

   (c) $x^2 y''(x) - xy'(x) + y(x) = 0$;

   (d) $x^2 y''(x) + xy'(x) - y(x) = 0$.

5. Determine whether the following boundary-value problems are self-adjoint:

   (a) $y''(x) + y(x) = 0$, with $y(a) = 0$ and $y'(b) + hy(b) = 0$, $h \geq 0$;

   (b) $y''(x) - y(x) = 0$, with $y'(a) = 0$ and $y'(b) + hy(b) = 0$, $h > 0$;

(c) $xy''(x) + y'(x) - y(x) = 0, y'(a) + h_1 y(a) = 0, y'(b) + h_2 y(b) = 0$, when both $h_1$and h_2 are not zero at the same time;

(d) $xy''(x) + y'(x) = 0$, with $y(a) = y(b)$ and $ay'(a) = by'(b)$;

(e) $(x - a)y''(x) + y'(x) - y(x) = 0$, with $|y(a)| < \infty \lim_{x\to a} |y(x)| < \infty$ and $y'(b) + hy(b) = 0, h > 0$;

(f) $y''(x) + y(x) = 0$, with $y'(0) + y(0) + y(a) = 0$ and $y'(0) - y(0) + y'(a) = 0$.

6. Construct the Green's function for the following problem:

$$y''(x) + 3y'(x) - 10y(x) = 0, \quad y(0) = 0, \quad \lim_{x\to\infty} |y(x)| < \infty.$$

Now reduce the problem to self-adjoint form and construct its Green's function. Observe how this affects the symmetry of the Green's function.

7. Construct the Green's function for the problem

$$y''(x) + k^2 y(x) = 0, \quad y(0) = 0,\ y'(1) = 0,$$

using the approach discussed in the proof of Theorem 1.1. Compare this with the method used in Theorem 1.4.

8. Use Lagrange's method to construct the Green's functions for the following boundary-value problems:

(a) $(xy'(x))' = 0$, with $\lim_{x\to 0} |y(x)| < \infty$, $y(a) = 0$;

(b) $y''(x) - k^2 y(x) = 0$, with $y'(0) - hy(0) = 0$, $y(a) = 0$, $h > 0$;

(c) $y^{\text{IV}}(x) = 0$, with $y(0) = y'(0) = 0$ and $y(a) = y'(a) = 0$;

(d) $y^{\text{IV}}(x) = 0$, with $y(0) = y''(0) - ky'(0) = 0$, $k > 0$ and $y'(a) = y'''(a) = 0$;

(e) $y^{\text{IV}}(x) = 0$, with $y''(0) = y'''(0) = 0$ and $y(a) = y'(a) = 0$;

(f) $y^{\text{IV}}(x) = 0$, with $y(0) = y''(0) = 0$ and $y''(a) = y'''(a) - ky(a) = 0$, $k > 0$;

(g) $y^{\text{IV}}(x) = 0$, with $y'(0) = y'''(0) + ky(0) = 0$, $k > 0$ and $y'(a) = y'''(a) = 0$;

(h) $y^{\text{IV}}(x) = 0$, with $y'(0) = y''(0) = 0$ and $y''(a) + ky'(a) = y'''(a) = 0$, $k > 0$.

9. Based on Theorem 1.4, calculate solutions for the following boundary-value problems by utilizing the corresponding Green's functions:

(a) $y''(x) + y(x) = 5\exp(2x)$, with $y(0) = 0$, $y'(a) - 2y(a) = 0$;

(b) $y''(x) - y(x) = 2x^2 - 1$, with $y'(0) = 0$, $y'(a) - y(a) = 0$;

(c) $y''(x) + 2y'(x) + y(x) = 2\sin x$, $y'(0) = 0$, $y'(a) = 0$;

(d) $y^{\mathrm{IV}}(x) = 3x^2$, with $y(0) = y'(0) = 0$, $y(a) = y'(a) = 0$;

(e) $y^{\mathrm{IV}}(x) = 720x$, with $y(0) = y''(0) = 0$, $y(a) = y''(a) + 3y'(a) = 0$;

(f) $y^{\mathrm{IV}}(x) = (\pi^4/16a^4)\cos(\pi x/2a)$, with $y(0) = y''(0) = 0$, $y'(a) = y'''(a) = 0$;

(g) $y^{\mathrm{IV}}(x) - 2y''(x) + y(x) = 12x\exp(-x)$, with $y'(0) = y''(0) = 0$, $\lim_{x\to\infty}|y(x)| < \infty$, $\lim_{x\to\infty}|y'(x)| < \infty$.

Chapter 2
The Laplace Equation

The material in constitutes a necessary background for the present chapter, where we present a detailed discussion on procedures for the construction of Green's functions for partial differential equations. The focus will be on a comprehensive review of three standard methods that can potentially be (and actually are) used for this purpose in the case of different boundary-value problems for the two- and three-dimensional Laplace equation. The first two of them are the method of images, which is reviewed in Section 2.1, and the method of conformal mapping covered in Section 2.2. The area for productive application of these methods is small and the number of closed analytical formulas for Green's functions, which we can obtain, is strictly limited.

The method of eigenfunction expansion, the detailed review of which is included in Section 2.3, represents the last of the three standard methods that are usually recommended within the field. Applying this method, we obtain Green's functions in a series form unsuitable to a direct computer implementation. This drawback has been generally recognized and considered indisputable.

This pessimistic opinion regarding the computational potential of the method of eigenfunction expansion is widespread and might look well-founded for applied mathematical physics. Indeed, it is evident that, for the two-dimensional Laplace equation, the logarithmic singularity of a Green's function makes its series representations non-uniformly convergent, which constrains direct computer implementations. Nowadays however, this pessimistic opinion must be revised. Outcomes of intensive research on this method undertaken in recent decades [21, 26, 42, 43, 45, 47, 48, 50, 51] significantly reinforced its potential power. In Section 2.3, we familiarize the reader with the details of these developments.

In Section 2.4 we review an unusual class of problems: we analyze potential fields as generated by point sources on surfaces of revolution. We consider several boundary-value problems on different fragments of spherical and toroidal surfaces. It will be shown that, for several such problems, our version of the method of eigenfunction expansion is efficient, providing us with closed analytical representations of Green's functions.

2.1 Method of Images

We begin with a review of the *method of images*. It is treated in nearly every textbook on partial differential equations. The idea behind the method is quite transparent, its algorithm is pretty much straightforward, but the range of its efficient applicability is very restricted. Only a few closed form Green's functions, as expressed in terms of elementary functions, can be obtained by the method of images. The objective of the method is to build up a regular component $R(P, Q)$ of the Green's function

$$G(P, Q) = -\frac{1}{2\pi} \ln |P - Q| + R(P, Q), \quad P, Q \in \Omega, \tag{2.1}$$

of the well-posed boundary-value problem

$$\nabla^2 u(P) = 0, \quad P \in \Omega, \tag{2.2}$$

$$T[u(P)] = 0, \quad P \in L, \tag{2.3}$$

for the Laplace equation on a simply-connected region Ω, bounded by a piecewise smooth contour L.

In the present book, we apply conventional terminology [18, 29, 37, 57, 67, 77] to the problem setting in (2.2) and (2.3). That is, the latter is said to be the *Dirichlet problem* if T represents the identity operator, $T \equiv I$. The *Neumann problem* corresponds to $T \equiv \partial/\partial n$, where n is the normal direction to the boundary L. The case of $T \equiv \partial/\partial n - \beta$, where β is a function of the coordinates of P, is usually referred to as the *Robin problem* which some sources call the *mixed problem*.

Recall that the singular component $-\frac{1}{2\pi} \ln |P - Q|$ of $G(P, Q)$ is interpreted, in applied sciences, as the response at a field (observation) point P to a unit source placed in an arbitrary point Q. With this in mind, in the method of images, we proceed to express the regular component $R(P, Q)$ of $G(P, Q)$ as a sum of a finite number of unit sources and sinks placed in points $Q_1^*, Q_2^*, \ldots, Q_m^*$outside the considered region Ω. None of those sources and sinks can, according to the definition of the Green's function, be located inside Ω. This makes the regular component

$$R(P, Q) = \sum_{j=1}^{m} \pm \frac{1}{2\pi} \ln |P - Q_j^*| \tag{2.4}$$

a harmonic function at any point P in Ω (since all the source points Q_j^* are outside Ω). The plus sign in (2.4) corresponds to a sink whilst the minus sign represents a source. Clearly, $G(P, Q)$ with this regular component $R(P, Q)$, is a harmonic function at any point $P \in \Omega$, with the exception of $P = Q$. Additionally, the boundary condition in (2.3) can be satisfied by choosing appropriate locations for $Q_1^*, Q_2^*, \ldots, Q_m^*$: the trace of the function $-T[\frac{1}{2\pi} \ln |P - Q|]$ on the boundary line L will be canceled out by $T[R(P, Q)]$.

We will explore the algorithm of the method of images in detail, with the following series of illustrative examples.

Example 2.1. Consider the classical case of the Dirichlet problem on the upper half-plane $\Omega(x, y) = \{-\infty < x < \infty, y > 0\}$. and construct its Green's function.

The field generated by the unit source at a point $Q(\xi, \eta)$ represents the singular component

$$-\frac{1}{4\pi} \ln\left((x-\xi)^2 + (y-\eta)^2\right)$$

of the Green's function. It can be canceled out, on the boundary, by a single unit sink placed in the point $Q^*(\xi, -\eta)$ located at the lower half-plane and symmetric with respect to $Q(\xi, \eta)$ about the boundary $y = 0$. With the field generated by this sink given as

$$\frac{1}{4\pi} \ln\left((x-\xi)^2 + (y+\eta)^2\right),$$

we find the Green's function of the Dirichlet problem for the upper half-plane as

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{(x-\xi)^2 + (y+\eta)^2}{(x-\xi)^2 + (y-\eta)^2}. \tag{2.5}$$

Example 2.2. As our next example, we consider another classical case of the Dirichlet problem for the quarter-plane $\Omega(r, \varphi) = \{0 < r < \infty, 0 < \varphi < \pi/2\}$. We refer to this region as the infinite circular sector with angle $\pi/2$.

Since in polar coordinates, the distance between two points (r_1, φ_1) and (r_2, φ_2) is defined as

$$\sqrt{r_1^2 - 2r_1r_2\cos(\varphi_1 - \varphi_2) + r_2^2}$$

the singular component of the Green's function $G(r, \varphi; \varrho, \psi)$ we are looking for, becomes

$$-\frac{1}{4\pi} \ln\left(r^2 - 2r\varrho\cos(\varphi - \psi) + \varrho^2\right) \tag{2.6}$$

which represents the generated field of the unit source acting at (ϱ, ψ) in observation point $M(r, \varphi) \in \Omega$. This is depicted in Figure 2.1 with the plus sign placed in $A(\varrho, \psi) \in \Omega$.

In order to cancel out the trace of the function in (2.6) (or, in other words, to satisfy the Dirichlet condition) on the boundary $y = 0$, we place the unit sink (labeled with the minus sign in Figure 2.1) at $D(\varrho, 2\pi - \psi)$. The field generated by this sink is given by

$$\frac{1}{4\pi} \ln(r^2 - 2r\varrho\cos(\varphi - (2\pi - \psi)) + \varrho^2). \tag{2.7}$$

Similarly, with the unit sink at $B(\varrho, \pi - \psi)$, whose generated field is defined as

$$\frac{1}{4\pi} \ln(r^2 - 2r\varrho\cos(\varphi - (\pi - \psi)) + \varrho^2), \tag{2.8}$$

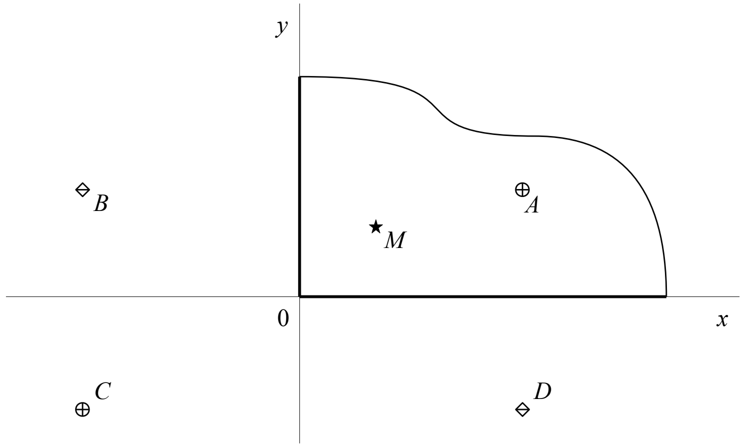

Figure 2.1. Derivation of the Green's function for the quarter-plane.

we cancel out the trace of (2.6) on the boundary $x = 0$, whereas to cancel the traces of the functions in (2.7) and (2.8) on $x = 0$ and $y = 0$, respectively, the unit source is required at $C(\varrho, \pi + \psi)$, which generates the field

$$-\frac{1}{4\pi}\ln(r^2 - 2r\varrho\cos(\varphi - (\pi + \psi)) + \varrho^2). \tag{2.9}$$

Hence, the Green's function of the Dirichlet problem stated on the infinite circular sector $\{0 < r < \infty, 0 < \varphi < \pi/2\}$ represents the sum of the components in (2.6), (2.7), (2.8), and (2.9). That is

$$G(r, \varphi; \varrho, \psi) = \frac{1}{4\pi}\ln\prod_{n=1}^{2}\frac{r^2 - 2r\varrho\cos(\varphi - (n\pi - \psi)) + \varrho^2}{r^2 - 2r\varrho\cos(\varphi - ((n-1)\pi + \psi)) + \varrho^2}$$

which reduces, after a trivial transformation, to a compact finite product-free form

$$G(r, \varphi; \varrho, \psi) = \frac{1}{4\pi}\ln\frac{(r^2 + \varrho^2)^2 - 4r^2\varrho^2\cos^2(\varphi + \psi)}{(r^2 + \varrho^2)^2 - 4r^2\varrho^2\cos^2(\varphi - \psi)}. \tag{2.10}$$

Example 2.3. Note that, for the infinite circular sector $\Omega(r, \varphi) = \{0 < r < \infty, 0 < \varphi < \pi/2\}$, if compensatory sources and sinks are placed, differently from the one just described in Example 2.2, the method of images enables us to construct the Green's function for a specific mixed boundary-value problem.

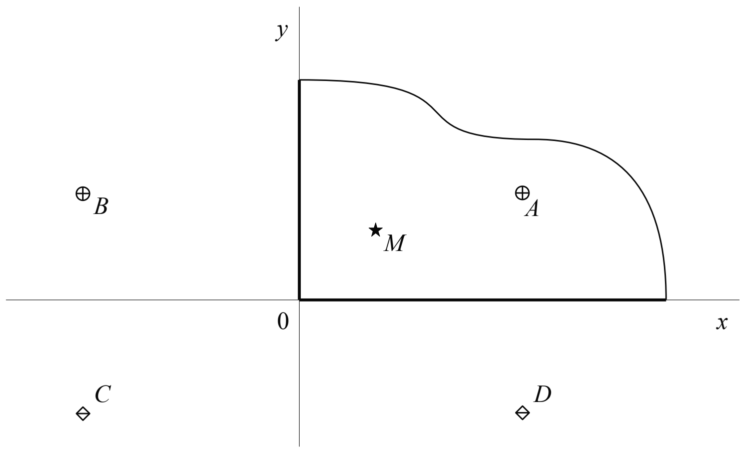

Figure 2.2. Dirichlet–Neumann problem for quarter-plane.

Proceeding now, in compliance with the scheme depicted in Figure 2.2, we obtain the Green's function

$$G(r,\varphi;\varrho,\psi) = \frac{1}{4\pi}\ln\frac{r^2-2r\varrho\cos(\varphi-(2\pi-\psi))+\varrho^2}{r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2} \times \frac{r^2-2r\varrho\cos(\varphi-(\pi+\psi))+\varrho^2}{r^2-2r\varrho\cos(\varphi-(\pi-\psi))+\varrho^2}$$

of the Dirichlet–Neumann boundary-value problem for the infinite circular sector of $\pi/2$, with Dirichlet and Neumann boundary conditions imposed on the boundary segments $y=0$ and $x=0$, respectively. The above expression transforms to

$$G(r,\varphi;\varrho,\psi) = \frac{1}{4\pi}\ln\frac{r^2-2r\varrho\cos(\varphi+\psi)+\varrho^2}{r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2} \times \frac{r^2+2r\varrho\cos(\varphi-\psi)+\varrho^2}{r^2+2r\varrho\cos(\varphi+\psi)+\varrho^2}. \tag{2.11}$$

Although the method of images turns out to be productive for several boundary-value problems on infinite circular sectors, it is not, however, supposed to be productive for several others. In a series of examples that follow, one can find both successful and unsuccessful applications of the method.

Example 2.4. Consider the Dirichlet problem on the infinite circular sector with angle $\pi/3$, that is $\Omega(r,\varphi) = \{0 < r < \infty, 0 < \varphi < \pi/3\}$. To construct its Green's function, we encourage the reader to consult the scheme depicted in Figure 2.3.

In order to cancel out the field generated by the singular component

$$-\frac{1}{4\pi}\ln\left(r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2\right)$$

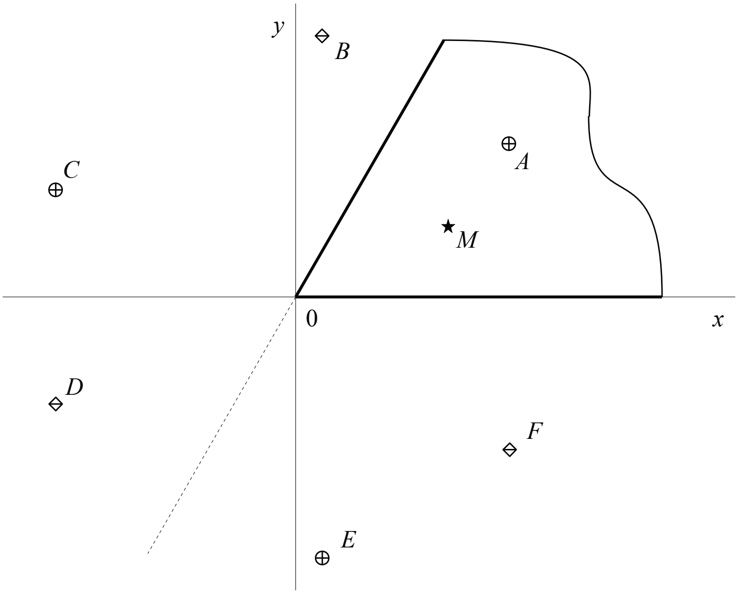

Figure 2.3. Dirichlet problem for an infinite circular sector with angle $\pi/3$.

of the Green's function on the boundary fragment $\varphi = 0$, we place a compensatory unit sink at $F(\varrho, 2\pi - \psi)$, while another unit sink is required at $B(\varrho, 2\pi/3 - \psi)$ in order to satisfy the Dirichlet condition on $\varphi = \pi/3$. To compensate the trace of the latter sink on the boundary fragment $\varphi = 0$, a unit source is required at $E(\varrho, 4\pi/3 + \psi)$. The trace of the latter source is compensated on $\varphi = \pi/3$ with the unit sink at $D(\varrho, 4\pi/3 - \psi)$, while the trace of this sink is compensated on $\varphi = 0$ with the unit source placed in $C(\varrho, 2\pi/3 + \psi)$.

Thus, the regular component $R(r, \varphi; \varrho, \psi)$ of the Green's function of the Dirichlet problem for the circular sector of $\pi/3$ can be formed as the aggregate of five compensatory sources and sinks located outside Ω, as illustrated in Figure 2.3.

This implies that the Green's function itself can ultimately be obtained by adding the singular component to $R(r, \varphi; \varrho, \psi)$. That is,

$$G(r, \varphi; \varrho, \psi) = \frac{1}{4\pi} \ln \prod_{n=1}^{3} \frac{r^2 - 2r\varrho \cos(\varphi - (\frac{2n\pi}{3} - \psi)) + \varrho^2}{r^2 - 2r\varrho \cos(\varphi - (\frac{2(n-1)\pi}{3} + \psi)) + \varrho^2}. \tag{2.12}$$

Contrary to the case of the mixed problem that we covered in Example 2.3, where the Green's function was successfully constructed for the infinite sector of $\pi/2$, the method of images fails in the case of a mixed problem on the infinite sector of $\pi/3$. We refer the reader to the next example, for proof of this assertion.

Example 2.5. We note the failure of the method of images, when applying it to case of the Green's function for the Dirichlet–Neumann problem on the infinite sector with angle $\pi/3$. The outline of our procedure is depicted in Figure 2.4.

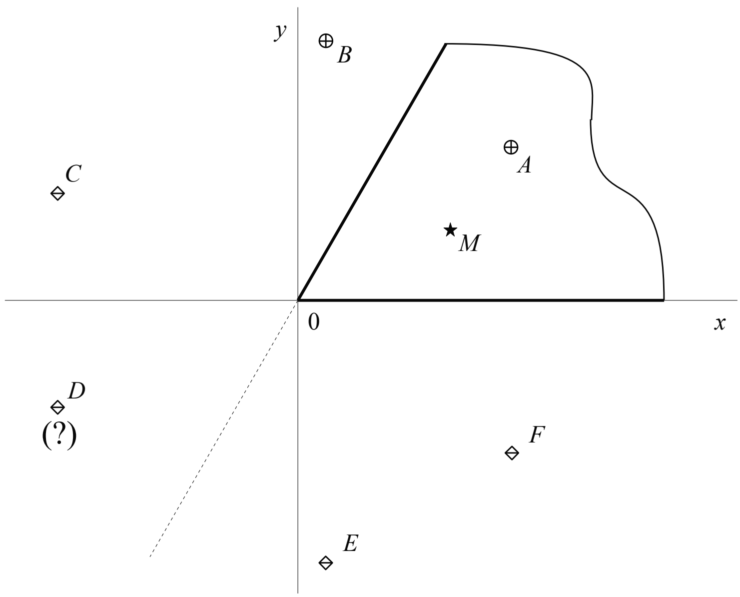

Figure 2.4. Failure of the method of images for a mixed problem.

Clearly, the Dirichlet condition on $\varphi = 0$ is satisfied by placing a unit sink in $F(\varrho, 2\pi - \psi)$. To allow this sink to satisfy the Neumann condition on $\varphi = \pi/3$, we also require one at $C(\varrho, 2\pi/3 + \psi)$. As to the Neumann condition on $\varphi = \pi/3$, the unit source in $A(\varrho, \psi)$ must be augmented with the unit source at $B(\varrho, 2\pi/3 - \psi)$, which, in turn, should be paired with a unit sink placed in $E(\varrho, 4\pi/3 + \psi)$. The latter sink must be paired with the unit sink at $D(\varrho, 4\pi/3 - \psi)$ for the Neumann condition to be satisfied on $\varphi = \pi/3$. If we now take a look at the two sinks in $C(\varrho, 2\pi/3 + \psi)$ and $D(\varrho, 4\pi/3 - \psi)$, we find that they don't support the Dirichlet condition on the boundary fragment $\varphi = 0$. This indicates the method's failure for this mixed problem.

We will discuss further examples, extending the range of possible problems, on an infinite circular sector, for which the method of images turns out to be efficient.

Example 2.6. Consider the case of a Dirichlet problem on the infinite circular sector with angle $\pi/4$, $\Omega(r, \varphi) = \{0 < r < \infty, 0 < \varphi < \pi/4\}$.

The scheme depicted in Figure 2.5, along with the experience gained from the previous examples, allows the reader to follow the procedure in detail and helps us obtain the Green's function for the Dirichlet problem on the circular sector $\Omega(r, \varphi) =$

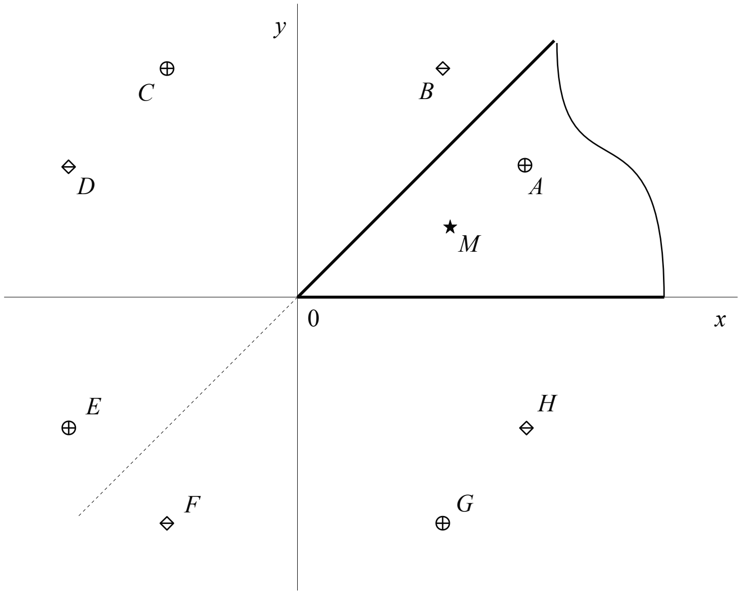

Figure 2.5. Dirichlet problem on the infinite circular sector $\pi/4$.

$\{0 < r < \infty, 0 < \varphi < \pi/4\}$. We can write it as

$$G(r,\varphi;\varrho,\psi) = \frac{1}{4\pi} \ln \prod_{n=1}^{4} \frac{r^2 - 2r\varrho\cos(\varphi - (\frac{n\pi}{2} - \psi)) + \varrho^2}{r^2 - 2r\varrho\cos(\varphi - (\frac{(n-1)\pi}{2} + \psi)) + \varrho^2}. \tag{2.13}$$

Example 2.7. The Green's function for the mixed problem on the infinite sector with angle $\pi/4$ can also be obtained using the method of images. To justify this assertion, consider the statement with Dirichlet and Neumann conditions imposed on the boundary segments $\varphi = 0$ and $\varphi = \pi/4$, respectively.

In order to outline the procedure for the method of images, we examine the scheme shown in Figure 2.6. Combining the field generated by eight sources and sinks in total, that this problem results in, we find the Green's function that we are looking for in the compact form

$$G(r,\varphi;\varrho,\psi) = \frac{1}{4\pi} \ln \prod_{n=1}^{2} \frac{r^2 - 2r\varrho\cos(\varphi - (\frac{(2n-1)\pi}{2} + \psi)) + \varrho^2}{r^2 - 2r\varrho\cos(\varphi - (\frac{(2n-1)\pi}{2} - \psi)) + \varrho^2} \times \frac{r^2 - 2r\varrho\cos(\varphi - (n\pi - \psi)) + \varrho^2}{r^2 - 2r\varrho\cos(\varphi - ((n-1)\pi + \psi)) + \varrho^2}. \tag{2.14}$$

Example 2.8. In the Dirichlet problem on the infinite sector with angle $\pi/6$, the method of images results in twelve unit sources and sinks, the aggregate of which

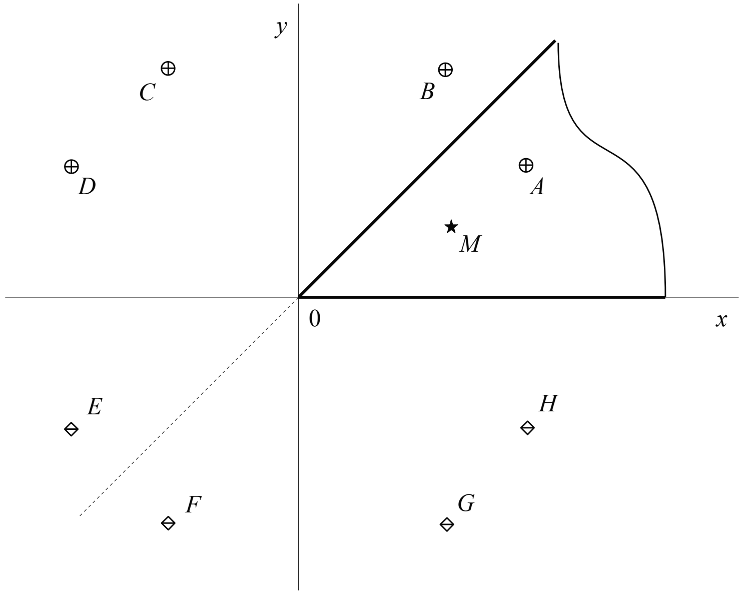

Figure 2.6. Dirichlet–Neumann problem on the infinite circular sector $\pi/4$.

represents the Green's function of our interest, in the form

$$G(r, \varphi; \varrho, \psi) = \frac{1}{4\pi} \ln \prod_{n=1}^{6} \frac{r^2 - 2r\varrho\cos(\varphi - (\frac{n\pi}{3} - \psi)) + \varrho^2}{r^2 - 2r\varrho\cos(\varphi - (\frac{(n-1)\pi}{3} + \psi)) + \varrho^2}. \qquad (2.15)$$

After analyzing the boundary-value problems, discussed so far for an infinite circular sector, we can come to a certain generalization. The following two examples, introduce the reader to details.

Example 2.9. Keeping in mind the Green's function of the Dirichlet problem we derived earlier, on the infinite sector with angle $\pi/2$, presented in (2.10), along with the one we obtained for the circular sector with angle $\pi/4$ (see (2.13)), we make the following generalization

$$G(r, \varphi; \varrho, \psi) = \frac{1}{4\pi} \ln \prod_{n=1}^{2^k} \frac{r^2 - 2r\varrho\cos(\varphi - (\frac{n\pi}{2^{k-1}} - \psi)) + \varrho^2}{r^2 - 2r\varrho\cos(\varphi - (\frac{(n-1)\pi}{2^{k-1}} + \psi)) + \varrho^2} \qquad (2.16)$$

representing the Green's function of the Dirichlet problem for the infinite sector with angle $\pi/2^k$, where $k = 0, 1, 2, \ldots$.

It is worth noting that in the case of $k = 0$, corresponding to the circular sector with angle π or, in other words, to the upper half-plane $y > 0$, we get from (2.16)

$$G(r, \varphi; \varrho, \psi) = \frac{1}{4\pi} \ln \frac{r^2 - 2r\varrho \cos(\varphi + \psi) + \varrho^2}{r^2 - 2r\varrho \cos(\varphi - \psi) + \varrho^2}$$

which represents Green's function we derived earlier (see (2.5)), here expressed in polar coordinates.

Example 2.10. Analysis of (2.12) and (2.15), obtained for the infinite sectors with the angle of $\pi/3$ and $\pi/6$, we can find the Green's function of the Dirichlet problem for the sector with angle $\pi/(3 \cdot 2^k)$

$$G(r, \varphi; \varrho, \psi) = \frac{1}{4\pi} \ln \prod_{n=1}^{3\cdot 2^k} \frac{r^2 - 2r\varrho \cos(\varphi - (\frac{2n\pi}{3\cdot 2^k} - \psi)) + \varrho^2}{r^2 - 2r\varrho \cos(\varphi - (\frac{2(n-1)\pi}{3\cdot 2^k} + \psi)) + \varrho^2}, \tag{2.17}$$

where $k = 0, 1, 2, \ldots$. Observe that $k = 0$ represents the infinite sector with the angle of $\pi/3$, while $k = 1$ represents the circular sector of $\pi/6$, and so on.

Earlier in this section, we gave an example of a problem for which the method of images may fail when applied to a mixed (Dirichlet–Neumann) boundary-value problem stated on an infinite sector. Note that the method is not necessarily effective even for the Dirichlet problem on a circular sector. We illustrate this with the following example.

Example 2.11. Consider the Dirichlet problem for the infinite circular sector $\Omega(r, \varphi) = \{0 < r < \infty, 0 < \varphi < 2\pi/3\}$ and attempt to construct its Green's function.

To show the failure of the method of images in this problem, we refer to the scheme shown in Figure 2.7. Let the unit source (which produces the singular component of the Green's function) be located at $A(\varrho, \psi) \in \Omega$. To compensate its trace on the fragment $\varphi = 0$ of the boundary Ω, place the compensating sink at $D(\varrho, 2\pi - \psi) \notin \Omega$. The trace of the latter on the boundary fragment $\varphi = 2\pi/3$ is, in turn, cancelled out by a unit source at $C(\varrho, 4\pi/3 + \psi) \notin \Omega$, whose trace on $\varphi = 0$ has to be compensated with a unit sink at $B(\varrho, 2\pi/3 - \psi)$, which is, located *inside* Ω. This is what leads to the failure of the method: compensatory sources and sinks cannot, according to the definition of the Green's function, be located inside Ω.

As we can see from this example the method of images can potentially fail when attempting to construct the Green's function for the Dirichlet problem on a circular sector, allowing cyclic symmetry. To study various other cases where the method fails, we refer the reader to the Chapter Exercises and encourage to apply the method procedure to other circular sectors (with for example angles $2\pi/5$ or $2\pi/7$) also allowing the cyclic symmetry.

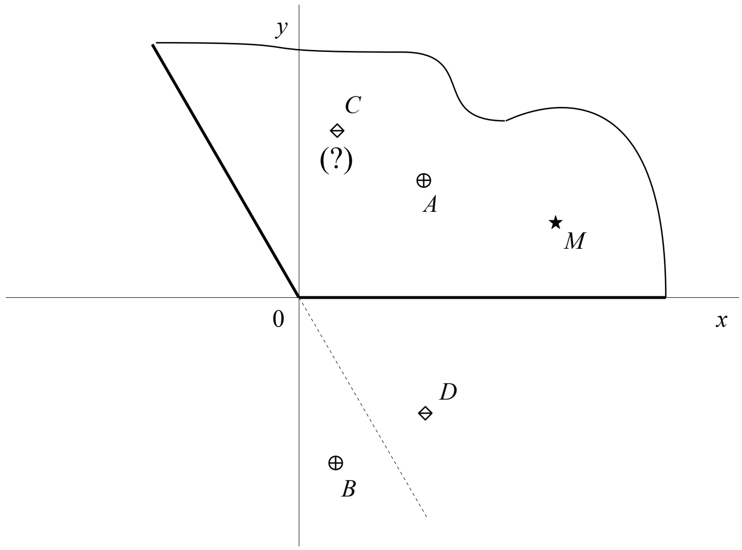

Figure 2.7. Failure of the method of images for a Dirichlet problem.

Based on the experience gained so far, it seems reasonable to suggest that the method of images turns out to be workable for Dirichlet problems on circular sectors with the angle of π/k, with k an integer. However, a word of caution is appropriate: what we presented so far, is merely to give a taste of the problem. The reader is strongly encouraged to find a way to prove it rigorously.

Example 2.12. In the next example of effective application of the method of images, we turn to the construction of Green's function for another classical case of the Dirichlet problem, on the disk with radius a.

We base our strategy for tackling the current problem using the method of images on an obvious fact, concerning the shape of equipotential lines in the two-dimensional field generated by a point source or a sink. Since these lines represent concentric circles centered at the generating point, we can reasonably state the following: for every location A of the unit source inside the disk, there exists a proper location B of the compensatory unit sink outside the disk so that the circumference of the disk is an equipotential line for the field generated by both the source and the sink.

Following the strategy just described, let the disk be centered at the origin of the polar coordinate system, and let the unit source generating the singular component

$$-\frac{1}{4\pi}\ln\left(r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2\right) \tag{2.18}$$

of the Green's function at $M(r,\varphi)$ be located at $A(\varrho,\psi)$ (see Figure 2.8). Let also $C(a,\varphi)$ be an arbitrary point on the circumference of the disk. It is evident that the

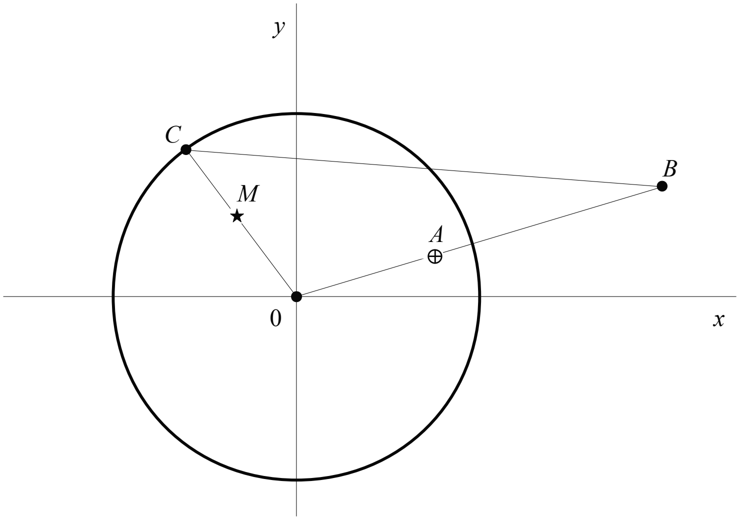

Figure 2.8. To the derivation of the Green's function for a disk.

point $B(\widetilde{\varrho}, \psi)$, where the compensatory unit sink

$$\frac{1}{4\pi} \ln \left(r^2 - 2r\widetilde{\varrho}\cos(\varphi - \psi) + \widetilde{\varrho}^2\right) \tag{2.19}$$

is located, must be on the extension of the radial line A. In other words, the angular component of B, ψ must be the same as of A. The radial component of B, $\widetilde{\varrho}$ can be determined from the condition that, when M approaches C ($r = a$), the sum of (2.18) and (2.19) becomes a constant, say λ. For the sake of convenience, we express λ as

$$\lambda = -\frac{1}{4\pi} \ln \mu. \tag{2.20}$$

This yields

$$\frac{1}{4\pi} \ln \frac{a^2 - 2a\widetilde{\varrho}\cos(\varphi - \psi) + \widetilde{\varrho}^2}{a^2 - 2a\varrho\cos(\varphi - \psi) + \varrho^2} = -\frac{1}{4\pi} \ln \mu$$

or

$$a^2 - 2a\varrho\cos(\varphi - \psi) + \varrho^2 = \mu \left(a^2 - 2a\widetilde{\varrho}\cos(\varphi - \psi) + \widetilde{\varrho}^2\right).$$

Making the substitution

$$\widetilde{\varrho} = \omega\varrho \tag{2.21}$$

we transform the above equation into

$$a^2 - 2a\varrho\cos(\varphi - \psi) + \varrho^2 = \mu\left(a^2 - 2a\omega\varrho\cos(\varphi - \psi) + \omega^2\varrho^2\right). \tag{2.22}$$

Clearly, equation (2.22) must hold for all values of $\varphi - \psi$, so, assuming for instance, $\varphi - \psi = \pi/2$ (which implies $\cos(\varphi - \psi) = 0$) we can reduce (2.22) to

$$a^2 + \varrho^2 = \mu\left(a^2 + \omega^2\varrho^2\right). \tag{2.23}$$

Subtracting (2.23) from (2.22), we now have

$$2a\varrho\cos(\varphi - \psi) = 2\mu a\omega\varrho\cos(\varphi - \psi).$$

This means that $\mu\omega = 1$. That is, the μ and ω each others' reciprocal. Substituting $\mu = 1/\omega$ into (2.23) yields

$$a^2 + \varrho^2 = \frac{1}{\omega}a^2 + \omega\varrho^2,$$

which can be rewritten as

$$a^2(\omega - 1) = \varrho^2\omega(\omega - 1). \tag{2.24}$$

Hence, $\omega = 1$ is one of the roots of the quadratic equation (2.24). It is evident that this root is meaningless, because the relation in (2.21) suggests, in this case, that compensatory point B (see Figure 2.8) is the same as A. The second root $\omega = a^2/\varrho^2$ of (2.24) implies

$$\widetilde{\varrho} = a^2/\varrho \quad \text{and} \quad \mu = \varrho^2/a^2.$$

Thus, we found the location where the point $B(a^2/\varrho, \psi)$ should be placed. Such a point is usually referred to as the image of A about the circumference of the disk. We also found the value of λ in (2.20)

$$\lambda = -\frac{1}{4\pi}\ln\frac{\varrho^2}{a^2}.$$

To complete the construction of the Green's function, observe that the unit sink at B generates the potential field

$$\frac{1}{4\pi}\ln\left(\frac{a^4}{\varrho^2} - 2r\frac{a^2}{\varrho}\cos(\varphi - \psi) + r^2\right)$$

at a point $M(r, \varphi)$ inside the disk. Hence, the potential field generated at $M(r, \varphi)$ by the unit source at A and the compensatory unit sink at B together, becomes

$$\frac{1}{4\pi}\ln\frac{a^4 - 2r\varrho a^2\cos(\varphi - \psi) + r^2\varrho^2}{\varrho^2\left(r^2 - 2r\varrho\cos(\varphi - \psi) + \varrho^2\right)}. \tag{2.25}$$

In other words, equation (2.25) presents a function that is harmonic everywhere inside the disk with the exception of the source point (ϱ, ψ). Additionally, the function in (2.25) takes on a constant value of $-\frac{1}{4\pi}\ln(\varrho^2/a^2)$ on the boundary of the disk. Thus, compensating the (2.25) with the opposite value of $-\frac{1}{4\pi}\ln(a^2/\varrho^2)$, we ultimately obtain the Green's function of the Dirichlet problem for the disk of radius a as

$$G(r,\varphi;\varrho,\psi) = \frac{1}{4\pi}\ln\frac{a^4 - 2r\varrho a^2\cos(\varphi-\psi) + r^2\varrho^2}{\varrho^2\left(r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2\right)} - \frac{1}{4\pi}\ln\frac{a^2}{\varrho^2}$$

which evidently reduces to

$$G(r,\varphi;\varrho,\psi) = \frac{1}{4\pi}\ln\frac{a^4 - 2r\varrho a^2\cos(\varphi-\psi) + r^2\varrho^2}{a^2\left(r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2\right)} \tag{2.26}$$

and can be rewritten as

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\ln\frac{|z\overline{\zeta} - a^2|}{a|z-\zeta|},$$

where we use complex variable notation $z = r(\cos\varphi + i\sin\varphi)$ and $\zeta = \varrho(\cos\psi + i\sin\psi)$ for the observation and the source point.

Note that all the Green's functions we have constructed so far, are expressed in a closed analytical form. In a series of examples that follow, we focus on a several other problems, for which the method of images allows us to obtain Green's functions expressed in terms of infinite products.

Example 2.13. We start with one of the classical settings, namely the Dirichlet problem on the infinite strip $\Omega(x,y) = \{-\infty < x < \infty, 0 < y < b\}$.

The closed analytical form of the Green's functions for this problem found in (2.63) and (2.64) of Section 2.3, is well known and available in every standard textbook [3, 17, 18, 22, 29, 45, 57] on partial differential equations. Later in this chapter, the reader will learn how it can be constructed by the methods of conformal mapping and eigenfunction expansion.

In what follows, we will show an alternative to the classical form of this Green's function by applying the method of images. To aid in understanding our procedure, we refer the reader to the scheme depicted in Figure 2.9, where we place a unit source S_0^+ in an arbitrary point $A(\xi,\eta)$ inside Ω. The field generated by the component S_0^+ in a point $M(x,y)$ represents the fundamental solution

$$G_0^+(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln\sqrt{(x-\xi)^2 + (y-\eta)^2}$$

of the Laplace equation.

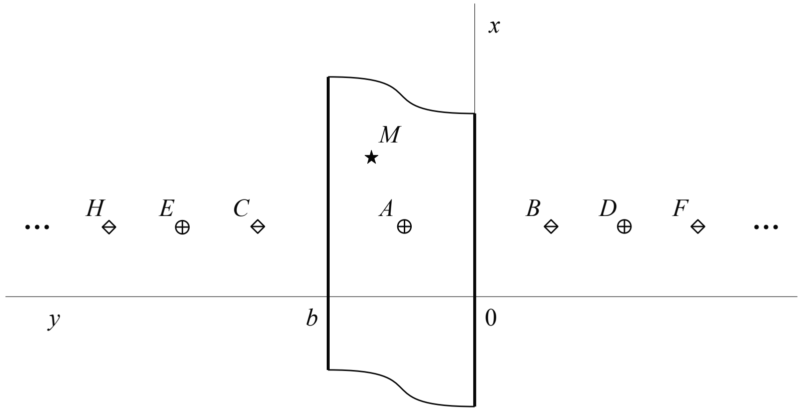

Figure 2.9. Derivation of the Green's function for the Dirichlet problem.

Clearly, the function $G_0^+(x, y; \xi, \eta)$ conflicts with the Dirichlet conditions on boundary fragments $y = 0$ and $y = b$ (it does not vanish). To compensate the traces of $G_0^+(x, y; \xi, \eta)$ on $y = 0$ and $y = b$, we place two unit sinks $S_{1,0}^-$ and $S_{1,b}^-$ in the points $B(\xi, -\eta)$ and $C(\xi, 2b-\eta)$, which represent the images of (ξ, η) about the lines $y = 0$ and $y = b$, respectively. The fields generated by these sinks in (x, y) evidently are

$$G_{1,0}^-(x, y; \xi, -\eta) = \frac{1}{2\pi} \ln \sqrt{(x-\xi)^2 + (y+\eta)^2}$$

and

$$G_{1,b}^-(x, y; \xi, 2b-\eta) = \frac{1}{2\pi} \ln \sqrt{(x-\xi)^2 + (y-(2b-\eta))^2}.$$

Traces of the functions $G_{1,0}^-(x, y; \xi, -\eta)$ and $G_{1,b}^-(x, y; \xi, 2b-\eta)$ on the boundary lines $y = 0$ and $y = b$ can, in turn, be compensated with the unit sources $S_{2,0}^+$ and $S_{2,b}^+$ which are located at $D(\xi, -2b+\eta)$ and $E(\xi, 2b+\eta)$. The combined fields generated at (x, y) are given as

$$G_{2,0}^+(x, y; \xi, -2b+\eta) = -\frac{1}{2\pi} \ln \sqrt{(x-\xi)^2 + (y-(-2b+\eta))^2}$$

and

$$G_{2,b}^+(x, y; \xi, 2b+\eta) = -\frac{1}{2\pi} \ln \sqrt{(x-\xi)^2 + (y-(2b+\eta))^2}.$$

Traces of the functions $G_{2,0}^+(x, y; \xi, -2b+\eta)$ and $G_{2,b}^+(x, y; \xi, 2b+\eta)$ on $y = 0$ and $y = b$ can then be cancelled out with unit sinks $S_{3,0}^-$ and $S_{3,b}^-$ located at $F(\xi, -2b-\eta)$ and $H(\xi, 4b-\eta)$, respectively.

Following the described procedure of properly placing compensatory unit sources alternating with unit sinks, the Green's function $G = G(x, y; \xi, \eta)$ of interest is obtained in infinite series form

$$G = G_0^+ + \sum_{i=1}^{\infty}(G_{2i-1,0}^- + G_{2i-1,b}^-) + \sum_{i=1}^{\infty}(G_{2i,0}^+ + G_{2i,b}^+).$$

Since the terms of this series represent logarithmic functions, the Nth partial sum

$$S_N(x, y; \xi, \eta) = G_0^+ + \sum_{i=1}^{N}(G_{2i-1,0}^- + G_{2i-1,b}^-) + \sum_{i=1}^{N}(G_{2i,0}^+ + G_{2i,b}^+)$$

can be written as the single logarithm

$$S_N(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \prod_{n=-N}^{N} \sqrt{\frac{(x-\xi)^2 + (y+\eta-2nb)^2}{(x-\xi)^2 + (y-\eta+2nb)^2}}.$$

After taking a limit as N approaches infinity, in the above expression

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \prod_{n=-\infty}^{\infty} \sqrt{\frac{(x-\xi)^2 + (y+\eta-2nb)^2}{(x-\xi)^2 + (y-\eta+2nb)^2}} \tag{2.27}$$

we finally obtain the infinite product representation of the Green's function for the Dirichlet problem on an infinite strip of width b.

Example 2.14. We now turn to another classical case, namely a mixed problem for the Laplace equation on the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$, with the Dirichlet condition imposed on $y = 0$ and the Neumann condition imposed on $y = b$. Later in the Chapter Exercises, the reader ist advised to construct a closed analytical expression for this Green's function.

Figure 2.10 illustrates the procedure of the methods of images, applied in a way similar to the one described earlier for the Dirichlet problem.

An alternative formula for the Green's function for the Dirichlet–Neumann problem (2.11) can be obtained as the aggregate field generated by an infinite sum of properly spaced unit sources and sinks. Their locations are chosen in accordance with the following pattern.

To compensate the trace of the fundamental solution $G_0^+(x, y; \xi, \eta)$ on the boundary line $y = 0$, a unit sink $S_{1,0}^-$ is placed in the point $B(\xi, -\eta)$, which generates a field at $M(x, y)$ given by

$$G_{1,0}^-(x, y; \xi, -\eta) = \frac{1}{2\pi} \ln \sqrt{(x-\xi)^2 + (y+\eta)^2}$$

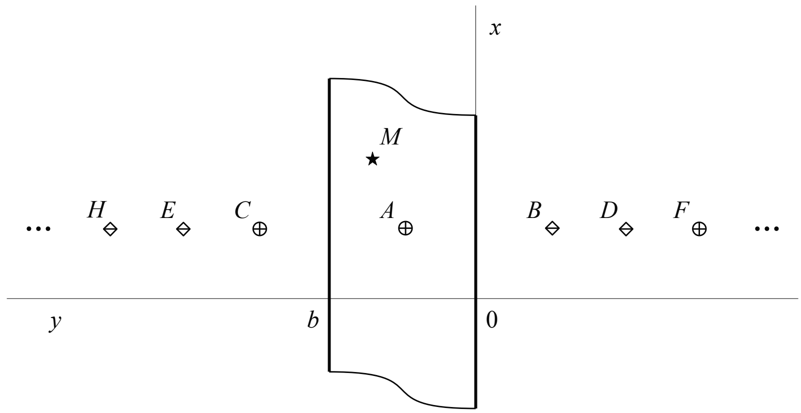

Figure 2.10. Derivation of the Green's function for the mixed problem.

The Neumann condition on $y = b$, can be satisfied by placing a unit source $S^+_{1,b}$ in point $C(\xi, 2b - \eta)$. This yields

$$G^+_{1,b}(x, y; \xi, 2b - \eta) = -\frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - (2b - \eta))^2}.$$

The trace of the function $G^+_{1,b}(x, y; \xi, 2b - \eta)$ on the boundary line $y = 0$ can, in turn, be compensated by a unit sink $S^-_{2,0}$ placed in $D(\xi, -2b + \eta)$, which generates a field at (x, y)

$$G^-_{2,0}(x, y; \xi, -2b + \eta) = \frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - (-2b + \eta))^2}$$

while the Neumann condition on $y = b$ can be satisfied by a unit sink $S^-_{2,b}$ located at $E(\xi, 2b + \eta)$, which at (x, y) generates the field

$$G^-_{2,b}(x, y; \xi, 2b + \eta) = \frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - (2b + \eta))^2}.$$

The trace of the function $G^-_{2,b}(x, y; \xi, 2b + \eta)$ on $y = 0$ can, in turn, be compensated by a unit source $S^+_{3,0}$ placed in $F(\xi, -2b - \eta)$, which generates the field

$$G^+_{3,0}(x, y; \xi, -2b - \eta) = -\frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y + (2b + \eta))^2}$$

while the Neumann condition on $y = b$ can be satisfied by placing a unit sink $S^-_{3,b}$ in $H(\xi, 4b - \eta)$, which at (x, y) generates the field

$$G^-_{3,b}(x, y; \xi, 4b - \eta) = \frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - (4b - \eta))^2}.$$

Continuing this process and following the approach described in Example 2.13, the sought-after Green's function is finally obtained in the following infinite product form

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \prod_{n=-\infty}^{\infty} \sqrt{\frac{(x-\xi)^2 + (y+\eta+4nb)^2}{(x-\xi)^2 + (y-\eta+4nb)^2}} \times \sqrt{\frac{(x-\xi)^2 + [y-\eta+2(2n+1)b]^2}{(x-\xi)^2 + [y+\eta+2(2n+1)b]^2}} \quad (2.28)$$

which is an alternative to the classical analytical form exhibited earlier in (2.11).

We will now apply the technique based on the method of images to obtain infinite product representations of the Green's functions of other classical boundary-value problems.

Example 2.15. Consider the Dirichlet problem on the semi-infinite strip $\Omega = \{(x, y) | 0 < x < \infty, 0 < y < b\}$. A closed analytical form of its Green's function (see (2.67) and (2.68) in Section 2.3) can be found in most canonical sources. For example, in [45], it was obtained with a modified version of the method of eigenfunction expansion.

In the following, we will derive an alternative infinite product form of the Green's function for our current problem with the aid of the method of images. To sketch its procedure in a way similar to that described earlier in detail in Examples 2.13 and 2.14, we refer the reader to follow the scheme depicted in Figure 2.11.

The potential field generated by a unit source acting at an arbitrary point $A(\xi, \eta)$ in Ω can be compensated on the edges $y = 0$ and $y = b$ with unit sources and sinks

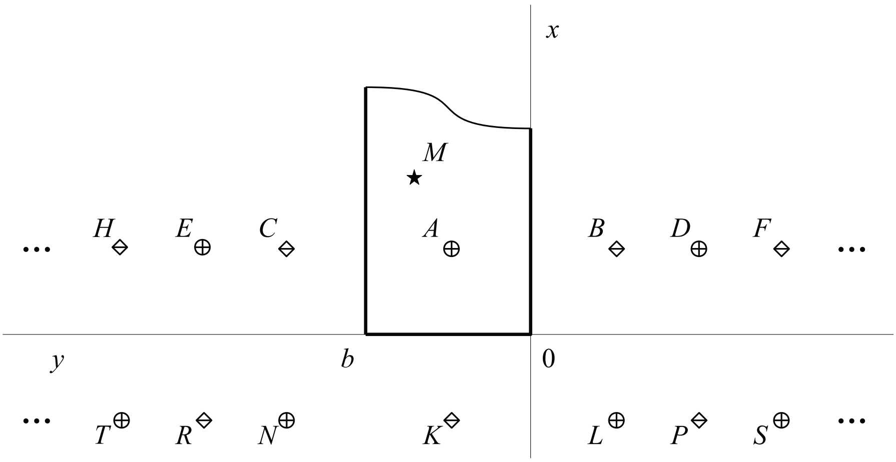

Figure 2.11. Derivation of an alternative form of the Green's function.

placed in the regular set of points $B(\xi,-\eta)$, $C(\xi,2b-\eta)$, $D(\xi,-2b+\eta)$, $E(\xi,2b+\eta)$, $F(\xi,-2b-\eta)$, $H(\xi,4b-\eta)$ etc., located outside of Ω. In other words, these sources and sinks allow us to satisfy the homogeneous Dirichlet boundary conditions imposed on the edges $y=0$ and $y=b$of Ω.

As to the boundary condition imposed on $x=0$, the influence of the sources and sinks acting at A, B, C, D, E, F, H etc. can, in turn, be compensated on that boundary line with unit sources and sinks, if we place them in another set of points $K(-\xi,\eta)$, $L(-\xi,-\eta)$, $N(-\xi,2b-\eta)$, $P(-\xi,-2b+\eta)$, $R(-\xi,2b+\eta)$, $S(-\xi,-2b-\eta)$, $T(-\xi,4b-\eta)$ etc., located outside of Ω. It is evident that the latter sources and sinks do not conflict with the boundary conditions on $y=0$ and $y=b$.

Thus, upon combining the influence of all canceling sources and sinks shown in Figure 2.11, one arrives at a form of the Green's function of the Dirichlet problem for the semi-infinite strip $\Omega=\{(x,y)|0<x<\infty, 0<y<b\}$, alternative to (2.68), expressed in the infinite product representation

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\prod_{n=-\infty}^{\infty}\sqrt{\frac{(x-\xi)^2+(y+\eta-2nb)^2}{(x-\xi)^2+(y-\eta+2nb)^2}} \times\sqrt{\frac{(x+\xi)^2+(y-\eta+2nb)^2}{(x+\xi)^2+(y+\eta-2nb)^2}}. \tag{2.29}$$

Example 2.16. As another example for the semi-infinite strip $\Omega=\{0<x<\infty, 0<y<b\}$, we consider a mixed boundary value problem: let Dirichlet conditions be imposed on the boundary fragments $y=0$ and $y=b$, whilst the Neumann condition is imposed on $x=0$.

The compact formula

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\sqrt{\frac{1-2\mathrm{e}^{\omega(x+\xi)}\cos\omega(y+\eta)+\mathrm{e}^{2\omega(x+\xi)}}{1-2\mathrm{e}^{\omega(x-\xi)}\cos\omega(y-\eta)+\mathrm{e}^{2\omega(x-\xi)}}} \times\sqrt{\frac{1-2\mathrm{e}^{\omega(x-\xi)}\cos\omega(y+\eta)+\mathrm{e}^{2\omega(x-\xi)}}{1-2\mathrm{e}^{\omega(x+\xi)}\cos\omega(y-\eta)+\mathrm{e}^{2\omega(x+\xi)}}},\quad \omega=\frac{\pi}{b}, \tag{2.30}$$

of the Green's function for our problem is described in [45], for example. An alternative formula to (2.30) can be derived with the aid of the procedure outlined in Figure 2.12.

As the previous example suggests, the traces of the fundamental solution (the field generated by the unit source acting at an arbitrary point $A(\xi,\eta)$ in Ω) on the edges $y=0$ and $y=b$ are compensated with unit sources and sinks, placed in $B(\xi,-\eta)$, $C(\xi,2b-\eta)$, $D(\xi,-2b+\eta)$, $E(\xi,2b+\eta)$, $F(\xi,-2b-\eta)$, $H(\xi,4b-\eta)$, etc., the set of points exterior to Ω.

To satisfy the Neumann condition for $x=0$, the influence of the sources and sinks in A, B, C, D, E, F, H etc. can be compensated for, similar to the Dirichlet problem,

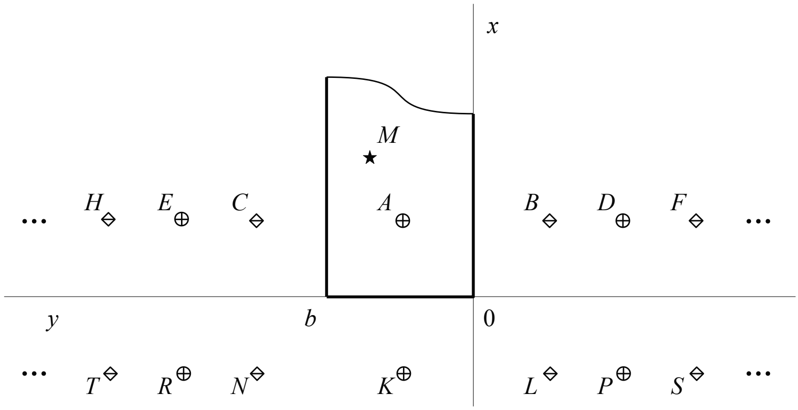

Figure 2.12. Derivation of an alternative formula for (2.30).

with unit sources and sinks at $K(-\xi, \eta)$, $L(-\xi, -\eta)$, $N(-\xi, 2b-\eta)$, $P(-\xi, -2b+\eta)$, $R(-\xi, 2b+\eta)$, $S(-\xi, -2b-\eta)$, $T(-\xi, 4b-\eta)$, etc., the set of points exterior to Ω. The order of sources and sinks is, however, different from the one suggested earlier for the Dirichlet problem.

Following the method of images, we arrives at the infinite-product-containing formula

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \prod_{n=-\infty}^{\infty} \sqrt{\frac{(x-\xi)^2 + (y+\eta-2nb)^2}{(x-\xi)^2 + (y-\eta+2nb)^2}} \times \sqrt{\frac{(x+\xi)^2 + (y+\eta-2nb)^2}{(x+\xi)^2 + (y-\eta+2nb)^2}} \tag{2.31}$$

of the Green's function for this mixed problem.

2.2 Conformal Mapping

Another method which has traditionally been employed for the construction of Green's functions for the two-dimensional Laplace equation, is the *method of conformal mapping* [15, 37, 66]. It is rooted into the classical topic of complex analysis. To gain an insight into its background, let $w(z, \zeta)$ represent a function of a complex variable mapping conformally a simply-connected region Ω, bounded by L, on the interior of a unit disk $|w| \leq 1$, while the point $z = \zeta$ maps onto the center $w = 0$ of the disk, that is $w(\zeta, \zeta) = 0$.

It is worth noting that conformal mapping of a simply-connected region onto a disk is not unique. Indeed, it is defined up to an arbitrary rotation around the disk's center.

As can be learned from complex analysis [15, 37], the Green's function to the Dirichlet problem

$$\nabla^2 u(P) = 0, \quad P \in \Omega, \tag{2.32}$$

$$u(P) = 0, \quad P \in L, \tag{2.33}$$

can be expressed in terms of the mapping function $w(z, \zeta)$ as

$$G(P, Q) = -\frac{1}{2\pi} \ln |w(z, \zeta)| \tag{2.34}$$

with $z = x + iy$ the observation point P, whilst $\zeta = \xi + i\eta$ is the source point Q.

This statement can be readily justified: we observe that since the function $w = w(z, \zeta)$ performs conformal mapping of Ω, it is an analytic function of z of Ω and $w(z, \zeta) \neq 0$ if $z \neq \zeta$, whilst $\frac{dw}{dz} \neq 0$ uniformly on Ω, with $z = \zeta$ included. Consequently, $z = \zeta$ is a simple pole for $w(z, \zeta)$. Hence, we can express the latter in the formula

$$w(z, \zeta) = (z - \zeta)\Phi(z, \zeta) \tag{2.35}$$

with $\Phi(z, \zeta)$ representing an analytic function of z of Ω, which is nonzero as $z = \zeta$, that is $\Phi(\zeta, \zeta) \neq 0$.

Since an analytic function of an analytic function is also analytic, we can state that the function

$$\ln \Phi(z, \zeta) = \ln |\Phi(z, \zeta)| + i \arg \Phi(z, \zeta) \tag{2.36}$$

is an analytic function of Ω. Hence, the real component $\ln |\Phi(z, \zeta)|$ in (2.36) represents a harmonic function of Ω and so, obviously, is

$$-\frac{1}{2\pi} \ln |\Phi(z, \zeta)|.$$

Hence, in light of the relation in (2.35), the function in (2.34) becomes

$$-\frac{1}{2\pi} \ln |w(z, \zeta)| = -\frac{1}{2\pi} \ln |z - \zeta| - \frac{1}{2\pi} \ln |\Phi(z, \zeta)|$$

which can be rewritten in the formula

$$-\frac{1}{2\pi} \ln |w(z, \zeta)| = -\frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - \eta)^2} - \frac{1}{2\pi} \ln |\Phi(z, \zeta)|.$$

We leave it as an easy exercise to the reader to show that the component

$$-\frac{1}{2\pi} \ln \sqrt{(x - \xi)^2 + (y - \eta)^2}$$

is a harmonic function of x and y almost everywhere on Ω. More specific, it is harmonic at every point $(x, y) \in \Omega$, except at $(x, y) = (\xi, \eta)$. This implies that the function in (2.34), as a function of x and y, is also harmonic everywhere in Ω, except at $(x, y) = (\xi, \eta)$.

We have shown that the function in (2.34) meets two of the three defining properties of a Green's function: it is harmonic everywhere in Ω , except at $(x, y) = (\xi, \eta)$ and contains a logarithmic singularity for $(x, y) \to (\xi, \eta)$. But what about the third defining property? Does the function in (2.34) vanish on the boundary L of Ω? The answer is *yes*, because from the fact that $w(z, \zeta)$ maps L onto the circumference of the unit disk, it follows that

$$|w(z, \zeta)| = 1 \quad \text{for } z \in L,$$

which leads us to

$$-\frac{1}{2\pi} \ln |w(z, \zeta)| = 0 \quad \text{for } z \in L.$$

Thus, equation (2.34) does indeed represent the Green's function to the Dirichlet problem described by (2.32) and (2.33).

In the following, the reader will finds several examples of the construction of Green's functions by the method of conformal mapping.

Example 2.17. Let us apply the method to the Dirichlet problem defined on the unit disk $|z| \leq 1$.

The family of functions $w(z, \zeta)$ that maps the unit disk onto itself conformally, with the point $z = \zeta$ being mapped onto the disk's center, is defined [15, 37] as

$$w(z, \zeta) = e^{i\beta} \cdot \frac{z - \zeta}{z\overline{\zeta} - 1},$$

where β is a real parameter characterizing the rotation of the disk around its center. For the sake of uniqueness, we neglect the rotation by assuming $\beta = 0$.

In compliance with (2.34), we arrive at the following expression

$$G(z, \zeta) = -\frac{1}{2\pi} \ln \left| \frac{z - \zeta}{z\overline{\zeta} - 1} \right| = \frac{1}{2\pi} \ln \left| \frac{z\overline{\zeta} - 1}{z - \zeta} \right| \tag{2.37}$$

for our Green's function.

Expressing the observation point z, and the source point ζ in polar coordinates

$$z = r(\cos\varphi + i \sin\varphi) \quad \text{and} \quad \zeta = \varrho(\cos\psi + i \sin\psi)$$

we can transform the numerator in the argument of the logarithm in (2.37) to

$$\begin{aligned} z\overline{\zeta} - 1 &= r(\cos\varphi + i\sin\varphi)\varrho(\cos\psi - i\sin\psi) - 1 \\ &= r\varrho[(\cos\varphi\cos\psi + \sin\varphi\sin\psi) + i(\cos\varphi\cos\psi + \sin\varphi\sin\psi)] - 1 \\ &= [r\varrho\cos(\varphi-\psi) - 1] + ir\varrho\sin(\varphi-\psi), \end{aligned}$$

the modulus of which is

$$\begin{aligned} |z\overline{\zeta} - 1| &= \sqrt{(r\varrho\cos(\varphi-\psi)-1)^2 + (r\varrho\sin(\varphi-\psi))^2} \\ &= \sqrt{r^2\varrho^2 - 2r\varrho\cos(\varphi-\psi) + 1}. \end{aligned} \tag{2.38}$$

The denominator in the argument of the logarithm in (2.37) represents the distance between z and ζ, which is

$$|z - \zeta| = \sqrt{r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2}. \tag{2.39}$$

Finally, after substituting (2.38) and (2.39) into (2.37), we obtain the Green's function for the Dirichlet problem on the unit disk

$$G(r,\varphi;\varrho,\psi) = \frac{1}{4\pi}\ln\frac{r^2\varrho^2 - 2r\varrho\cos(\varphi-\psi)+1}{r^2 - 2r\varrho\cos(\varphi-\psi)+\varrho^2}. \tag{2.40}$$

This may be compared to the one derived earlier, in Section 2.1 (see equation (2.26), with the radius a set to unity).

Example 2.18. We revisit one of the classical problems and will use the conformal mapping method to construct the Green's function of the Dirichlet problem on the infinite strip $\Omega = \{-\infty < x < \infty, 0 \le y \le \pi\}$.

From complex variable theory [15, 37], we learn that the family of functions

$$w(z,\zeta) = e^{i\beta}\frac{e^z - e^\zeta}{e^z - e^{\overline{\zeta}}} \tag{2.41}$$

maps the infinite strip Ω onto the unit disk conformally $|w| \le 1$, whilst the point $z = \zeta$ is mapped onto the disk's center $w = 0$. For the sake of uniqueness, assume the rotation parameter to be $\beta = 0$.

Before substituting this mapping function into (2.34), express the observation point and the source point in Cartesian coordinates

$$z = x + iy \quad \text{and} \quad \zeta = \xi + i\eta$$

and transform the modulus of the numerator in (2.41) by means of the classical Euler formula

$$|e^z - e^\zeta| = \sqrt{\mathrm{Re}^2(e^z - e^\zeta) + \mathrm{Im}^2(e^z - e^\zeta)},$$

where

$$\operatorname{Re}(\mathrm{e}^{z} - \mathrm{e}^{\zeta}) = \mathrm{e}^{x}\cos y - \mathrm{e}^{\xi}\cos\eta$$

and

$$\operatorname{Im}(\mathrm{e}^{z} - \mathrm{e}^{\zeta}) = \mathrm{e}^{x}\sin y - \mathrm{e}^{\xi}\sin\eta.$$

After some trivial steps of complex algebra we obtain

$$\begin{aligned}|\mathrm{e}^{z} - \mathrm{e}^{\zeta}| &= \sqrt{\mathrm{e}^{2x} + \mathrm{e}^{2\xi} - 2\mathrm{e}^{(x+\xi)}\cos(y-\eta)}\\ &= \mathrm{e}^{\xi}\cdot\sqrt{1 - 2\mathrm{e}^{(x-\xi)}\cos(y-\eta) + \mathrm{e}^{2(x-\xi)}}.\end{aligned}$$

The modulus of the denominator in (2.41) similarly transforms to

$$|\mathrm{e}^{z} - \mathrm{e}^{\overline{\zeta}}| = \mathrm{e}^{\xi}\cdot\sqrt{1 - 2\mathrm{e}^{(x-\xi)}\cos(y+\eta) + \mathrm{e}^{2(x-\xi)}}.$$

This yields the Green's function for our problem, namely

$$G(x,y;\xi,\eta) = \frac{1}{4\pi}\ln\frac{1 - 2\mathrm{e}^{(x-\xi)}\cos(y+\eta) + \mathrm{e}^{2(x-\xi)}}{1 - 2\mathrm{e}^{(x-\xi)}\cos(y-\eta) + \mathrm{e}^{2(x-\xi)}}. \tag{2.42}$$

An equivalent but more compact form of this Green's function can be obtained by multiplying both the numerator and the denominator in (2.42) with $\mathrm{e}^{(\xi-x)}$ yielding

$$G(x,y;\xi,\eta) = \frac{1}{4\pi}\ln\frac{\mathrm{e}^{(x-\xi)} + \mathrm{e}^{(\xi-x)} - 2\cos(y+\eta)}{\mathrm{e}^{(x-\xi)} + \mathrm{e}^{(\xi-x)} - 2\cos(y-\eta)}$$

which, after dividing both the numerator and the denominator of the argument of the logarithm by 2, transforms to

$$G(x,y;\xi,\eta) = \frac{1}{4\pi}\ln\frac{\cosh(x-\xi) - \cos(y+\eta)}{\cosh(x-\xi) - \cos(y-\eta)}. \tag{2.43}$$

Example 2.19. The half-plane $\Omega = \{-\infty < x < \infty, y \geq 0\}$ is mapped conformally onto the unit disk (with point $z = \zeta$ mapped onto the disk's center, $w = 0$) by a family of functions, one of which is [15, 37]

$$w(z,\zeta) = \frac{z-\zeta}{z-\overline{\zeta}}.$$

Thus, the Green's function for the Dirichlet problem for the Laplace equation on the half-plane is given by

$$G(z,\zeta) = \frac{1}{2\pi}\ln\left|\frac{z-\zeta}{z-\overline{\zeta}}\right|$$

which, in Cartesian coordinates, is

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{(x-\xi)^2 + (y+\eta)^2}{(x-\xi)^2 + (y-\eta)^2} \tag{2.44}$$

whereas in polar coordinates it is

$$G(r, \varphi; \varrho, \psi) = \frac{1}{4\pi} \ln \frac{r^2 - 2r\varrho\cos(\varphi+\psi) + \varrho^2}{r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2}. \tag{2.45}$$

Note that, in the previous section, we obtained the two above representations by the method of images.

Furthermore, note that in the examples considered so far in this section, conformal mapping of a given region onto the interior of a unit disk is performed by an elementary function. The next example is different; we aim to find the Green's function of the Dirichlet problem for a rectangle. However, conformal mapping of a rectangle onto a unit disk cannot be accomplished with an elementary function [15, 37, 66].

Example 2.20. As can be read in [66], the rectangle $\Omega = \{0 \le x \le a, 0 \le y \le b\}$ maps conformally onto the unit disk (with a point $z = \zeta$ mapped onto the disk's center $w = 0$) by the function

$$w(z, \zeta) = \frac{W(z-\zeta; \omega_1, \omega_2) \cdot W(z+\zeta; \omega_1, \omega_2)}{W(z-\overline{\zeta}; \omega_1, \omega_2) \cdot W(z+\overline{\zeta}; \omega_1, \omega_2)}$$

defined in terms of a special function $W(t; \omega_1, \omega_2)$ which is referred to [66] as the *Weierstrass elliptic function*. The parameters ω_1 and ω_2 in $W(t; \omega_1, \omega_2)$ are determined from the dimensions of the rectangle as $\omega_1 = 2a$ and $\omega_2 = 2ib$. Standard subroutines aiming at computing the Weierstrass function are not available in contemporary software, although this drawback can potentially be eliminated upon using, say, its series representation [66]

$$W(t; \omega_1, \omega_2) = \frac{1}{t^2} + \sum_{m,n=0}^{\infty} \left[\frac{1}{(t - 2m\omega_1 - 2n\omega_2)^2} - \frac{1}{(2m\omega_1 + 2n\omega_2)^2} \right],$$

where the summation indices m and n are not allowed to take simultaneous zero values.

Thus, the Green's function for the Dirichlet problem on the rectangle Ω is expressed in terms of the Weierstrass function as

$$G(z, \zeta) = \frac{1}{2\pi} \ln \left| \frac{W(z-\zeta; 2a, 2ib) \cdot W(z+\zeta; 2a, 2ib)}{W(z-\overline{\zeta}; 2a, 2ib) \cdot W(z+\overline{\zeta}; 2a, 2ib)} \right|. \tag{2.46}$$

In the next section, we will employ a method for the construction of Green's functions for the Laplace equation different from the two covered so far. We will consider an extensive list of boundary-value problems for a number of regions. Among others, we will revisit the Dirichlet problem on a rectangle. The reason for this is that the expression for $G(z,\zeta)$ in (2.46) is inconvenient for computer implementations, because subroutines for calculating the Weierstrass function are not included in standard computer software. Hence, in the following, we will derive a few more easily computable alternatives to (2.46) by using the method of eigenfunction expansion.

2.3 Method of Eigenfunction Expansion

In the remaining part of this chapter, we concentrate on giving the reader a comprehensive review of another method, which turns out to be extremely productive in the construction of Green's functions for boundary-value problems for many partial differential equations, namely the method of eigenfunction expansion, representing one of the most efficient and generally recommended methods in the field.

In particular, our review of the method of eigenfunction expansion aims at promoting constructive research in the derivation of different representations of Green's functions for applied partial differential equations.

2.3.1 Corollary of Green's Formula

Earlier in Chapter 2, the reader was familiarized with two standard methods, traditionally employed to construct of Green's functions for the two-dimensional Laplace equation. These are the method of images and the conformal mapping method. Chapter 1 dealt with ordinary differential equations and gave a background for the current chapter, where again the focus will be on partial differential equations. Another standard approach to the construction of Green's functions for the two-dimensional Laplace equation will be discussed herein. The method of eigenfunction expansion [29, 66] turns out to be workable for a quite broad class of problem settings.

The approach is based on a corollary of Green's second formula [3, 66], according to which the solution $u(P)$ of the well-posed boundary-value problem

$$\nabla^2 u(P) = -f(P), \quad P \in \Omega, \tag{2.47}$$

$$B[u(P)] = 0, \quad P \in L, \tag{2.48}$$

for the (inhomogeneous) Poisson equation can be expressed in terms of the Green's function $G(P,Q)$ for the corresponding homogeneous problem (the Laplace equation) as

$$u(P) = \iint_{\Omega} G(P,Q) f(Q) d\Omega(Q). \tag{2.49}$$

Hence, we aim at finding a solution to the problem, stated in equations (2.47) and (2.48). An important feature of this approach is that our major concern is not to simply obtain the solution by any means and in any form. Rather, we are required to be more specific in the selection of a way to tackle the problem. That is, the objective of our approach is to express the solution in the integral form of (2.49), the kernel of which does in fact represent the Green's function of the homogeneous boundary-value problem corresponding to (2.47) and (2.48). An algorithm for the method of eigenfunction expansion turns out to be effective in achieving the objective.

2.3.2 Cartesian Coordinates

In the following, details of the approach based on the method of eigenfunction expansion, as well es its specific features are clarified and explained whilst passing through a series of illustrating examples, where we tackle problems stated in Cartesian coordinates. Later in this chapter we will also consider a number of problem settings stated in polar coordinates. Later in this section, we will focus on the construction of Green's functions for several "exotic" boundary-value problems for the two-dimensional Laplace equation on surfaces of revolution.

Example 2.21. Recollect the construction of the Green's function of the Dirichlet problem for the Laplace equation on the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$, as treated earlier in this chapter.

In equations (2.27) and (2.43), we find two different formulas for this Green's function, which have been obtained by the method of images and by conformal mapping. The current example presents an alternative derivation using the method of eigenfunction expansion.

To lay the basis for our approach to the construction of Green's functions, we consider the boundary-value problem

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = -f(x,y), \quad (x,y) \in \Omega, \tag{2.50}$$

$$u(x,0) = u(x,b) = 0, \tag{2.51}$$

where $u(x,y)$ is required to be bounded as x approaches negative and positive infinity, whilst $f(x,y)$ is supposed to be integrable on Ω, implying, in other words, that the improper integral

$$\iint_\Omega f(x,y)\,d\Omega(x,y)$$

is convergent.

We represent the solution $u(x, y)$ of the problem described by (2.50) and (2.51) as a Fourier sine-series

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x) \sin \nu y, \tag{2.52}$$

where $\nu = n\pi/b$, with $n = 1, 2, 3, \ldots$. We now expand the right-hand side function $f(x, y)$ in (2.50) into a Fourier sine-series

$$f(x, y) = \sum_{n=1}^{\infty} f_n(x) \sin \nu y. \tag{2.53}$$

Once we substitute the expansions (2.52) and (2.53) into (2.50), we obtain

$$\sum_{n=1}^{\infty} \left[\frac{d^2 u_n(x)}{dx^2} - \nu^2 u_n(x) \right] \sin \nu y = -\sum_{n=1}^{\infty} f_n(x) \sin \nu y.$$

Equating the coefficients of the two series in the above relation yields the ordinary differential equation

$$\frac{d^2 u_n(x)}{dx^2} - \nu^2 u_n(x) = -f_n(x), \quad -\infty < x < \infty, \tag{2.54}$$

for the coefficients $u_n(x)$ of the series in (2.52). Clearly, the boundedness conditions

$$\lim_{x \to -\infty} |u_n(x)| < \infty, \quad \lim_{x \to \infty} |u_n(x)| < \infty \tag{2.55}$$

must be imposed on $u_n(x)$ to make the problem described by (2.54) and (2.55) well-posed.

To construct the Green's function of the above boundary-value problem, we may choose either the approach employing its defining properties, or the one based on the method of variation of parameters. We take the latter procedure, which was described in detail in Chapter 1. That is, we express the general solution to (2.54) as

$$u_n(x) = C_1(x)e^{\nu x} + C_2(x)e^{-\nu x} \tag{2.56}$$

which yields the well-posed system of linear algebraic equations

$$\begin{pmatrix} e^{\nu x} & e^{-\nu x} \\ \nu e^{\nu x} & -\nu e^{-\nu x} \end{pmatrix} \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -f(x) \end{pmatrix}$$

in $C_1'(x)$ and $C_2'(x)$. The solution to this system is

$$C_1'(x) = -\frac{1}{2\nu} e^{-\nu x} f_n(x), \quad C_2'(x) = \frac{1}{2\nu} e^{\nu x} f_n(x).$$

We find expressions for $C_1(x)$ and $C_2(x)$

$$C_1(x) = -\frac{1}{2\nu}\int_{-\infty}^{x} \mathrm{e}^{-\nu\xi} f_n(\xi)d\xi + D_1$$

and

$$C_2(x) = \frac{1}{2\nu}\int_{-\infty}^{x} \mathrm{e}^{\nu\xi} f_n(\xi)d\xi + D_2$$

after performing integration. Substituting these into (2.56) we obtain

$$u_n(x) = \mathrm{e}^{-\nu x}\left(\frac{1}{2\nu}\int_{-\infty}^{x} \mathrm{e}^{\nu\xi} f_n(\xi)d\xi + D_2\right) - \mathrm{e}^{\nu x}\left(\frac{1}{2\nu}\int_{-\infty}^{x} \mathrm{e}^{-\nu\xi} f_n(\xi)d\xi + D_1\right). \tag{2.57}$$

The first boundedness condition in (2.55) requires the factor of $\mathrm{e}^{-\nu x}$ in (2.57) to be zero as x approaches negative infinity, which yields $D_2 = 0$. The second condition in (2.55), in turn, implies that the factor of $\mathrm{e}^{\nu x}$ must be zero as x approaches infinity. This yields

$$D_1 = -\frac{1}{2\nu}\int_{-\infty}^{\infty} \mathrm{e}^{-\nu\xi} f_n(\xi)d\xi.$$

After substitution of D_1 and D_2 into (2.57), the solution of the boundary-value problem described by (2.54) and (2.55) appears as

$$u_n(x) = \int_{-\infty}^{\infty} \frac{1}{2\nu}\mathrm{e}^{\nu(x-\xi)} f_n(\xi)d\xi + \int_{-\infty}^{x} \frac{1}{2\nu}[\mathrm{e}^{\nu(\xi-x)} - \mathrm{e}^{\nu(x-\xi)}] f_n(\xi)d\xi$$

which reads as a single integral

$$u_n(x) = \int_{-\infty}^{\infty} g_n(x,\xi) f_n(\xi)d\xi \tag{2.58}$$

with a kernel written

$$g_n(x,\xi) = \frac{1}{2\nu}\mathrm{e}^{-\nu|x-\xi|}, \quad -\infty < x, \xi < \infty.$$

Hence, the above represents the Green's function of the homogeneous boundary-value problem corresponding to that in (2.54) and (2.55).

With the aid of the Euler–Fourier formula, the coefficient $f_n(\xi)$ in the series of (2.53) is expressed through the right-hand side function of the equation in (2.50) as

$$f_n(\xi) = \frac{2}{b}\int_0^b f(\xi,\eta)\sin\nu\eta d\eta.$$

After substituting this into (2.58) and subsequent substitution of $u_n(x)$ in (2.52), we obtain the solution of the problem described by (2.50) and (2.51):

$$u(x, y) = \int_0^b \int_{-\infty}^{\infty} \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\nu|x-\xi|}}{n} \sin \nu y \sin \nu \eta f(\xi, \eta) d\xi d\eta \tag{2.59}$$

which suggests that, in view of (2.49), the kernel

$$G(x, y; \xi, \eta) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\nu|x-\xi|}}{n} \sin \nu y \sin \nu \eta \tag{2.60}$$

of the integral representation from (2.59) represents the Green's function for the homogeneous boundary-value problem described by (2.50) and (2.51).

The series in (2.60) is non-uniformly convergent. In fact, due to the logarithmic singularity, it diverges when the observation point (x, y) coincides with the source point (ξ, η). This makes the above series form of the Green's function somewhat inconvenient for numerical implementation. However, we can radically improve the situation, because the series is in fact summable. To sum it up, we transform (2.60) into

$$\begin{aligned} G(x, y; \xi, \eta) &= \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{e^{-\nu|x-\xi|}}{n} [\cos \nu(y - \eta) - \cos \nu(y + \eta)] \\ &= \frac{1}{2\pi} \left[\sum_{n=1}^{\infty} \frac{e^{-\nu|x-\xi|}}{n} \cos \nu(y - \eta) - \sum_{n=1}^{\infty} \frac{e^{-\nu|x-\xi|}}{n} \cos \nu(y + \eta) \right] \end{aligned} \tag{2.61}$$

and recall the classical [1, 27, 37, 66] summation formula

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \cos n\vartheta = -\frac{1}{2} \ln(1 - 2r \cos \vartheta + r^2) \tag{2.62}$$

that holds for $r < 1$ and $0 \leq \vartheta < 2\pi$.

It is evident that the series in (2.61) are isomorphic to that in (2.62), and the constraints on p and ϑ are met. Indeed, it is clear that

$$e^{-\nu|x-\xi|} < 1$$

and

$$0 \leq \omega(y - \eta) < 2\pi \quad \text{and} \quad 0 \leq \omega(y + \eta) < 2\pi.$$

Hence, the series in (2.61) are entirely summable, which yields the analytical representation

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{1 - 2e^{\omega(x-\xi)} \cos \omega(y + \eta) + e^{2\omega(x-\xi)}}{1 - 2e^{\omega(x-\xi)} \cos \omega(y - \eta) + e^{2\omega(x-\xi)}} \tag{2.63}$$

for the Green's function to the homogeneous boundary-value problem corresponding to (2.50) and (2.51). Here $\omega = \pi/b$.

Here, we refer the reader back to equation (2.42), which was obtained (by the method of conformal mapping) as the Green's function for the Dirichlet problem on the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < \pi\}$ of width π. Clearly, if we assume $b = \pi$ (implying $\omega = 1$), then the expression in (2.63) reduces to that of (2.42).

Note that, similar to the conversion (2.42) into (2.43) that we made in Section 2.2, the expression in (2.63) converts to the more compact formula

$$G(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{\cosh \omega(x-\xi) - \cos \omega(y+\eta)}{\cosh \omega(x-\xi) - \cos \omega(y-\eta)}. \tag{2.64}$$

The equivalence of (2.63) and (2.64) can be verified by multiplying both the numerator and denominator in (2.63) by $e^{\omega(\xi-x)}$, and subsequent use of the Euler formula for the hyperbolic cosine function.

We can find another compact version of the Green's function shown in (2.63): upon introducing complex variable notation

$$z = x + iy \quad \text{and} \quad \zeta = \xi + i\eta$$

for the observation point (x, y) and the source point (ξ, η), and recalling the Euler formula

$$e^z = e^x(\cos y + i \sin y)$$

for the complex exponent, equation (2.63) coverts to the following compact formula

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{|1 - e^{\omega(z-\overline{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|}, \tag{2.65}$$

where the bar on ζ means the complex conjugate.

Example 2.22. Let us turn to another classical setting and construct the Green's function for the Dirichlet problem

$$u(x, 0) = u(x, b) = u(0, y) = 0 \tag{2.66}$$

for the Laplace equation on the semi-infinite strip $\Omega = \{0 < x < \infty, 0 < y < b\}$.

Following through with our technique, we will consider the boundary-value problem described by (2.50) and (2.66) on Ω, with the additional requirement that $u(x, y)$ stays bounded as x approaches infinity. The right-hand side function $f(x, y)$ in (2.50) is presumed to be integrable on Ω.

Similarly to the development in Example 2.21, we expand $u(x, y)$ and $f(x, y)$ in Fourier series of (2.52) and (2.53). This yields the boundary-value problem

$$\frac{d^2u_n(x)}{dx^2} - \nu^2 u_n(x) = -f_n(x), \quad 0 < x < \infty, \ \nu = n\pi/b,$$

$$u_n(0) = 0, \quad \lim_{x\to\infty} |u_n(x)| < \infty$$

for the coefficients $u_n(x)$ of the series in (2.52).

The Green's function $g_n(x, \xi)$ for the above problem can be obtained by following the recommendations in Chapter 1. Using our current notation, we write it down as

$$g_n(x, \xi) = \frac{1}{2\nu}[\mathrm{e}^{-\nu|x-\xi|} - \mathrm{e}^{-\nu(x+\xi)}], \quad \text{for } 0 \le x, \xi \le \infty.$$

Analogous to Example 2.21, we finally obtain a series expansion of the Green's function for the homogeneous boundary-value problem corresponding to that in (2.50) and (2.66)

$$G(x, y; \xi, \eta) = \frac{2}{b}\sum_{n=1}^{\infty} g_n(x, \xi)\sin\nu y \sin\nu\eta,$$

and after employing the summation formula from (2.62), the above representation transforms into closed analytical form

$$G(x, y; \xi, \eta) = \frac{1}{2\pi}\ln\frac{|1 - \mathrm{e}^{\omega(z-\overline{\zeta})}||1 - \mathrm{e}^{\omega(z+\overline{\zeta})}|}{|1 - \mathrm{e}^{\omega(z-\zeta)}||1 - \mathrm{e}^{\omega(z+\zeta)}|} \tag{2.67}$$

with $\omega = \pi/b$. It can be easily verified that the above expression is equivalent with

$$G(x, y; \xi, \eta) = \frac{1}{4\pi}\ln\left(\frac{\cosh\omega(x+\xi) - \cos\omega(y-\eta)}{\cosh\omega(x-\xi) - \cos\omega(y-\eta)} \times \frac{\cosh\omega(x-\xi) - \cos\omega(y+\eta)}{\cosh\omega(x+\xi) - \cos\omega(y+\eta)}\right), \tag{2.68}$$

which is generally presented in the literature [57] as the Green's function of the Dirichlet problem on the semi-infinite strip.

Up to this point in our development, we used the method of eigenfunction expansion as an alternative approach to the construction of those Green's functions which could also be obtained with other methods. In the example that follows we will instead consider a mixed boundary-value problem, for which the method of eigenfunction expansion probably represents the only way to obtain its Green's function.

Example 2.23. Construct the Green's function for the boundary-value problem

$$\frac{\partial u(0,y)}{\partial x} - \beta u(0,y) = 0, \quad u(x,0) = u(x,b) = 0, \quad \beta \geq 0, \tag{2.69}$$

for the Laplace equation on the semi-infinite strip $\Omega = \{0 < x < \infty, 0 < y < b\}$.

To do so we consider, following our approach, the Poisson equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = -f(x,y), \quad (x,y) \in \Omega, \tag{2.70}$$

subject to the conditions in (2.69).

By virtue of the Fourier sine-series expansions

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x) \sin \nu y, \quad \nu = \frac{n\pi}{b},$$

and

$$f(x,y) = \sum_{n=1}^{\infty} f_n(x) \sin \nu y$$

we obtain the boundary-value problem

$$\frac{d^2 u_n(x)}{dx^2} - \nu^2 u_n(x) = -f_n(x), \quad 0 < x < \infty,$$
$$\frac{du_n(0)}{dx} - \beta u_n(0) = 0, \quad \lim_{x\to\infty,} |u_n(x)| < \infty$$

for the coefficients $u_n(x)$ of the above series expansion of $u(x,y)$.

Following the procedure of the method of variation of parameters (as described earlier in Chapter 1), the Green's function $g_n(x,\xi)$ for the above problem is

$$g_n(x,\xi) = \frac{1}{2\nu}[\mathrm{e}^{-\nu|x-\xi|} + \beta^* \mathrm{e}^{-\nu(x+\xi)}], \quad \text{for } 0 \leq x, \xi \leq \infty, \tag{2.71}$$

where the parameter β^* is defined as $(\nu - \beta)/(\nu + \beta)$.

The solution of the mixed boundary-value problem described by (2.69) and (2.70) is obtained from (2.71), in terms of $g_n(x,\xi)$ as

$$u(x,y) = \int_0^b \int_0^\infty \frac{2}{b} \sum_{n=1}^{\infty} g_n(x,\xi) \sin \nu y \sin \nu \eta f(\xi,\eta) d\xi d\eta. \tag{2.72}$$

This implies that the kernel of the integral in (2.72)

$$G(x,y;\xi,\eta) = \frac{2}{b} \sum_{n=1}^{\infty} g_n(x,\xi) \sin \nu y \sin \nu \eta \tag{2.73}$$

represents the Green's function of the mixed boundary-value problem described by (2.70) for the Laplace equation.

Hence, the problem stated in this example is formally solved; we obtained the sought-after Green's function. However, a question about the computability of (2.73) remains open: is it conducive to direct computer evaluation? The answer to this question is negative. In contrast to closed analytical forms, like those obtained earlier in Examples 2.21 and 2.22, equation (2.73) is not suitable for immediate computer implementation, because the series in equation (2.73) does not (and cannot) converge uniformly. We have already touched upon this phenomenon earlier in this book: any series-only representation of a Green's function for the two-dimensional Laplace equation cannot uniformly converge due to the presence of the logarithmic singularity in the Green's function.

To give the reader a sense of the level of accuracy attainable by the series expansion of the Green's function in (2.73), Figure 2.13 exhibits profiles of $G(x, y; \xi, \eta)$ for different (10th, 25th and 100th) partial sums of the series, using parameter values $b = 1$, $\beta = 0.5$ The source point was fixed at $(0.1, 0.5)$ and the profile of $G(x, 0.3; 0.1, 0.5)$ was depicted in the vicinity of the source point $[0, 0.4]$.

From Figure 2.13, we might conclude that, although increasing order of the partial sum provides a reasonable improvement, the approximation in the immediate vicinity of the source point still remains quite bad. In other words, any attempt to approximate the Green's function with a partial sum of (2.73) is ineffective, at least in the vicinity of the source point (ξ, η).

With the enhancement of the practicality of equation (2.73) in mind, we turn back to the cases that we covered earlier in Examples 2.21 and 2.22, where we managed to successfully sum series expansions of Green's functions (observe, for example, how

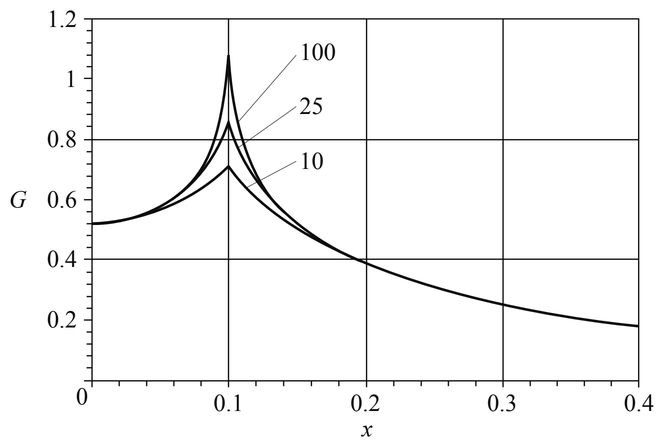

Figure 2.13. Profiles of different partial sums of the series in (2.73).

the series in (2.60) converges to the closed analytical form in (2.63)). Contrary to those cases, the series in (2.73) cannot be summed.

Hence, since truncation of the series does not work, we require extra effort to find a way to enhance its computability or, in other words, to improve its convergence. One possible way of doing this was proposed in [43]. The idea behind it is to split the expression for $g_n(x,\xi)$ in (2.71) into two parts, one of which contains the components responsible for the singularity and allows a complete summation of the series in (2.73), whilst the other part leads to a uniformly convergent series. In doing so, we recall the coefficient $g_n(x,\xi)$ from (2.71)

$$g_n(x,\xi) = \frac{1}{2\nu}\left(\mathrm{e}^{-\nu|x-\xi|} + \frac{\nu-\beta}{\nu+\beta}\mathrm{e}^{-\nu(x+\xi)}\right) \quad \text{for } 0 < x, \xi < \infty$$

and write the factor $(\nu-\beta)/(\nu+\beta)$ of the exponential function $\mathrm{e}^{-\nu(x+\xi)}$ as

$$\frac{\nu-\beta}{\nu+\beta} = 1 - \frac{2\beta}{\nu+\beta},$$

which yields

$$g_n(x,\xi) = \frac{1}{2\nu}\left(\mathrm{e}^{-\nu|x-\xi|} + \mathrm{e}^{-\nu(x+\xi)} - \frac{2\beta}{\nu+\beta}\mathrm{e}^{-\nu(x+\xi)}\right).$$

Upon substitution into (2.73), we can rewrite it as

$$G(x,y;\xi,\eta) = \frac{1}{b}\sum_{n=1}^{\infty}\frac{1}{\nu}[\mathrm{e}^{-\nu|x-\xi|} + \mathrm{e}^{-\nu(x+\xi)}]\sin\nu y\sin\nu\eta$$
$$- \frac{2\beta}{b}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\nu(x+\xi)}}{\nu(\nu+\beta)}\sin\nu y\sin\nu\eta, \quad \nu = \frac{n\pi}{b}.$$

The first of the above two series is summable, which can be accomplished in exactly same way as in Examples 2.21 and 2.22. The second series does not allow summation. However, it is uniformly convergent and, since it converges at the relatively high rate of $1/n^2$, we may leave it in its current form without significantly impeding the computability of the whole expression.

Thus, a computer-friendly formula for the Green's function to the mixed boundary-value problem described by (2.70) for the Laplace equation on the semi-infinite strip $\Omega = \{0 < x < \infty, 0 < y < b\}$ becomes

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\frac{|1-\mathrm{e}^{\omega(z+\zeta)}|\,|1-\mathrm{e}^{\omega(z-\overline{\zeta})}|}{|1-\mathrm{e}^{\omega(z-\zeta)}|\,|1-\mathrm{e}^{\omega(z+\overline{\zeta})}|}$$
$$- \frac{2\beta}{b}\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\nu(x+\xi)}}{\nu(\nu+\beta)}\sin\nu y\sin\nu\eta, \quad \omega = \frac{\pi}{b}. \tag{2.74}$$

Notice that if $\beta = 0$, then the above expression reduces to a closed analytical form for the Green's function

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{|1 - e^{\omega(z+\zeta)}|\,|1 - e^{\omega(z-\overline{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|\,|1 - e^{\omega(z+\overline{\zeta})}|} \tag{2.75}$$

for the boundary-value problem

$$\frac{\partial u(0, y)}{\partial x} = 0, \quad u(x, 0) = u(x, b) = 0$$

stated for the Laplace equation on the semi-infinite strip $\Omega = \{0 < x < \infty, 0 < y < b\}$.

We turn again to the expression of the Green's function shown in (2.74). As we already pointed out, its series component converges at a high rate. To be more specific, we might estimate its Nth remainder as

$$R_N(x, y; \xi, \eta) = \sum_{n=N+1}^{\infty} \frac{e^{-\nu(x+\xi)}}{\nu(\nu + \beta)} \sin \nu y \sin \nu \eta. \tag{2.76}$$

First observe that the exponential and trigonometric factors of the general term in this series never exceed unity. Furthermore, taking into account that the parameter β is non-negative (see the constraint in equation (2.70)), we arrive at the following estimate for the absolute value of R_N

$$|R_N(x, y; \xi, \eta)| \le \sum_{n=N+1}^{\infty} \frac{1}{\nu(\nu + \beta)}$$

$$\le \sum_{n=N+1}^{\infty} \frac{1}{\nu^2} = \frac{b^2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2} = \frac{b^2}{\pi^2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{N} \frac{1}{n^2} \right).$$

We can sum the infinite series in parentheses [3, 27, 37] yielding

$$|R_N(x, y; \xi, \eta)| \le \frac{b^2}{\pi^2} \left(\frac{\pi^2}{6} - \sum_{n=1}^{N} \frac{1}{n^2} \right). \tag{2.77}$$

Three issues might be highlighted with regard to the above inequality. First, it is quite compact and very simple to use. Second, it brings a uniform estimate and is, therefore, valid at any point on Ω. Finally, it follows from the way the above inequality was derived, that it provides a relatively coarse estimate. The latter issue makes it advisable to revisit the analysis of the remainder in (2.76). In doing so, let us set its trigonometric factors to unity and express the parameter ν in terms of the summation index n. This yields

$$|R_N(x, y; \xi, \eta)| \le \sum_{n=N+1}^{\infty} \frac{e^{-\nu(x+\xi)}}{\nu(\nu + \beta)} = \frac{b^2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{e^{-\nu(x+\xi)}}{n(n + \beta_0)},$$

where $\beta_0 = \beta b/\pi$. In the case of $\beta_0 \geq 1$, the above might be improved upon. That is

$$|R_N(x, y; \xi, \eta)| \leq \frac{b^2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{e^{-\nu(x+\xi)}}{n(n+1)} = \frac{b^2}{\pi^2} \left[\sum_{n=1}^{\infty} \frac{e^{-\nu(x+\xi)}}{n(n+1)} - \sum_{n=1}^{N} \frac{e^{-\nu(x+\xi)}}{n(n+1)} \right].$$

Note that the infinite series in the brackets is summable. Thus, using the standard summation formula [3, 27, 37]

$$\sum_{n=1}^{\infty} \frac{p^n}{n(n+1)} = 1 - \frac{1-p}{p} \ln \frac{1}{1-p}, \quad p^2 < 1,$$

where $p = e^{-\nu(x+\xi)}$ and $\omega = \pi/b$, we arrive at the following estimate for the remainder in (2.76)

$$|R_N(x, y; \xi, \eta)| \leq \frac{b^2}{\pi^2} \left[1 + (e^{\omega(x+\xi)} - 1) \ln(1 - e^{-\omega(x+\xi)}) - \sum_{n=1}^{N} \frac{e^{-\nu(x+\xi)}}{n(n+1)} \right]. \tag{2.78}$$

Contrary to (2.77), the estimate in (2.78) is nonuniform: its right-hand side depends upon the observation and the source point to which the estimate is applied. This makes it more flexible for practical computation; it allows the user to vary the truncation of the series in (2.74) in different zones of Ω in order to keep a certain desired accuracy level for the entire region.

The improvement that has been made by this transformation of the *series-only* form of the Green's function in (2.73) turns out to be very remarkable. It can be especially

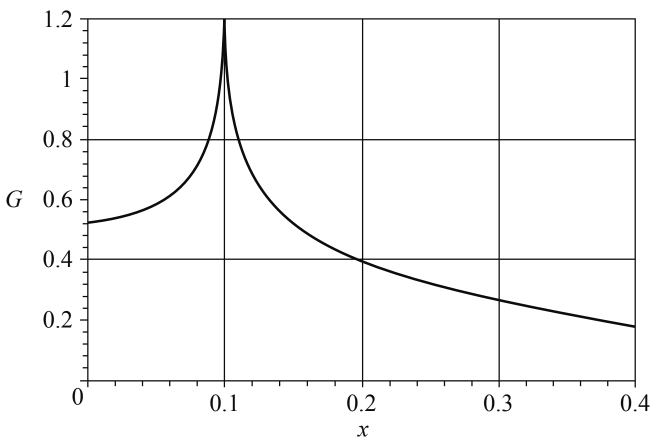

Figure 2.14. Convergence of equation (2.74).

appreciated after comparing the profile depicted in Figure 2.14 with those shown earlier in Figure 2.13. The representation in (2.74), with only the 10th partial sum of its series component is accounted for.

Example 2.24. We will now use the method of eigenfunction expansion to construct the Green's function of the Dirichlet problem for the Laplace equation on a rectangle.

We already encountered this problem earlier in this chapter, when we obtained a representation (see equation (2.46)) of its Green's function by the method of conformal mapping. Equation (2.46) is expressed in terms of a special (Weierstrass) function which is not yet tabulated, making it inconvenient for computer implementation.

The objective of the current example is to derive an alternative formula for the Green's function for the Laplace equation on a rectangle. In other words, we aim at finding its most easily computable form. In doing so, consider the boundary-value problem

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = -f(x,y), \quad (x,y) \in \Omega, \tag{2.79}$$

$$u(x,0) = u(x,b) = u(0,y) = u(a,y) = 0 \tag{2.80}$$

on the rectangle $\Omega = \{0 < x < a, 0 < y < b\}$, where $f(x,y)$ is presumed to be integrable (continuous) on the closure of Ω.

We assume the reader is familiar with the fact, shown in courses on differential equations (see, for example, [3, 16, 18, 29, 57, 66, 77]), that components in the set of functions

$$U_{m,n}(x,y) = \sin \mu x \sin \nu y$$

with $\mu = m\pi/a$ and $\nu = n\pi/b$, with $m,n = 1,2,3,\ldots$, represent eigenfunctions of the Dirichlet problem for the Laplace operator on the rectangle Ω. Indeed, one can directly check out that every component of the set $U_{m,n}(x,y)$ satisfies conditions (2.80) and, if the parameter λ is defined in terms of indices m and n as $\lambda^2 = \mu^2 + \nu^2$, is also a solution of the Helmholtz equation

$$\frac{\partial^2 U_{m,n}(x,y)}{\partial x^2} + \frac{\partial^2 U_{m,n}(x,y)}{\partial y^2} + \lambda^2 U_{m,n}(x,y) = 0, \quad (x,y) \in \Omega.$$

This motivates our strategy for solving the problem described by (2.79) and (2.80), where we write its solution $u(x,y)$ in the eigenfunction expansion (double Fourier sine-series) form

$$u(x,y) = \sum_{m,n=1}^{\infty} u_{m,n} \sin \mu x \sin \nu y \tag{2.81}$$

and expand the right-hand side function $f(x, y)$ in (2.79) into the double Fourier sine-series

$$f(x, y) = \sum_{m,n=1}^{\infty} f_{m,n} \sin \mu x \sin \nu y. \tag{2.82}$$

Once the expansions (2.81) and (2.82) are substituted into (2.79), we have

$$-\sum_{m,n=1}^{\infty} \left(\mu^2 + \nu^2\right) u_{m,n} \sin \mu x \sin \nu y = -\sum_{m,n=1}^{\infty} f_{m,n} \sin \mu x \sin \nu y.$$

Equating the left-hand and right-hand side coefficients of the series in the above relation yields

$$u_{m,n} = \frac{f_{m,n}}{\mu^2 + \nu^2}.$$

With the aid of the Euler–Fourier formula, the coefficients $f_{m,n}$ of the series (2.82) are expressed as

$$f_{m,n} = \frac{4}{ab} \int_0^b \int_0^a f(\xi, \eta) \sin \mu \xi \sin \nu \eta d\xi d\eta.$$

After substituting $f_{m,n}$ into the above formula for the coefficients $u_{m,n}$ and then substituting $u_{m,n}$ in (2.81), we obtain the solution of the problem described by (2.79) and (2.80) in the form

$$u(x, y) = \int_0^b \int_0^a \frac{4}{ab} \sum_{m,n=1}^{\infty} \frac{\sin \mu x \sin \nu y \sin \mu \xi \sin \nu \eta}{\mu^2 + \nu^2} f(\xi, \eta) d\xi d\eta.$$

Since the solution to the problem described by (2.79) and (2.80) is expressed in the integral form of (2.49), its kernel

$$G(x, y; \xi, \eta) = \frac{4}{ab} \sum_{m,n=1}^{\infty} \frac{\sin \mu x \sin \nu y \sin \mu \xi \sin \nu \eta}{\mu^2 + \nu^2} \tag{2.83}$$

represents the Green's function of the Dirichlet problem on the rectangle $\Omega = \{0 < x < a, 0 < y < b\}$.

It is evident that critical issue for the double-series in (2.83) is its computability. We address this issue in the following analysis: we assume, for the sake of simplicity, $a = \pi$ and $b = \pi$, which transforms (2.83) into

$$G(x, y; \xi, \eta) = \frac{4}{\pi^2} \sum_{m,n=1}^{\infty} \frac{\sin mx \sin ny \sin m\xi \sin n\eta}{m^2 + n^2} \tag{2.84}$$

the Green's function for the square $\Omega = \{0 < x < \pi, 0 < y < \pi\}$.

To examine the rate of convergence for the series in (2.84), we depict in Figure 2.15, profiles of its M, Nth partial sum for various values of the truncation parameters M and N. The x coordinate of the field point is fixed at $x = \pi/2$, whilst the source point (ξ, η) is chosen as $(\pi/2, 2)$.

Two important observations can be drawn from Figure 2.15. Both of them indicate a low computational potential for the expression in (2.84). First, the logarithmic singularity is poorly approximated when we truncate the series. Second, there is a high-frequency oscillation which dramatically reduces its practicality.

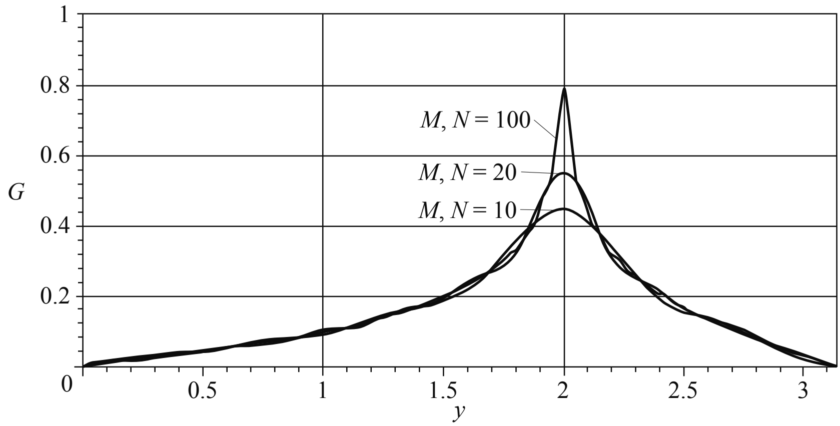

Figure 2.15. Convergence of the series in (2.84).

Note that the oscillation cannot be eliminated entirely, even in the case of $M = 100$ and $N = 100$. This implies, in particular, that the accuracy in computing derivatives of the Green's function (which are frequently required in applications) must be even much lower compared to that of the function itself.

Hence, We need to take steps to further enhance the computational potential of equation (2.84). In [45, 48], for example, it was proposed to rearrange the double-summation:

$$\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{\sin mx \sin m\xi}{m^2+n^2}\right)\sin ny \sin n\eta$$
$$= \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{2}\sum_{m=1}^{\infty}\frac{\cos m(x-\xi)-\cos m(x+\xi)}{m^2+n^2}\right)\sin ny \sin n\eta.$$

Splitting the m-series into two parts, the above transforms to

$$\frac{2}{\pi^2}\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{\cos m(x-\xi)}{m^2+n^2}-\frac{\cos m(x+\xi)}{m^2+n^2}\right)\sin ny \sin n\eta. \tag{2.85}$$

In compliance with the standard summation formula [1, 27, 37]

$$\sum_{m=1}^{\infty} \frac{\cos m\beta}{m^2+\alpha^2} = \frac{\pi}{2\alpha}\frac{\cosh\alpha(\pi-\beta)}{\sinh\alpha\pi} - \frac{1}{2\alpha^2},$$

where β is supposed to be bounded as $0 < \beta < 2\pi$, each of the m-series in (2.85) is analytically summable. After performing the summation, we reduce the double-series in (2.84) to

$$\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cosh n(\pi - |x-\xi|) - \cosh n(\pi-(x+\xi))}{n\sinh n\pi}\sin ny \sin n\eta$$
$$= \frac{2}{\pi}\sum_{n=1}^{\infty} T_n(x,\xi)\sin ny\sin n\eta \tag{2.86}$$

with the coefficient $T_n(x,\xi)$ defined as

$$T_n(x,\xi) = \frac{1}{n\sinh n\pi}\begin{cases}\sinh n(\pi-x)\sinh n\xi, & x \ge \xi,\\ \sinh n(\pi-\xi)\sinh nx, & x \le \xi.\end{cases}$$

To analyze the convergence of the single-series formula of the Green's function in (2.86) we depict, in Figure 2.16, profiles of its Nth partial sum in a manner similar to that of Figure 2.15. Comparison of the data presented in Figures 2.15 and 2.16 clearly indicates that the single-series expression of the Green's function works slightly better in the approximation of the basic logarithmic singularity. On the other hand, a high-frequency oscillation is still there and does not show significant improvement. This becomes even more notable for the single-series formula. Hence, none of the two series formulas for the Green's function in (2.84) and in (2.86) is computationally efficient, leaving room for further improvement.

A significant step in that direction can be provided by accelerating convergence of the series in (2.86). This can be done by operating on either branch of the coefficient $T_n(x,\xi)$. To check this, we choose the one valid for $x \ge \xi$, and transform the series as

$$\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sinh n\pi\cosh nx - \sinh nx\cosh n\pi}{n\sinh n\pi}\sinh n\xi\sin ny\sin n\eta$$
$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cosh nx - \sinh nx\coth n\pi}{n}\sinh n\xi\sin ny\sin n\eta.$$

Adding and subtracting the $\sinh nx$ in the numerator, we rewrite the above as

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[\cosh nx - \sinh nx + \sinh nx(1-\coth n\pi)]\sinh n\xi\sin ny\sin n\eta$$

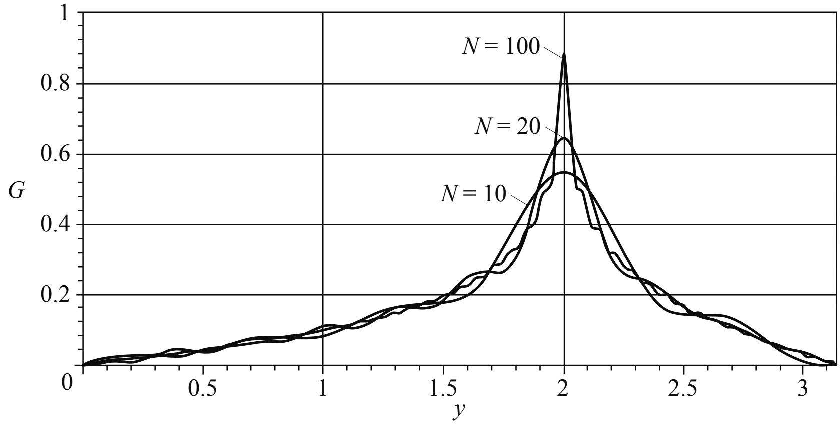

Figure 2.16. Convergence of the series in (2.86).

from which, after removing the brackets, we get

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cosh nx - \sinh nx)\sinh n\xi \sin ny \sin n\eta$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sinh nx(1-\coth n\pi)\sinh n\xi \sin ny \sin n\eta.$$

It can be readily shown that the first of the above two series turns out to be analytically summable. To proceed with the summation, we convert its hyperbolic functions into exponential form, which yields

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{e}^{-nx}\frac{\mathrm{e}^{n\xi}-\mathrm{e}^{-n\xi}}{2}\sin ny \sin n\eta$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-\coth n\pi)\sinh nx \sinh n\xi \sin ny \sin n\eta$$

transforming, after some elementary algebra, to

$$\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}[\mathrm{e}^{-n(x-\xi)}-\mathrm{e}^{-n(x+\xi)}][\cos n(y-\eta)-\cos n(y+\eta)]$$

$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sinh nx \sinh n\xi}{n\mathrm{e}^{n\pi}\sinh n\pi}\sin ny \sin n\eta.$$

When we remove the brackets from the first of the above two series, it breaks up into four pieces, each of which allows analytical summation in compliance with the standard summation formula in (2.62). This then converts the Green's function in (2.86) to

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\sqrt{\frac{1-2\mathrm{e}^{-(x-\xi)}\cos(y+\eta)+\mathrm{e}^{-2(x-\xi)}}{1-2\mathrm{e}^{-(x-\xi)}\cos(y-\eta)+\mathrm{e}^{-2(x-\xi)}}}$$
$$+\frac{1}{2\pi}\ln\sqrt{\frac{1-2\mathrm{e}^{-(x+\xi)}\cos(y-\eta)+\mathrm{e}^{-2(x+\xi)}}{1-2\mathrm{e}^{-(x+\xi)}\cos(y+\eta)+\mathrm{e}^{-2(x+\xi)}}}$$
$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sinh nx\sinh n\xi}{n\mathrm{e}^{n\pi}\sinh n\pi}\sin ny\sin n\eta.$$

After some elementary transformations, the logarithmic terms reduce to a more compact form

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\ln\frac{|1-\mathrm{e}^{(z-\bar{\zeta})}|\,|1-\mathrm{e}^{(z+\bar{\zeta})}|}{|1-\mathrm{e}^{(z-\zeta)}|\,|1-\mathrm{e}^{(z+\zeta)}|}$$
$$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sinh nx\sinh n\xi}{n\mathrm{e}^{n\pi}\sinh n\pi}\sin ny\sin n\eta, \tag{2.87}$$

where complex variable notation

$$z=x+iy \quad \text{and} \quad \zeta=\xi+i\eta$$

as introduced earlier, is used for the field point (x,y) and the source point (ξ,η).

The computational superiority (2.87) over equations (2.84) and (2.86) cannot be disputed, mainly because in (2.87), the basic logarithmic singularity of the Green's function is analytically expressed and contained in the term

$$\frac{1}{2\pi}\ln\frac{1}{|1-\mathrm{e}^{(z-\zeta)}|}. \tag{2.88}$$

To verify this, we expand the exponent $\mathrm{e}^{(z-\zeta)}$ in a Taylor series and substitute it into (2.88). This yields

$$\frac{1}{2\pi}\ln\frac{1}{|1-\mathrm{e}^{(z-\zeta)}|}=-\frac{1}{2\pi}\ln\left|(z-\zeta)+\frac{1}{2!}(z-\zeta)^2+\frac{1}{3!}(z-\zeta)^3+\cdots\right|$$
$$=-\frac{1}{2\pi}\ln\left(|z-\zeta|\left|1+\frac{1}{2!}(z-\zeta)+\frac{1}{3!}(z-\zeta)^2+\cdots\right|\right)$$
$$=\frac{1}{2\pi}\ln\frac{1}{z-\zeta}-\frac{1}{2\pi}\ln\left|1+\frac{1}{2!}(z-\zeta)+\frac{1}{3!}(z-\zeta)^2+\cdots\right|$$

with the first logarithmic term giving the fundamental solution of the Laplace equation, whereas the second logarithm is a regular function which vanishes for $z=\zeta$.

It is worth noting that, although in (2.87) the basic logarithmic singularity is explicit, the representation as a whole has a notable computational drawback: its rate

Field	Truncation parameter, N				
point, x/π	10	20	50	100	200
0.185	0.0299291	0.0299291	0.0299291	0.0299291	0.0299291
0.385	0.0767002	0.0767002	0.0767002	0.0767002	0.0767002
0.585	0.1752671	0.1752671	0.1752671	0.1752671	0.1752671
0.785	0.3851038	0.3851034	0.3851033	0.3851033	0.3851033
0.985	0.0171972	0.0171947	0.0171937	0.0171936	0.0171936

Table 2.1. Convergence of (2.87) for the source point $(3\pi/4, \pi/2)$.

Field	Truncation parameter, N				
point, x/π	10	20	50	100	200
0.185	0.0010733	0.0010733	0.0010733	0.0010733	0.0010733
0.385	0.0027066	0.0027066	0.0027066	0.0027066	0.0027066
0.585	0.0057374	0.0057367	0.0057363	0.0057363	0.0057363
0.785	0.0137082	0.0136948	0.0136931	0.0136928	0.0136928
0.985	0.3066126	0.2703686	0.2567230	0.2560739	0.2560668

Table 2.2. Convergence of (2.87) for the source point $(0.99\pi, \pi/2)$.

of convergence varies with the location of the field and the source points. In other words, the series in (2.87) converges non-uniformly. Indeed, the series converges at a high rate unless both (x, y) and (ξ, η) are in the immediate vicinity of the boundary segment $x = \pi$. This feature of the series expansion of Green's functions could be referred to as the *near-boundary singularity*. The data in Tables 2.1 and 2.2 are presented to illustrate the near-boundary singularity of (2.87).

The data in Table 2.1 are obtained for a source point located far away from the boundary segment $x = \pi$, whereas in Table 2.2, it is in the immediate vicinity of the boundary. The data in Table 2.1 are nearly independent of the truncation parameter, indicating high convergence of the series. The data in Table 2.2 are, in contrast, significantly affected by N revealing a poor convergence when both the field point and the source point approach the boundary.

Convergence of the series in (2.87) can be further improved when we apply elementary transformation, reducing it to a uniformly convergent series:

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{|1 - \mathrm{e}^{(z-\bar{\zeta})}|\,|1 - \mathrm{e}^{(z+\bar{\zeta})}|}{|1 - \mathrm{e}^{(z-\zeta)}|\,|1 - \mathrm{e}^{(z+\zeta)}|} + \frac{1}{4\pi} \ln \frac{|1 - \mathrm{e}^{(z_1+\bar{\zeta}_1)}|\,|1 - \mathrm{e}^{(z_2+\bar{\zeta}_2)}|}{|1 - \mathrm{e}^{(z_1+\zeta_1)}|\,|1 - \mathrm{e}^{(z_2+\zeta_2)}|}$$
$$+ \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\mathrm{e}^{n\pi} \cosh n(x - \xi) - \cosh n\pi \cosh n(x + \xi)}{n\mathrm{e}^{2n\pi} \sinh n\pi} \sin ny \sin n\eta, \tag{2.89}$$

where

$$z_1 = (x + \pi) + iy, \quad \zeta_1 = (\xi + \pi) + i\eta,$$
$$z_2 = (x - \pi) + iy, \quad \zeta_2 = (\xi - \pi) + i\eta.$$

To ensure accurate computation for (2.89), we estimate its series remainder as

$$|R_N(x, y; \xi, \eta)| = \left| \sum_{n=N+1}^{\infty} \frac{e^{n\pi} \cosh n(x - \xi) - \cosh n\pi \cosh n(x + \xi)}{n e^{2n\pi} \sinh n\pi} \sin ny \sin n\eta \right|$$
$$\leq \left| \sum_{n=N+1}^{\infty} \frac{e^{n\pi} \cosh n(x - \xi) - \cosh n\pi \cosh n(x + \xi)}{n e^{2n\pi} \sinh n\pi} \right|.$$

Since the second additive term in the numerator is never negative, we have

$$|R_N(x, y; \xi, \eta)| \leq \left| \sum_{n=N+1}^{\infty} \frac{e^{n\pi} \cosh n(x - \xi)}{n e^{2n\pi} \sinh n\pi} \right| \leq \sum_{n=N+1}^{\infty} \frac{e^{-n\pi} \cosh nx}{n \sinh n\pi}$$
$$\leq \sum_{n=N+1}^{\infty} \frac{e^{-n\pi}}{n} = \sum_{n=1}^{\infty} \frac{e^{-n\pi}}{n} - \sum_{n=1}^{N} \frac{e^{-n\pi}}{n}.$$

It appears that the infinite series in the above is analytically summable [3, 27, 37], which leads us to

$$|R_N(x, y; \xi, \eta)| \leq \ln \frac{1}{1 - e^{-\pi}} - \sum_{n=1}^{N} \frac{e^{-n\pi}}{n}$$

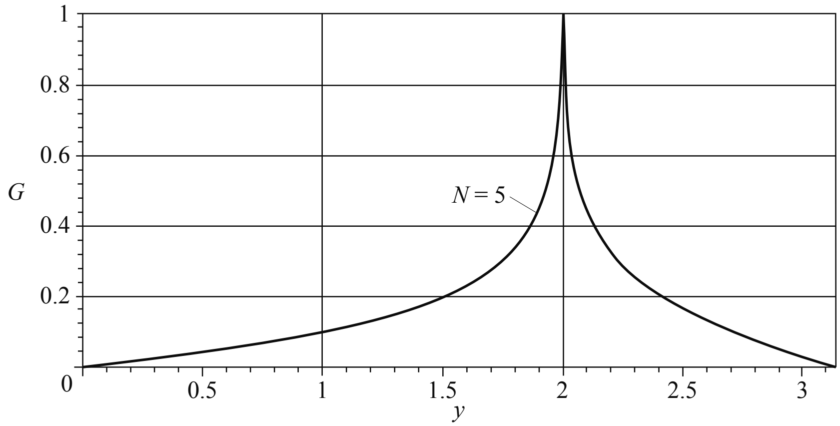

Figure 2.17. Convergence of the series in (2.89).

indicating an extremely high convergence rate of the series in (2.89), with an attainable error level of the order of, say, 10^{-8}, for any location of the field and the source point, with a truncation parameter as low as $N = 5$. We illustrate the superiority of (2.89) compared to all the others in Figure 2.17, with $G(\pi/2, y; \pi/2, 2)$ analogous to Figure 2.15 and Figure 2.16.

We find that the form for the Green's function of the Dirichlet problem for the Laplace equation, as given by (2.89), turns out to be far more computer-friendly than those of (2.84), (2.86) and (2.87). Two features of (2.89) justify this assertion: (i) the analytical form of the basic logarithmic singularity and (ii) the uniform convergence of the series term. The latter feature allows complete elimination of the high-frequency oscillation, by truncating the series to its lower partial sums. The fifth partial sum is depicted in Figure 2.17.

2.3.3 Polar Coordinates

We start this subsection with a problem that we have already encountered twice in the present monograph. Earlier in this chapter, we constructed the classical expression of the Green's function

$$G(r,\varphi;\varrho,\psi) = \frac{1}{4\pi} \ln \frac{a^4 - 2r\varrho a^2 \cos(\varphi-\psi) + r^2\varrho^2}{a^2\left(r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2\right)} \tag{2.90}$$

for the Dirichlet problem on a disk of radius a, using the methods of images and conformal mapping (see equations (2.26) and (2.40)). We will now use the eigenfunction expansion method for the derivation of (2.90). Notice, however, that the range of successful implementation of this method is not limited to this setting, but much wider. We will apply it later to the construction of Green's functions for other problems, for which both the methods of images and conformal mapping fail.

Example 2.25. Consider the Poisson equation written in polar coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u(r,\varphi)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u(r,\varphi)}{\partial \varphi^2} = -f(r,\varphi), \quad (r,\varphi) \in \Omega, \tag{2.91}$$

on a disk $\Omega = \{0 < r < a, 0 \le \varphi < 2\pi\}$ with radius a subject to the boundary conditions

$$\lim_{r\to 0} |u(r,\varphi)| < \infty \quad \text{and} \quad u(a,\varphi) = 0. \tag{2.92}$$

We require the solution to be bounded as r approaches zero, due to the appearance of the equation in (2.91): $r = 0$ represents its singular point and therefore a standard boundary condition cannot be imposed.

As we have already seen, our objective in the method of eigenfunction expansion is to express the solution of the problem described by (2.91) and (2.92) in integral form, which in this case reads

$$u(r,\varphi) = \int_0^{2\pi} \int_0^a G(r,\varphi;\varrho,\psi) f(\varrho,\psi)\varrho d\varrho d\psi, \tag{2.93}$$

delivering the Green's function $G(r,\varphi;\varrho,\psi)$ we are interested in.

Taking into account the 2π-periodicity with respect to the variable φ of the solution of the problem described by (2.91) and (2.92), we expand the $u(r,\varphi)$ in the following trigonometric Fourier series

$$u(r,\varphi) = \frac{1}{2}u_0(r) + \sum_{n=1}^{\infty} \left(u_n^c(r)\cos n\varphi + u_n^s(r)\sin n\varphi\right). \tag{2.94}$$

We can also represent $f(r,\varphi)$ in the right-hand side of (2.91) by the Fourier series

$$f(r,\varphi) = \frac{1}{2}f_0(r) + \sum_{n=1}^{\infty} \left(f_n^c(r)\cos n\varphi + f_n^s(r)\sin n\varphi\right). \tag{2.95}$$

Upon substitution of the expansions from (2.94) and (2.95) into (2.91) and equating the corresponding coefficients of the series on both sides, we derive the following linear ordinary differential equation

$$\frac{d}{dr}\left(r\frac{du_n(r)}{dr}\right) - \frac{n^2}{r^2}u_n(r) = -rf_n(r), \quad n = 0,1,2,\ldots, \tag{2.96}$$

for coefficients $u_n(r)$ of the expansion in (2.94). At this point in our development, we conveniently omit the superscripts in $u_n(r)$ and $f_n(r)$, which will be recalled later on.

The equations (2.92) imply that the solution $u_n(r)$ of (2.96) must be subject to the boundary conditions

$$\lim_{r\to 0}|u_n(r)| < \infty \quad \text{and} \quad u_n(a) = 0. \tag{2.97}$$

It is worth noting that the fundamental set of solutions of the homogeneous equation corresponding to (2.96) for the case of $n = 0$ differs of that for the case of $n \geq 1$. This means that while constructing the Green's function to the boundary-value problem described by (2.96) and (2.97), these two cases must be considered separately.

In the case that $n = 0$, the boundary-value problem described by (2.96) and (2.97) simplifies to

$$\frac{d}{dr}\left(r\frac{du_0(r)}{dr}\right) = -rf_0(r), \tag{2.98}$$

$$\lim_{r\to 0}|u_0(r)| < \infty \quad \text{and} \quad u_0(a) = 0 \tag{2.99}$$

with $u(r) = \ln r$ and $u(r) = 1$ representing the fundamental set of solutions for the homogeneous equation corresponding to (2.98). Hence, we can write the general solution for (2.98)

$$u_0(r) = C_1(r)\ln r + C_2(r). \tag{2.100}$$

Substituting this into (2.98) and following the method of variation of parameters, we obtain

$$C_1'(r) = -rf_0(r) \quad \text{and} \quad C_2'(r) = r\ln rf_0(r),$$

with the integration resulting in

$$C_1(r) = -\int_0^r \varrho f_0(\varrho)d\varrho + D_1, \quad C_1(r) = \int_0^r \varrho \ln \varrho f_0(\varrho)d\varrho + D_2.$$

Once we substitute the above expressions into (2.100) and combine the integral terms, we find the general solution to (2.98)

$$u_0(r) = \int_0^r \ln\left(\frac{\varrho}{r}\right) f_0(\varrho)\varrho d\varrho + D_1 \ln r + D_2.$$

The constants

$$D_1 = 0 \quad \text{and} \quad D_2 = -\int_0^a \ln\left(\frac{\varrho}{a}\right) f_0(\varrho)\varrho d\varrho$$

are obtained by taking the boundary conditions in (2.99) into account. Upon substituting the constants into the above equation for $u_0(r)$, the solution of the boundary-value problem described by (2.98) and (2.99) becomes

$$u_0(r) = \int_0^r \ln\left(\frac{\varrho}{r}\right) f_0(\varrho)\varrho d\varrho - \int_0^a \ln\left(\frac{\varrho}{a}\right) f_0(\varrho)\varrho d\varrho,$$

which can be rewritten in as a single integral

$$u_0(r) = \int_0^a g_0(r,\varrho) f_0(\varrho)\varrho d\varrho, \tag{2.101}$$

with the kernel

$$g_0(r,\varrho) = -\begin{cases} \ln(\varrho/a), & \text{for } r \le \varrho, \\ \ln(r/a), & \text{for } r \ge \varrho, \end{cases} \tag{2.102}$$

representing the Green's function of the homogeneous problem corresponding to (2.98) and (2.99).

We now turn our attention to the case of $n \geq 1$ in (2.96), i.e. we consider the boundary-value problem described by (2.96) and (2.97) as it is. Since the governing equation is of the Cauchy–Euler type [20, 37, 66], we can form its fundamental set of solutions using the functions $u(r) = r^n$ and $u(r) = r^{-n}$. This yields the general solution for (2.96)

$$u_n(r) = C_1(r)r^n + C_2(r)r^{-n}.$$

After proceeding with the method of the variation of parameters routine, this reduces to

$$u_n(r) = \int_0^r \frac{1}{2n}\left[\left(\frac{\varrho}{r}\right)^n - \left(\frac{r}{\varrho}\right)^n\right] f_n(\varrho)\varrho d\varrho + D_1 r^n + D_2 r^{-n}. \tag{2.103}$$

The boundary conditions in (2.97) now yield

$$D_2 = 0 \quad \text{and} \quad D_1 = \int_0^a \frac{1}{2n}\left[\left(\frac{1}{\varrho}\right)^n - \left(\frac{\varrho}{a^2}\right)^n\right] f_n(\varrho)\varrho d\varrho.$$

Upon substituting these into (2.96), we obtain

$$u_n(r) = \int_0^r \frac{1}{2n}\left[\left(\frac{\varrho}{r}\right)^n - \left(\frac{r}{\varrho}\right)^n\right] f_n(\varrho)\varrho d\varrho + \int_0^a \frac{1}{2n}\left[\left(\frac{r}{\varrho}\right)^n - \left(\frac{r\varrho}{a^2}\right)^n\right] f_n(\varrho)\varrho d\varrho,$$

or using a more compact notation,

$$u_n(r) = \int_0^a g_n(r,\varrho) f_n(\varrho)\varrho d\varrho \tag{2.104}$$

with the kernel $g_n(r,\varrho)$ defined in two segments. For $r \leq \varrho$, it is

$$g_n(r,\varrho) = \frac{1}{2n}\left[\left(\frac{r}{\varrho}\right)^n - \left(\frac{r\varrho}{a^2}\right)^n\right]$$

whilst for $r \geq \varrho$, we find

$$g_n(r,\varrho) = \frac{1}{2n}\left[\left(\frac{\varrho}{r}\right)^n - \left(\frac{r\varrho}{a^2}\right)^n\right].$$

We will now address the coefficients $u_n^c(r)$ and $u_n^s(r)$ of the expansion in (2.94). The expression in (2.104) suggests that the cosine-coefficients and the sine-coefficients of the Fourier series in (2.94) can be written in terms of the Green's functions $g_n(r,\varrho)$ and $g_0(r,\varrho)$ which we found to be

$$u_n^c(r) = \int_0^a g_n(r,\varrho) f_n^c(\varrho)\varrho d\varrho, \quad n = 0, 1, 2, \ldots, \tag{2.105}$$

and

$$u_n^s(r) = \int_0^a g_n(r,\varrho) f_n^s(\varrho)\varrho d\varrho, \quad n = 1, 2, 3, \ldots, \tag{2.106}$$

where, in compliance with the Euler–Fourier formulas

$$f_n^c(\varrho) = \frac{1}{\pi}\int_0^{2\pi} f(\varrho,\psi)\cos n\psi d\psi, \quad n = 0, 1, 2, \ldots, \tag{2.107}$$

and

$$f_n^s(r) = \frac{1}{\pi}\int_0^{2\pi} f(\varrho,\psi)\sin n\psi d\psi, \quad n = 1, 2, 3, \ldots. \tag{2.108}$$

Upon substituting the $f_n^c(\varrho)$ and $f_n^s(\varrho)$ from (2.107) and (2.108) into (2.101), (2.105), and (2.106), followed by substitution of $u_0(r)$, $u_n^c(r)$, and $u_n^s(r)$ into (2.94), we obtain the solution of the boundary-value problem stated by (2.91) and (2.92):

$$u(r,\varphi) = \int_0^{2\pi}\int_0^a \left\{\frac{1}{\pi}\left[\frac{g_0(r,\varrho)}{2} + \sum_{n=1}^{\infty} g_n(r,\varrho)\cos n\varphi\cos n\psi \right.\right.$$
$$\left.\left. + \sum_{n=1}^{\infty} g_n(r,\varrho)\sin n\varphi\sin n\psi\right]\right\} f(\varrho,\psi)\varrho d\varrho d\psi,$$

which can be written in compact form after combining the two trigonometric series to obtain

$$u(r,\varphi) = \int_0^{2\pi}\int_0^a \left\{\frac{1}{\pi}\left[\frac{g_0(r,\varrho)}{2} + \sum_{n=1}^{\infty} g_n(r,\varrho)\cos n(\varphi-\psi)\right]\right\} f(\varrho,\psi)\varrho d\varrho d\psi. \tag{2.109}$$

Because the expression $\varrho d\varrho d\psi$ represents an element of area in polar coordinates, we observe that the solution of the boundary-value problem described by (2.91) and (2.92) is indeed obtained in the form of (2.93): the integration in (2.109) is over the entire disk Ω, which allows us to conclude that the kernel in the above integral

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left[g_0(r,\varrho) + 2\sum_{n=1}^{\infty} g_n(r,\varrho)\cos n(\varphi-\psi)\right] \tag{2.110}$$

represents the Green's function of the Dirichlet problem for the Laplace equation on a disk of radius a.

Proceeding with the summation of the series term in $G(r,\varphi;\varrho,\psi)$, we note that either branches of $g_0(r,\varrho)$ and $g_n(r,\varrho)$ can be used. Taking, for example the branches valid for $r \le \varrho$ and substituting them into (2.110), we have

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left\{-\ln\left(\frac{\varrho}{a}\right) + \sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{\varrho}\right)^n - \left(\frac{r\varrho}{a^2}\right)^n\right]\cos n(\varphi-\psi)\right\}.$$

Recalling the summation formula from (2.62), we arrive at

$$G(r,\varphi;\varrho,\psi)=\frac{1}{2\pi}\left\{-\ln\left(\frac{\varrho}{a}\right)-\frac{1}{2}\ln\left(1-2\left(\frac{r}{\varrho}\right)\cos(\varphi-\psi)+\left(\frac{r}{\varrho}\right)^2\right)\right.$$
$$\left.+\frac{1}{2}\ln\left(1-2\left(\frac{r\varrho}{a^2}\right)\cos(\varphi-\psi)+\left(\frac{r\varrho}{a^2}\right)^2\right)\right\}$$

which, after combining all the logarithmic terms into one, reads as a familiar formula

$$G(r,\varphi;\varrho,\psi)=\frac{1}{4\pi}\ln\frac{a^4-2r\varrho a^2\cos(\varphi-\psi)+r^2\varrho^2}{a^2\left(r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2\right)}$$

of the Green's function for the Dirichlet problem on the disk of radius a.

In the following example, we will focus on another boundary-value problem on a disk. We will present the reader with a derivation yielding a computer-friendly formula for a Green's function, as an alternative to its classical formula, the computer implementation of which does not look very promising.

Example 2.26. We turn our attention to the mixed (Robin) boundary-value problem

$$\frac{\partial u(a,\varphi)}{\partial r}+\beta u(a,\varphi)=0,\quad \beta>0, \tag{2.111}$$

on the disk $\Omega=\{0<r<a, 0\le\varphi<2\pi\}$. Recall that, due to the singularity of the Laplace operator at the point $r=0$, the boundedness condition

$$\lim_{r\to 0}|u(r,\varphi)|<\infty$$

also applies, making the problem well-posed.

We find the following classical expression

$$G(r,\varphi;\varrho,\psi)=\frac{1}{2\pi}\left[\frac{1}{\beta}+\frac{1}{2}\ln\frac{1-2r\varrho\cos(\varphi-\psi)+r^2\varrho^2}{r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2}\right]$$
$$+\frac{1}{\pi}\operatorname{Re}\left((r\varrho e^{i(\varphi-\psi)})^{-\beta}\int_0^{r\varrho e^{i(\varphi-\psi)}}\frac{\zeta^{\beta-1}d\zeta}{1-\zeta}\right) \tag{2.112}$$

for the Green's function of the Robin problem stated in (2.111), for a unit circle in [66]. It can be clearly seen that computer implementations of the expression in (2.112) are complicated by the integral component of the regular part; the formula in (2.112) is not computer-friendly. Hence, we aim, in the present example, to obtain a different, readily computable form of this Green's function.

Following again the method of eigenfunction expansion, we expand the solution $u(r,\varphi)$ of (2.91) and (2.111), and the right-hand side function $f(r,\varphi)$ of (2.91) in the Fourier series of (2.94) and (2.95), respectively. In our context, this yields the boundary-value problem

$$\frac{d}{dr}\left(r\frac{du_n(r)}{dr}\right) - \frac{n^2}{r^2}u_n(r) = -rf_n(r), \quad n = 0, 1, 2, \ldots, \tag{2.113}$$

$$\lim_{r\to 0} |u_n(r)| < \infty \quad \text{and} \quad \frac{du_n(a)}{dr} + \beta u_n(a) = 0 \tag{2.114}$$

for the coefficients $u_n(r)$ of the expansion (2.94).

Similarly to the treatment of Example 2.25, we deal with the cases of $n = 0$ and $n \geq 1$ separately. For $n = 0$, the Green's function of the homogeneous boundary-value problem corresponding to that in (2.113) and (2.114) is

$$g_0(x, s) = \frac{1}{a\beta} - \ln\left(\frac{\varrho}{a}\right), \quad r \leq \varrho,$$

whilst $n \geq 1$ yields

$$g_n(r, \varrho) = \frac{1}{2n}\left[\left(\frac{r}{\varrho}\right)^n + \left(\frac{r\varrho}{a^2}\right)^n\right] - \frac{a\beta}{n(n + a\beta)}\left(\frac{r\varrho}{a^2}\right)^n \quad \text{for } r \leq \varrho.$$

Note that the branches of $g_0(r, \varrho)$ and $g_n(r, \varrho)$ valid for $\varrho \leq r$ can be obtained by interchanging r with ϱ.

This leads to the Green's function for the homogeneous version of (2.91) and (2.111), which has the form

$$G(r, \varphi; \varrho, \psi) = \frac{1}{2\pi}\left\{\frac{1}{a\beta} - \ln\left(\frac{\varrho}{a}\right) + \sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{\varrho}\right)^n + \left(\frac{r\varrho}{a^2}\right)^n - \frac{2a\beta}{(n + a\beta)}\left(\frac{r\varrho}{a^2}\right)^n\right]\cos n(\varphi - \psi)\right\}$$

Using the standard summation formula in (2.62), we can partially sum the series, converting the above formula to

$$G(r, \varphi; \varrho, \psi) = \frac{1}{2\pi}\left[\frac{1}{a\beta} - \ln\left(\frac{\varrho}{a}\right) - \frac{1}{2}\ln\left(1 - 2\left(\frac{r}{\varrho}\right)\cos(\varphi - \psi) + \left(\frac{r}{\varrho}\right)^2\right) - \frac{1}{2}\ln\left(1 - 2\left(\frac{r\varrho}{a^2}\right)\cos(\varphi - \psi) + \left(\frac{r\varrho}{a^2}\right)^2\right) - \sum_{n=1}^{\infty}\frac{2a\beta}{n(n + a\beta)}\left(\frac{r\varrho}{a^2}\right)^n\cos n(\varphi - \psi)\right].$$

After some trivial algebra, we can reduce this to a more compact expression

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left[\frac{1}{a\beta} + \ln\frac{a^3}{|z-\zeta|\,|z\bar{\zeta}-a^2|} - \sum_{n=1}^{\infty}\frac{2a\beta}{n(n+a\beta)}\left(\frac{r\varrho}{a^2}\right)^n \cos n(\varphi-\psi)\right]. \tag{2.115}$$

Clearly, the series in (2.115) converges at the rate of $1/n^2$ making this formula convenient for computer implementation.

It is worth noting that the boundary condition in (2.111) reduces to the Dirichlet type if β goes to infinity. In compliance with this, the limit of (2.115) for β going to infinity must represent the Green's function for the Dirichlet problem on a disk of radius a: by taking the limit in (2.115), we arrive at

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left[\ln\frac{a^3}{|z-\zeta|\,|z\bar{\zeta}-a^2|} - \sum_{n=1}^{\infty}\frac{2}{n}\left(\frac{r\varrho}{a^2}\right)^n \cos n(\varphi-\psi)\right], \tag{2.116}$$

where the series sums to

$$\sum_{n=1}^{\infty}\frac{2}{n}\left(\frac{r\varrho}{a^2}\right)^n \cos n(\varphi-\psi) = -2\ln\frac{|z\bar{\zeta}-a^2|}{a^2},$$

transforming (2.116) to the familiar formula

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\ln\frac{|z\bar{\zeta}-a^2|}{a|z-\zeta|}$$

of the Green's function for the Dirichlet problem on a disk of radius a.

There are boundary-value problems for the Laplace equation, for which Green's functions are hard to find in existing literature on partial differential equations. Even if the required Green's function, is given, it is most likely not presented in a computer-friendly formula. In the following examples, we turn our attention to several problems defined an annular region. Analysis of the literature suggests that computer-friendly formulas for Green's functions for those problems are not available at all.

Example 2.27. We aim at an easily computable form of the Green's function for the Dirichlet problem on the annular region $\Omega = \{a < r < b, 0 \le \varphi < 2\pi\}$. We start out by imposing the boundary conditions

$$u(a,\varphi) = 0, \quad u(b,\varphi) = 0 \tag{2.117}$$

on the boundary of Ω.

We now expand $u(r, \varphi)$ and $f(r, \varphi)$ in (2.91) in the Fourier series shown in (2.94) and (2.95). This yields the boundary-value problem

$$u_n(a) = 0 \quad \text{and} \quad u_n(b) = 0 \tag{2.118}$$

for equation (2.96).

For $n = 0$, the problem described by (2.96) and (2.118) transforms to

$$\frac{d}{dr}\left(r\frac{du_0(r)}{dr}\right) = -rf_0(r), \tag{2.119}$$

$$u_0(a) = 0 \quad \text{and} \quad u_0(b) = 0 \tag{2.120}$$

and we find the solution of the above problem described in the integral form as

$$u_0(r) = \int_a^b g_0(r, \varrho) f_0(\varrho)\varrho d\varrho \tag{2.121}$$

with the kernel

$$g_0(r, \varrho) = \frac{1}{\ln(b/a)} \begin{cases} \ln(r/a)\ln(b/\varrho), & \text{for } r \le \varrho, \\ \ln(\varrho/a)\ln(b/r), & \text{for } r \ge \varrho, \end{cases} \tag{2.122}$$

representing the Green's function of the homogeneous problem corresponding to (2.119) and (2.120).

Now, for $n \ge 1$ in (2.96) and (2.118), we arrive at

$$u_n(r) = \int_a^b g_n(r, \varrho) f_n(\varrho)\varrho d\varrho \tag{2.123}$$

with the kernel

$$g_n(r, \varrho) = \frac{r^{-n}\varrho^{-n}}{2n(b^{2n} - a^{2n})} \begin{cases} (b^{2n} - \varrho^{2n})(r^{2n} - a^{2n}), & \text{for } r \le \varrho, \\ (b^{2n} - r^{2n})(\varrho^{2n} - a^{2n}), & \text{for } r \ge \varrho, \end{cases} \tag{2.124}$$

representing the Green's function of the homogeneous problem corresponding to (2.96) and (2.118).

Upon substituting from (2.121) and (2.123) into the expansion (2.94), the solution to the boundary-value problem described by (2.91) and (2.117) reduces to the double integral

$$u(r, \varphi) = \int_0^{2\pi} \int_a^b \frac{1}{\pi} \left[\frac{g_0(r, \varrho)}{2} + \sum_{n=1}^{\infty} g_n(r, \varrho) \cos n\varphi \cos n\psi \right.$$
$$\left. + \sum_{n=1}^{\infty} g_n(r, \varrho) \sin n\varphi \sin n\psi \right] f(\varrho, \psi)\varrho d\varrho d\psi$$

which can be rewritten as

$$u(r,\varphi)=\int_0^{2\pi}\int_a^b \frac{1}{\pi}\left[\frac{g_0(r,\varrho)}{2}+\sum_{n=1}^{\infty} g_n(r,\varrho)\cos n(\varphi-\psi)\right] f(\varrho,\psi)\varrho d\varrho d\psi. \tag{2.125}$$

As soon as the solution of (2.91) and (2.117) appears in the integral form of (2.49), we can assert that the kernel

$$G(r,\varphi;\varrho,\psi)=\frac{1}{2\pi}\left[g_0(r,\varrho)+2\sum_{n=1}^{\infty} g_n(r,\varrho)\cos n(\varphi-\psi)\right] \tag{2.126}$$

in (2.125) represents the Green's function for the Dirichlet problem for the Laplace equation on the annular region $\Omega=\{a<r<b, 0\leq\varphi<2\pi\}$.

Close analysis reveals that the current form of equation (2.126) does not guarantee a high accuracy when computing the Green's function, due to the appearance of the coefficient $g_n(r,\varrho)$. This drawback is caused by two different types of singularity occuring in the series in (2.126): (i) the principal logarithmic singularity, which appears whenever the field point (r,φ) approaches the source point (ϱ,ψ) and (ii) the near-boundary singularity appearing whenever both field and source point approach either the inner $r=a$ or the outer $r=b$ segment of the boundary.

In Figure 2.18, we depict the accuracy level attainable in a direct assessment of the expansion in. It shows the profile $G(r,\varphi;2.0,4\pi/9)$ of the Green's function for the annular region with $a=1.0$ and $b=3.0$. The series in (2.126) is truncated to its tenth partial sum, which is clearly not sufficient for getting a reasonable approximation.

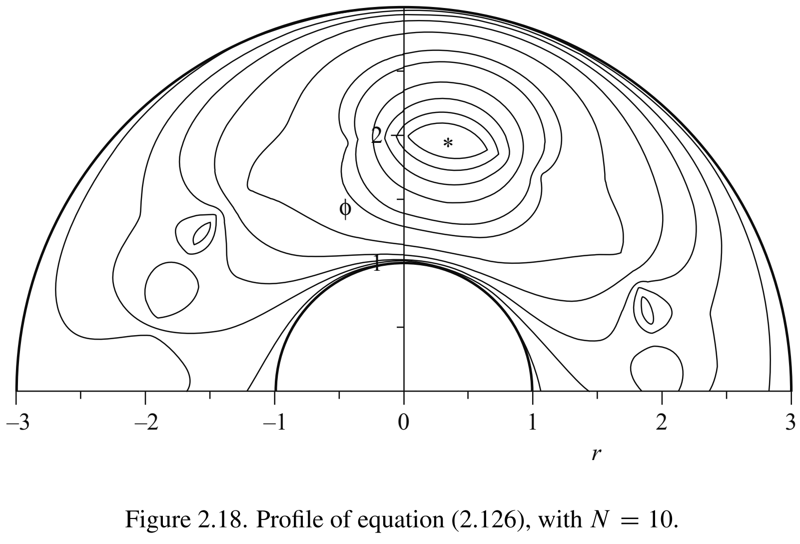

Figure 2.18. Profile of equation (2.126), with $N=10$.

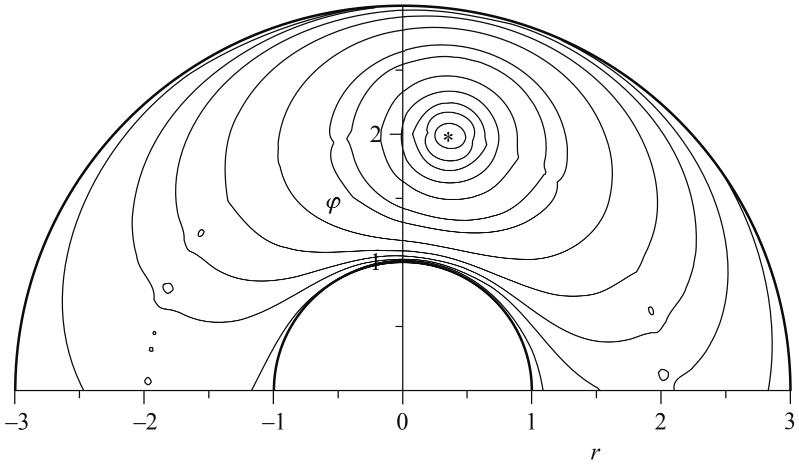

Figure 2.19. Profile of equation (2.126), with $N = 100$.

To find out if the order of the partial sum affects the accuracy level that can be attained by the expansion in (2.126), we refer to Figure 2.19. Analogous to Figure 2.18, the profile $G(r, \varphi; 2.0, 4\pi/9)$ of the Green's function is depicted, this time up to the 100th partial sum of the series in (2.126). Clearly, such a radical increase in the order of the partial sum somewhat improves the overall accuracy level, but it still remains low in the immediate vicinity of the angular coordinate ψ of the source point.

Based on the data in Figure 2.18 and Figure 2.19 we reach a convincing conclusion with regard to application of the series in (2.126): we can hardly consider the costly way of including higher partial sums in computing the non-uniformly convergent series in (2.126) productive.

Hence, we have to develop an alternative strategy, which can resolve the issue. It turns out that we could choose an effective method to improve convergence of the series prior, to its computer implementation. This can be achieved through analyzing the coefficient $g_n(r, \varrho)$ in (2.124). We can select either of its branches. Taking the one, for $r \le \varrho$, we subtract and add the term $1/b^{2n}$ to its factor of $1/(b^{2n} - a^{2n})$ and group the terms in the brackets as shown

$$g_n(r, \varrho) = \frac{1}{2n(r\varrho)^n}\left[\left(\frac{1}{b^{2n} - a^{2n}} - \frac{1}{b^{2n}}\right) + \frac{1}{b^{2n}}\right](b^{2n} - \varrho^{2n})(r^{2n} - a^{2n})$$
$$= \frac{1}{2n(r\varrho)^n}\left[\frac{a^{2n}}{b^{2n}(b^{2n} - a^{2n})} + \frac{1}{b^{2n}}\right](b^{2n} - \varrho^{2n})(r^{2n} - a^{2n}).$$

With this alternative form of the coefficient $g_n(r, \varrho)$, the series in (2.126) splits into two series with different rate of convergence. The first series, the one associated with the term

$$\frac{a^{2n}}{b^{2n}(b^{2n}-a^{2n})},$$

converges uniformly, whilst the other series, the one associated with the term $1/b^{2n}$, allows complete summation, despite converging non-uniformly. This can be accomplished with the use of the standard summation formula displayed earlier in (2.62). The summation yields a computer-friendly form of the Green's function

$$\begin{aligned}G(r,\varphi;\varrho,\psi) = {} & \frac{1}{2\pi}\left[g_0(r,\varrho)+2\sum_{n=1}^{\infty} g_n^*(r,\varrho)\cos n(\varphi-\psi)\right]\\ & + \frac{1}{4\pi}\ln\frac{a^4-2a^2r\varrho\cos(\varphi-\psi)+r^2\varrho^2}{r^2-2r\varrho\cos(\varphi-\psi)+\varrho^2}\\ & + \frac{1}{4\pi}\ln\frac{b^4-2b^2r\varrho\cos(\varphi-\psi)+r^2\varrho^2}{b^4r^2-2a^2b^2r\varrho\cos(\varphi-\psi)+a^4\varrho^2}\end{aligned} \tag{2.127}$$

with $g_n^*(r,\varrho)$

$$g_n^*(r,\varrho) = \frac{a^{2n}(b^{2n}-\varrho^{2n})(r^{2n}-a^{2n})}{2n(b^2r\varrho)^n(b^{2n}-a^{2n})}. \tag{2.128}$$

Note that the expression for $G(r,\varphi;\varrho,\psi)$ in (2.127) is valid for $r \le \varrho$ and cannot be used directly for $r \ge \varrho$. The following transformations must be made to convert the expression in (2.127) to a formula valid for $r \ge \varrho$: (i) select the corresponding branch of $g_0(r,\varrho)$, valid for $r \ge \varrho$. (ii) Interchange the variables r and ϱ in the $g_n^*(r,\varrho)$ in (2.128). (iii) Replace the denominator of the second logarithmic term in (2.127) with

$$b^4\varrho^2-2a^2b^2r\varrho\cos(\varphi-\psi)+a^4r^2.$$

Finally, we now use complex variable notation for the arguments of the logarithmic terms in (2.127) to rewrite the Green's function of the Dirichlet problem on an annular region with radii a and b as

$$\begin{aligned}G(r,\varphi;\varrho,\psi) = {} & \frac{1}{2\pi}\left[g_0(r,\varrho)+2\sum_{n=1}^{\infty} g_n^*(r,\varrho)\cos n(\varphi-\psi)\right]\\ & + \frac{1}{2\pi}\ln\frac{|a^2-z\overline{\zeta}|\,|b^2-z\overline{\zeta}|}{|z-\zeta|\,|b^2z-a^2\zeta|},\end{aligned} \tag{2.129}$$

where the factor $|b^2z-a^2\zeta|$ in the denominator holds for $r \le \varrho$, while for $r \ge \varrho$ it should be replaced with $|b^2\zeta-a^2z|$.

It can be easily shown that the (2.129) must be notably more efficient compared with (2.127). Two features back up this assertion. First, the principal singularity is

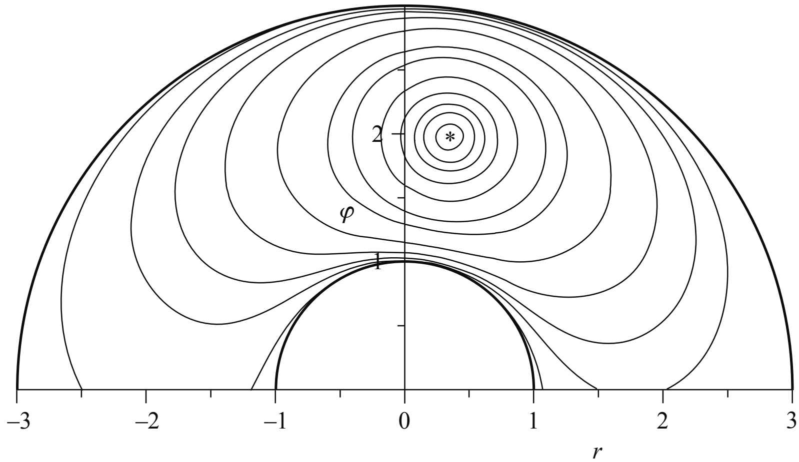

Figure 2.20. Profile of equation (2.129), with $N = 10$.

expressed analytically. Second, the series in (2.129) converges uniformly allowing an accurate assessment at relatively low computational cost. The smooth graph in Figure 2.20 supports the efficient computability of the above formula for the Green's function.

In Figure 2.20, we depict the profile $G(r, \varphi; 2.0, 4\pi/9)$ of the Green's function for the annular region with radii $a = 1.0$ and $b = 3.0$, with the series in (2.129) truncated to its tenth partial sum.

Example 2.28. We now turn to a mixed boundary-value problem, and will consider the "Dirichlet–Neumann" setting

$$u(a, \varphi) = 0, \quad \frac{\partial u(b, \varphi)}{\partial r} = 0 \tag{2.130}$$

for the Laplace equation on the annular region $\Omega = \{a < r < b, 0 \leq \varphi < 2\pi\}$.

Following the method of eigenfunction expansion for the problem described by (2.96) and (2.130), we arrive at (2.126), the series representation for the Green's function to the corresponding homogeneous boundary-value problem. We can find expressions for $g_0(r, \varrho)$ and $g_n(r, \varrho)$ of the series in (2.126), valid for $r \leq \varrho$ to be

$$g_0(r, \varrho) = \ln \frac{r}{a} \quad \text{and} \quad g_n(r, \varrho) = \frac{(b^{2n} + \varrho^{2n})(r^{2n} - a^{2n})}{2n(r\varrho)^n(b^{2n} + a^{2n})},$$

and their formulas valid for $r \geq \varrho$ can be obtained from those above by interchanging the variables r and ϱ.

Upon substituting $g_0(r,\varrho)$ and $g_n(r,\varrho)$ into (2.126) and some algebra, we obtain the computer-friendly formula

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left\{\ln\frac{|b^2z - a^2\zeta|\,|a^2 - z\overline{\zeta}|}{a|z|\,|z-\zeta|\,|b^2 - z\overline{\zeta}|} + \sum_{n=1}^{\infty} g_n^*(r,\varrho)\cos n(\varphi-\psi)\right\} \tag{2.131}$$

of the Green's function to the "Dirichlet–Neumann" problem for the annular region with radii a and b. The $r \le \varrho$ branch for $g_n^*(r,\varrho)$ is

$$g_n^*(r,\varrho) = \frac{a^{2n}(b^{2n}+\varrho^{2n})(a^{2n}-r^{2n})}{n(b^2r\varrho)^n(b^{2n}+a^{2n})}$$

whilst to find its $r \ge \varrho$ branch, the variables r and ϱ must be exchanged. Note also that the factors $|z|$ and $|b^2z - a^2\zeta|$ in the argument of the logarithmic term in (2.131) are valid for $r \le \varrho$, while for $r \ge \varrho$ they should be replaced with $|\zeta|$ and $|b^2\zeta - a^2z|$, respectively.

The series (2.131) converges uniformly, which allows accurate and immediate assessment of the Green's function by truncating the series to the Nth partial sum. We leave it as an exercise to the reader to verify this assertion by examining the coefficient $g_n^*(r,\varrho)$ and to obtain graphical confirmation of the uniform convergence by depicting a few profiles of equation (2.131) for different values of the truncation parameter N.

Example 2.29. We consider another mixed boundary-value problem for the Laplace equation, namely the "Neumann–Dirichlet"

$$\frac{\partial u(b,\varphi)}{\partial r} = 0, \quad u(a,\varphi) = 0 \tag{2.132}$$

problem on the annular region $\Omega = \{a < r < b, 0 \le \varphi < 2\pi\}$.

We construct the Green's function to the homogeneous boundary-value problem for (2.96) and (2.132), by following the eigenfunction expansion method-based procedure and arrive again at the series representation in (2.126). Expressions for the series coefficients $g_0(r,\varrho)$ and $g_n(r,\varrho)$ valid for $r \le \varrho$, are found as

$$g_0(r,\varrho) = \ln\frac{b}{\varrho} \quad \text{and} \quad g_n(r,\varrho) = \frac{(\varrho^{2n}-b^{2n})(r^{2n}+a^{2n})}{2n(r\varrho)^n(b^{2n}+a^{2n})},$$

whilst for $r \ge \varrho$ the variables r and ϱ must be interchanged.

Analogous to the derivation in the previous example, we now substitute the expressions for the components $g_0(r,\varrho)$ and $g_n(r,\varrho)$ into (2.126). After some algebra, we obtain the sought-after branch $r \le \varrho$ of the Green's function in the computer-friendly formula

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left\{\ln\frac{|\zeta|\,|b^2z - a^2\zeta|\,|b^2 - z\overline{\zeta}|}{b^3|z-\zeta|\,|a^2 - z\overline{\zeta}|} + \sum_{n=1}^{\infty} g_n^*(r,\varrho)\cos n(\varphi-\psi)\right\} \tag{2.133}$$

with

$$g_n^*(r,\varrho) = \frac{a^{2n}(\varrho^{2n} - b^{2n})(a^{2n} + r^{2n})}{n(b^2 r\varrho)^n (b^{2n} + a^{2n})}. \tag{2.134}$$

Note that to obtain the branch of $G(r,\varphi;\varrho,\psi)$ for $r \geq \varrho$, we must exchange r and ϱ in (2.134), whilst $|\zeta|$ and $|b^2 z - a^2\zeta|$ in the argument of the logarithmic term in (2.133) must be replaced with $|z|$ and $|b^2\zeta - a^2 z|$, respectively.

Uniform convergence of (2.133) can again be justified by analysis of its coefficient $g_n^*(r,\varrho)$. We again recommend the reader to plot several profiles of the Green's function when experimenting with different values of the truncation parameter N.

2.3.4 Surfaces of Revolution

In this chapter, we have by now analyzed a vast number of boundary-value problems for the two-dimensional Laplace equation. All the problems we considered have been defined, rather conventionally, in either Cartesian or polar coordinates. To our mind, this allows for a reasonably systematic presentation of the suggested techniques for constructing Green's functions. Generally, most if not all problems, where Green's functions are provided in the literature, are defined in these two coordinate systems.

In the current subsection we go beyond these conventions. It extends the range of successful implementations of the method of eigenfunction expansion to the construction of Green's functions for potential problems defined on surfaces of revolution. This class of problems may look exotic, but it is not. In fact, it is of great importance in engineering and science to determine various physical potential fields occurring in thin shell structures, where the energy flow normal to their surface is small and can therefore be neglected.

In the following, we present a few problems for regions on spherical and toroidal surfaces. However, this by no means indicates a narrow applicability of this technique. Within the scope of the suggested technique, conical, paraboloidal, hyperboloidal, and other surfaces of revolution can also be considered by the suggested technique.

We assume a spherical surface of radius a to be determined by the following parametric equations

$$x = a\sin\varphi\cos\vartheta, \quad y = a\sin\varphi\sin\vartheta, \quad z = a\cos\varphi. \tag{2.135}$$

These relations uniquely define a point (x, y, z) on the surface in terms of the geographical coordinates φ and ϑ that represent the latitude and the longitude, respectively.

Going through the change of the independent variables determined by (2.135), the Cartesian three-dimensional Laplace operator

$$\Delta(x, y, z) \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

transforms into the two-dimensional differential form written as

$$\Delta(\varphi, \vartheta) \equiv \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\vartheta^2}.$$

It is evident that the above formula can also be obtained from the three-dimensional Laplace operator

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial\vartheta^2},$$

written in spherical coordinates r, φ and ϑ, if we let the variable r be a constant so that we can neglect derivatives with respect to this variable.

Example 2.30. For our first example of a spherical surface, we display the following mixed boundary-value problem:

$$\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u(\varphi,\vartheta)}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 u(\varphi,\vartheta)}{\partial\vartheta^2} = -f(\varphi,\vartheta), \quad (\varphi,\vartheta) \in \Omega, \tag{2.136}$$

$$|u|_{\varphi=0} < \infty, \qquad \left.\frac{\partial u}{\partial\varphi}\right|_{\varphi=\alpha} = 0, \quad u|_{\vartheta=0,\beta} = 0 \tag{2.137}$$

on the spherical triangle $\Omega = \{(\varphi,\vartheta) : 0 < \varphi < \alpha, 0 < \vartheta < \beta\}$, with parameters α and β constrained to $\alpha < \pi$ and $\beta < 2\pi$.

To construct the required Green's function, we expand the solution $u(\varphi,\vartheta)$ of (2.136) and (2.137) in the Fourier sine-series

$$u(\varphi,\vartheta) = \sum_{n=1}^{\infty} u_n(\varphi)\sin\nu\vartheta, \quad \nu = \frac{n\pi}{\beta}, \tag{2.138}$$

whilst the right-hand side in (2.136) is expanded in the series

$$f(\varphi,\vartheta) = \sum_{n=1}^{\infty} f_n(\varphi)\sin\nu\vartheta.$$

Substituting the expansions of $u(\varphi,\vartheta)$ and $f(\varphi,\vartheta)$ into the (2.136) and (2.137) and equating the corresponding coefficients of the Fourier series on both sides of (2.136), we obtain the following linear ordinary differential equation

$$\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{du_n(\varphi)}{d\varphi}\right) - \frac{\nu^2}{\sin^2\varphi}u_n(\varphi) = -f_n(\varphi)$$

for the coefficients $u_n(\varphi)$ of the series in (2.138). After introducing the integrating factor of $\sin\varphi$, the equation reduces to the self-adjoint form

$$\frac{d}{d\varphi}\left(\sin\varphi\frac{du_n(\varphi)}{d\varphi}\right)-\frac{\nu^2}{\sin\varphi}u_n(\varphi)=-f_n(\varphi)\sin\varphi \tag{2.139}$$

which, according to (2.137), must be subject to the boundary conditions

$$|u_n(0)|<\infty,\quad \frac{du_n(\alpha)}{d\varphi}=0,\quad n=1,2,3,\ldots. \tag{2.140}$$

We can indirectly obtain a fundamental set of solutions to the homogeneous equation corresponding to that in (2.139) by changing the independent variable φ in (2.139) and introducing a new one

$$\omega=\ln\left(\tan\frac{\varphi}{2}\right). \tag{2.141}$$

This yields

$$\frac{d\omega}{d\varphi}=\frac{1}{2}\frac{\sec^2\varphi/2}{\tan\varphi/2}=\frac{1}{2\sin\varphi/2\cos\varphi/2}=\frac{1}{\sin\varphi}$$

and

$$\frac{d}{d\varphi}=\frac{d}{d\omega}\frac{d\omega}{d\varphi}=\frac{1}{\sin\varphi}\frac{d}{d\omega}$$

or

$$\sin\varphi\frac{d}{d\varphi}=\frac{d}{d\omega}.$$

We now have

$$\frac{d}{d\varphi}\left(\sin\varphi\frac{d}{d\varphi}\right)=\frac{d^2}{d\omega^2}\frac{d\omega}{d\varphi}=\frac{1}{\sin\varphi}\frac{d^2}{d\omega^2}.$$

With this in mind, we convert the differential operator of equation (2.139) to

$$\frac{d}{d\varphi}\left(\sin\varphi\frac{d}{d\varphi}\right)-\frac{\nu^2}{\sin\varphi}\equiv\frac{1}{\sin\varphi}\left(\frac{d^2}{d\omega^2}-\nu^2\right).$$

This suggests that the fundamental set of solutions for (2.139) can be represented by the exponential functions

$$\exp(\nu\omega)\quad\text{and}\quad\exp(-\nu\omega).$$

Substituting this back, in compliance with (2.141), the fundamental set of solutions for the homogeneous equation corresponding to (2.139) reduces to

$$\tan^{\nu}\frac{\varphi}{2} \quad \text{and} \quad \cot^{\nu}\frac{\varphi}{2}. \tag{2.142}$$

With the fundamental set of solutions just derived in hand, we are now in a position to follow the procedure and construct the Green's function $g_n(\varphi, \psi)$ for the homogeneous problem corresponding to (2.139) and (2.140). In doing so, we obtain the branch of $g_n(\varphi, \psi)$ for $\varphi \leq \psi$ as

$$g_n(\varphi, \psi) = \frac{1}{2\nu}\left\{\left[\tan\frac{\varphi}{2}\tan\frac{\psi}{2}\cot^2\frac{\alpha}{2}\right]^{\nu} + \left[\tan\frac{\varphi}{2}\cot\frac{\psi}{2}\right]^{\nu}\right\}. \tag{2.143}$$

The fact that boundary-value problem described by (2.139) and (2.140) is self-adjoint suggests that the other branch of $g_n(\varphi, \psi)$ valid for $\varphi \geq \psi$ can be obtained from that in (2.143) by exchanging φ and ψ.

Proceeding with our routine, as employed in the preceding sections, we write the Green's function of the homogeneous boundary-value problem corresponding to (2.136) and (2.137) as

$$G(\varphi, \vartheta; \psi, \tau) = \frac{1}{\beta}\sum_{n=1}^{\infty} g_n(\varphi, \psi)[\cos\nu(\vartheta + \tau) - \cos\nu(\vartheta - \tau)]. \tag{2.144}$$

The above trigonometric series is summable. To carry out the summation, we rewrite (2.144) as

$$\begin{aligned} G(\varphi, \vartheta; \psi, \tau) = \frac{1}{2\pi}\Bigg\{&\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{\Phi}{\Psi}\right)^n + \left(\frac{\Phi\Psi}{A^2}\right)^n\right]\cos n\gamma \\ &- \sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{\Phi}{\Psi}\right)^n - \left(\frac{\Phi\Psi}{A^2}\right)^n\right]\cos n\delta\Bigg\}, \end{aligned} \tag{2.145}$$

where, for compactness, we have introduced

$$\Phi = \tan^{\pi/\beta}\frac{\varphi}{2}, \quad \Psi = \tan^{\pi/\beta}\frac{\psi}{2}, \quad A = \tan^{\pi/\beta}\frac{\alpha}{2}$$

and

$$\gamma = \frac{\pi}{\beta}(\vartheta - \tau), \quad \delta = \frac{\pi}{\beta}(\vartheta + \tau).$$

Notice that φ and ψ in (2.145) satisfy the relation $\varphi \leq \psi$. From this, it follows that the series in (2.145) meet the conditions necessary to apply the standard summation formula from (2.62) which, in turn, justifies the summability of the series.

After carrying out the summation and doing some elementary algebra, we finally obtain the Green's function for the homogeneous problem corresponding to (2.136) and (2.137) in the following closed form

$$G(\varphi,\vartheta;\psi,\tau)=\frac{1}{2\pi}\left(\ln\sqrt{\frac{\Phi^2-2\Phi\Psi\cos\delta+\Psi^2}{\Phi^2-2\Phi\Psi\cos\gamma+\Psi^2}}\right.$$
$$\left.+\ln\sqrt{\frac{A^4-2A^2\Phi\Psi\cos\delta+\Phi^2\Psi^2}{A^4-2A^2\Phi\Psi\cos\gamma+\Phi^2\Psi^2}}\right). \tag{2.146}$$

It is evident that (2.146) is compact enough and its immediate computer implementation presents no problems whatsoever.

The example that we just provided lends us confidence in the productivity of the suggested technique for the construction of Green's functions for a variety of boundary value problems for potential functions, on surfaces of revolution.

In the following example, we present the reader with another problem on a spherical surface.

Example 2.31. We here present a brief description of the procedure for constructing the Green's function for a Dirichlet problem corresponding to the Laplace equation on the spherical segment $\Omega=\{(\varphi,\vartheta):0<\varphi<\alpha, 0\le\vartheta<2\pi\}$, with α being limited to $\alpha<\pi$.

To construct the required Green's function by following our approach, we formulate a corresponding boundary-value problem for the Poisson equation in (2.136) on the segment Ω. The solution $u(\varphi,\vartheta)$ of that problem must be a 2π-periodic function of the ϑ variable subject to the boundary conditions

$$|u(\varphi,\vartheta)|_{\varphi=0}<\infty,\quad u(\alpha,\vartheta)=0. \tag{2.147}$$

Due to the 2π-periodicity in (2.136) and (2.147), the functions $u(\varphi,\vartheta)$ and $f(\varphi,\vartheta)$ can be expanded in the general Fourier series

$$u(\varphi,\vartheta)=\frac{1}{2}u_0(\varphi)+\sum_{n=1}^{\infty}(u_n^c(\varphi)\cos n\vartheta+u_n^s(\varphi)\sin n\vartheta) \tag{2.148}$$

and

$$f(\varphi,\vartheta)=\frac{1}{2}f_0(\varphi)+\sum_{n=1}^{\infty}(f_n^c(\varphi)\cos n\vartheta+f_n^s(\varphi)\sin n\vartheta).$$

Following the algorithm described in Example 2.30, we substitute the above expansions into the original formulation. This yields the following set of self-adjoint boundary value problems

$$\frac{d}{d\varphi}\left(\sin\varphi\frac{du_n(\varphi)}{d\varphi}\right) - \frac{n^2}{\sin\varphi}u_n(\varphi) = -f_n(\varphi)\sin\varphi, \tag{2.149}$$

$$|u_n(0)| < \infty, \quad u_n(\alpha) = 0, \quad n = 0, 1, 2, \ldots, \tag{2.150}$$

for the coefficients $u_n(\varphi)$ of the series in (2.148). The sought-after Green's function, is finally expressed as the series

$$G(\varphi, \vartheta; \psi, \tau) = \frac{1}{2\pi}\left[g_0(\varphi, \psi) + 2\sum_{n=1}^{\infty} g_n(\varphi, \psi)\cos n(\vartheta - \tau)\right] \tag{2.151}$$

with coefficients $g_0(\varphi, \psi)$ and $g_n(\varphi, \psi)$ representing the Green's functions for the boundary value problems in (2.149) and (2.150). As we have already learned, in constructing these Green's functions, the case of $n = 0$ should be treated individually, because its fundamental set of solutions (see [33])

$$\ln\left(\tan\frac{\varphi}{2}\right) \quad \text{and} \quad 1$$

is different from that of the general case of $n \geq 1$ derived earlier in Example 2.30 and displayed in (2.142).

Hence, obtaining the Green's functions $g_0(\varphi, \psi)$ and $g_n(\varphi, \psi)$ is not a problem. And once they are constructed, we can sum the series in (2.151) completely, ultimately yielding the Green's function for the Dirichlet problem for the spherical segment in closed form

$$G(\varphi, \vartheta; \psi, \tau) = \frac{1}{2\pi}\ln\left(\frac{A_0\sqrt{\Phi_0^2 - 2\Phi_0\Psi_0\cos(\vartheta - \tau) + \Psi_0^2}}{\sqrt{A_0^4 - 2A_0^2\Phi_0\Psi_0\cos(\vartheta - \tau) + \Phi_0^2\Psi_0^2}}\right), \tag{2.152}$$

where the following notations

$$A_0 = \tan\frac{\alpha}{2}, \quad \Phi_0 = \tan\frac{\varphi}{2}, \quad \Psi_0 = \tan\frac{\psi}{2}$$

were introduced for compactness.

To further illustrate the efficiency of our version of the method of eigenfunction expansion for the construction of Green's functions for problems involving potentials on surfaces of revolution, we will now consider a boundary-value problem on a region of a toroidal surface.

Let a and R represent the radius of the meridian cross-section of the circular toroidal surface and the distance between the center of the meridian cross-section and the axis of revolution, respectively, with $a < R$ (see Figure 2.21).

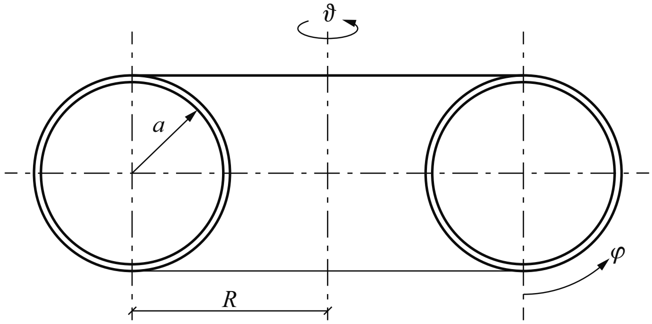

Figure 2.21. Geometry of a toroidal shell.

In the following, we assume a point (x, y, z) on the toroidal surface is to be determined by the geographical coordinates φ (the latitude) and ϑ (the longitude) in accordance with the parameterization

$$x = D(\varphi)\cos\vartheta, \quad y = D(\varphi)\sin\vartheta, \quad z = a\cos\varphi, \tag{2.153}$$

where $D(\varphi) = R + a\sin\varphi$.

The change of variables due to the parametrization in (2.153) transforms the three-dimensional Cartesian Laplace operator to the following two-dimensional form

$$\Delta(\varphi,\vartheta) \equiv \frac{1}{D(\varphi)}\frac{\partial}{\partial\varphi}\left(D(\varphi)\frac{\partial}{\partial\varphi}\right) + \frac{a^2}{D^2(\varphi)}\frac{\partial^2}{\partial\vartheta^2},$$

representing a two-dimensional Laplace operator written in toroidal coordinates φ and ϑ.

Example 2.32. Consider a toroidal sector Ω which is closed in the direction of φ but open in the direction of ϑ. That is, $\Omega = \{(\varphi,\vartheta) : 0 < \varphi \le 2\pi, 0 < \vartheta < \beta\}$, with the β constrained to $\beta < 2\pi$. We define the Dirichlet problem

$$\frac{\partial}{\partial\varphi}\left(D(\varphi)\frac{\partial u(\varphi,\vartheta)}{\partial\varphi}\right) + \frac{a^2}{D(\varphi)}\frac{\partial^2 u(\varphi,\vartheta)}{\partial\vartheta^2} = -f(\varphi,\vartheta)D(\varphi), \tag{2.154}$$

$$u|_{\vartheta=0} = 0, \quad u|_{\vartheta=\beta} = 0 \tag{2.155}$$

on Ω. Due to the Ω being closed in the direction of φ, the above formulation is augmented with the relations

$$u|_{\varphi=0} = u|_{\varphi=2\pi}, \quad \left.\frac{\partial u}{\partial\varphi}\right|_{\varphi=0} = \left.\frac{\partial u}{\partial\varphi}\right|_{\varphi=2\pi}. \tag{2.156}$$

The above relations represent conditions for the 2π-periodicity of the solution in the direction of φ.

The solution $u(\varphi, \vartheta)$ of (2.154)–(2.156) is written as the following trigonometric expansion

$$u(\varphi, \vartheta) = \sum_{n=1}^{\infty} u_n(\varphi) \sin \nu\vartheta, \quad \nu = \frac{n\pi}{\beta},$$

which allows for $u(\varphi, \vartheta)$ to satisfy the boundary conditions in (2.155).

Now, expand the right-hand side of (2.154), $f(\varphi, \vartheta)$, into

$$f(\varphi, \vartheta) = \sum_{n=1}^{\infty} f_n(\varphi) \sin \nu\vartheta.$$

This yields the self-adjoint boundary value problem

$$\frac{d}{d\varphi}\left(D(\varphi)\frac{du_n(\varphi)}{d\varphi}\right) - \frac{a^2\nu^2}{D(\varphi)}u_n(\varphi) = -D(\varphi) f_n(\varphi), \tag{2.157}$$

$$u_n(0) = u_n(2\pi), \quad \frac{du_n(0)}{d\varphi} = \frac{du_n(2\pi)}{d\varphi}, \quad n = 1, 2, 3, \ldots, \tag{2.158}$$

for the coefficients $u_n(\varphi)$ of the above series for $u(\varphi, \vartheta)$.

A fundamental set of solutions, required for the construction of the Green's function $g_n(\varphi, \psi)$ for the homogeneous boundary value problem corresponding to (2.157) and (2.158), is represented by the exponential functions (see [33])

$$u_n^{(1)}(\varphi) = \exp\left(\frac{2a\nu}{\sqrt{R^2 - a^2}} \arctan\left(\frac{a + R\tan(\varphi/2)}{\sqrt{R^2 - a^2}}\right)\right)$$

and

$$u_n^{(2)}(\varphi) = \exp\left(-\frac{2a\nu}{\sqrt{R^2 - a^2}} \arctan\left(\frac{a + R\tan(\varphi/2)}{\sqrt{R^2 - a^2}}\right)\right).$$

Constructing $g_n(\varphi, \psi)$, based on $u_n^{(1)}(\varphi)$ and $u_n^{(2)}(\varphi)$ is a routine procedure. Once obtained, the series representation

$$G(\varphi, \vartheta; \psi, \tau) = \frac{2}{\beta}\sum_{n=1}^{\infty} g_n(\varphi, \psi) \sin \nu\vartheta \sin \nu\tau$$

of the Green's function $G(\varphi, \vartheta; \psi, \tau)$ for the Dirichlet problem for the potential equation on the toroidal sector is derived within the scope of our approach. The above series is entirely summable with the aid of the summation formula in (2.62). Omitting the details, we arrive at the following closed form of $G(\varphi, \vartheta; \psi, \tau)$

$$G(\varphi, \vartheta; \psi, \tau) = \frac{1}{2\pi} \ln \sqrt{\frac{\Phi_1^2(\varphi) - 2\Phi_1(\varphi)\Phi_1(\psi)\cos\delta + \Phi_1^2(\psi)}{\Phi_1^2(\varphi) - 2\Phi_1(\varphi)\Phi_1(\psi)\cos\gamma + \Phi_1^2(\psi)}}, \tag{2.159}$$

where

$$\Phi_1(t) = \exp\left(\frac{2\pi a}{\beta\sqrt{R^2 - a^2}} \arctan\left(\frac{a + R\tan(t/2)}{\sqrt{R^2 - a^2}}\right)\right)$$

with γ and δ defined as

$$\gamma = \frac{\pi}{\beta}(\vartheta - \tau), \quad \delta = \frac{\pi}{\beta}(\vartheta + \tau).$$

Our main achievement in the current section is the development of a method, based on eigenfunction expansion, for constructing Green's functions for boundary-value problems for the two-dimensional Laplace equation written in geographical coordinates, defining a point on a surface of revolution.

Later in this book, when dealing with potential fields on joined surfaces of revolution (see Chapter 6), we will take advantage of the experience we gained in the current section and will revisit this class of problems. The Green's function formalism, which we have been dealing with in this chapter, will be extended to special systems of partial differential equations. This extension will be carried out in a manner similar to the one implemented in Chapter 1 for special systems of ordinary differential equations. In doing so, the notion of Green's function, as introduced in the current chapter, will be adapted to special PDE systems.

2.4 Three-Dimensional Problems

The primary intention of this book is to deal with a variety of two-dimensional partial differential equations. In a few places in the book we turn to higher-dimensional problems. The current section in particular, is designed to show the reader how several of the conventional methods, which turn out to be efficient in the construction of two-dimensional Green's functions, can also be applied successfully to three-dimensional problems.

Of the three conventional methods that we used in this chapter to treat the two-dimensional Laplace, only two are workable for treatment of three-dimensional problems. Clearly, the method of conformal mapping cannot be considered, but the method of images and the eigenfunction expansion approach show considerable potential.

Let the boundary-value problem

$$\nabla^2 u(P) = 0, \quad P \in D, \tag{2.160}$$

$$T[u(P)] = 0, \quad P \in S, \tag{2.161}$$

for the three-dimensional Laplace equation be well-posed within a simply-connected region D bounded with a piecewise smooth surface S.

In introducing the Green's function for the above problem, we denote the observation (field) and the source point with P and Q, respectively. It is well known [3, 37, 53, 66] that the fundamental solution of the three-dimensional Laplace equation is different than the two-dimensional case. For example, in Cartesian coordinates, it is expressed in terms of the distance $|P - Q|$ between $P(x, y, z)$ and $Q(\xi, \eta, \zeta)$ as

$$\frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \tag{2.162}$$

and represents a harmonic function at every point $(x, y, z) \neq (\xi, \eta, \zeta)$.

The Green's function for (2.160) and (2.161) can consequently be written as

$$G(P, Q) = \frac{1}{4\pi|P - Q|} + R(P, Q), \quad P, Q \in D, \tag{2.163}$$

with $R(P, Q)$ representing the regular component of $G(P, Q)$.

In order to get a three-dimensional Green's function we need to find its regular component $R(P, Q)$, similar to the two-dimensional situation. A number of 3-D Green's functions can be found with the aid of the method of images. Examples that follow illustrate this assertion. We start with the simplest of them.

Example 2.33. Construct the Green's function for the Dirichlet problem for the Laplace equation in the half-space $D = \{z > 0\}$.

After placing a unit source, which generates a field defined by (2.162), at an arbitrary point $Q(\xi, \eta, \zeta)$ belonging to D, we compensate its trace on the boundary plane $z = 0$ with the unit sink

$$-\frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}} \tag{2.164}$$

located at $Q^*(\xi, \eta, -\zeta)$, symmetric with $Q(\xi, \eta, \zeta)$ with respect to the plane $z = 0$. Clearly, (2.164) is harmonic everywhere in D, and the sum of (2.162) and (2.164)

$$\begin{aligned} G(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi}\Bigg(&\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \\ &- \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}}\Bigg) \end{aligned} \tag{2.165}$$

is a harmonic function of the coordinates of the field point P everywhere in D, except for $(x, y, z) = (\xi, \eta, \zeta)$. Thus, it contains the singularity of the fundamental solution, and vanishes on the boundary of D. In other words, it does, indeed, represent the sought-after Green's function.

Example 2.34. Construct the Green's function for the Dirichlet problem for the three-dimensional Laplace equation in a sphere with radius a.

Analogous to the approach for the Dirichlet problem in a disk, described in Section 2.1, we exploit the fact that the shape of an equipotential surface in the three-dimensional field generated by a point source or a point sink is a concentric sphere centered at the generating point. That is, for any location of the source inside the sphere, there exists a proper location for a compensatory sink outside the sphere so that the face of the sphere is a surface of zero potential for the field generated by both the source and the sink.

In order to tackle this problem, it is convenient to introduce spherical coordinates. With the field point $P(r, \vartheta, \varphi)$ and the source point $Q(\rho, \chi, \psi)$, we find the following expression for the Green's function of the Dirichlet problem on a sphere of radius a

$$G(P, Q) = \frac{1}{4\pi}\left(\frac{1}{|P-Q|} - \frac{1}{|P-Q^*|}\right) \tag{2.166}$$

with the distance $|P-Q|$ defined as

$$\sqrt{r^2 - 2r\rho\cos\gamma + \rho^2}$$

whilst the expression

$$\frac{1}{a}\sqrt{r^2\rho^2 - 2a^2 r\rho\cos\gamma + a^4}$$

defines the distance $|P-Q^*|$ between the field point P and the compensatory sink point Q^*, located outside of the sphere. Note that γ represents the angle formed by vectors $\overrightarrow{P}$ and $\overrightarrow{Q}$, and $\cos\gamma$ is defined in terms of the spherical angle coordinates of P and Q as

$$\sin\vartheta\sin\chi\cos(\varphi-\psi) + \cos\vartheta\cos\chi.$$

Example 2.35. Construct the Green's function of the Dirichlet problem in the infinite layer $D = \{0 < z < h\}$ of width h.

A closed analytical formula for this Green's function is not available. However, the method of images allows us to devise a series expression. The approach described earlier in Example 2.13 turns out to be helpful here as well. That is, if we place the unit source

$$G_0(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \tag{2.167}$$

at an arbitrary point (ξ, η, ζ) inside D, then traces of (2.167) on the boundary planes $z = 0$ and $z = h$ can be compensated by the unit sinks

$$G_0(x, y, z; \xi, \eta, -\zeta) = -\frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}}$$

and

$$G_0(x, y, z; \xi, \eta, 2h - \zeta) = -\frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-2h)^2}}$$

placed in the points $(\xi, \eta, -\zeta)$ and $(\xi, \eta, 2h - \zeta)$, respectively.

Clearly, $G_0(x, y, z; \xi, \eta, -\zeta)$ and $G_0(x, y, z; \xi, \eta, 2h - \zeta)$ leave nonzero traces on the boundary planes $z = 0$ and $z = h$. To cancel those, we place the unit sources

$$G_0(x, y, z; \xi, \eta, -2h + \zeta) = \frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta+2h)^2}}$$

and

$$G_0(x, y, z; \xi, \eta, 2h + \zeta) = \frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta-2h)^2}}$$

at $(\xi, \eta, -2h + \zeta)$ and $(\xi, \eta, 2h + \zeta)$, respectively.

Traces of $G_0(x, y, z; \xi, \eta, -2h + \zeta)$ and $G_0(x, y, z; \xi, \eta, 2h + \zeta)$ on the boundary planes can now be canceled out with the unit sinks

$$G_0(x, y, z; \xi, \eta, -2h - \zeta) = -\frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta+2h)^2}}$$

and

$$G_0(x, y, z; \xi, \eta, 4h - \zeta) = -\frac{1}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-4h)^2}},$$

placed in $(\xi, \eta, -2h - \zeta)$ and $(\xi, \eta, 4h - \zeta)$, respectively.

Following familiar pattern, we place appropriate pairs of compensatory sources and sinks N times, and sum up their influence as shown

$$\frac{1}{4\pi}\sum_{n=-N}^{N}\left[G_0(x, y, z; \xi, \eta, \zeta - 2nh) - G_0(x, y, z; \xi, \eta, 2nh - \zeta)\right].$$

After taking the limit for N approaching infinity, we obtain the infinite series

$$G(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi}\sum_{n=-\infty}^{\infty}\left(\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta+2nh)^2}} - \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-2nh)^2}}\right) \tag{2.168}$$

which represents the sought-after Green's function.

It is evident that the first additive component

$$\frac{1}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}$$

in the $n = 0$ term of (2.168) represents the fundamental solution to the three-dimensional Laplace equation. This implies that if the $n = 0$ term is omitted then the series in (2.168) should be uniformly convergent in D.

To complete the discussion in this section, we turn our attention to the method of eigenfunction expansion, and present an appetizer for its application in constructing Green's functions for the three-dimensional Laplace equation.

Example 2.36. Construct the Green's function for the homogeneous Dirichlet problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=-f(x,y,z), \quad (x,y,z)\in D, \tag{2.169}$$

$$u(x,y,0)=u(x,y,h)=u(x,0,z)=u(x,b,z)=0, \tag{2.170}$$

$$u(0,y,z)=0, \quad \lim_{x\to\infty}|u(x,y,z)|<\infty \tag{2.171}$$

in the semi-infinite bar $D = \{0 < x < \infty, 0 < y < b, 0 < z < h\}$ with rectangular cross-section.

Note that if we find a solution of (2.169)-(2.171) of the form

$$u(x,y,z)=\int_0^h\int_0^b\int_0^\infty K(x,y,z;\xi,\eta,\zeta)f(\xi,\eta,\zeta)dD(\xi,\eta,\zeta), \tag{2.172}$$

then the kernel $K(x, y, z; \xi, \eta, \zeta)$ in the above integral represents the Green's function for the corresponding homogeneous problem.

Taking into account the boundary conditions of (2.170), we expand the functions $u = u(x, y, z)$ and $f(x, y, z)$ in the double Fourier series

$$u(x,y,z)=\sum_{m,n=1}^{\infty}u_{mn}(x)\sin\mu y\sin\nu z \tag{2.173}$$

and

$$f(x,y,z)=\sum_{m,n=1}^{\infty}f_{mn}(x)\sin\mu y\sin\nu z \tag{2.174}$$

with μ and ν defined in terms of the summation indices of the above series as $\mu = m\pi/b$ and $\nu = n\pi/h$.

The completing steps of our derivation are analogous to the two-dimensional case. developed in detail earlier in this chapter: upon substituting the expansions (2.173) and (2.174) into (2.169), we arrive at the boundary-value problem

$$\frac{d^2u_{mn}(x)}{dx^2} - \lambda^2 u_{mn}(x) = -f_{mn}(x), \quad \lambda^2 = \mu^2 + \nu^2, \tag{2.175}$$

$$u_{mn}(0) = 0, \quad \lim_{x\to\infty} |u_{mn}(x)| < \infty \tag{2.176}$$

for the coefficients $u_{mn}(x)$ of the series in (2.173).

If $g_{mn}(x,\xi)$ represents the Green's function for the homogeneous problem corresponding to (2.175) and (2.176), then the solution to the problem itself can be written as

$$u_{mn}(x) = \int_0^\infty g_{mn}(x,\xi) f_{mn}(\xi) d\xi. \tag{2.177}$$

From Chapter 1 (see Example 1.4) we learn that, using our current notation, $g_{mn}(x,\xi)$ reads

$$g_{mn}(x,\xi) = \frac{1}{2\lambda}(\mathrm{e}^{-\lambda|x-\xi|} - \mathrm{e}^{-\lambda(x+\xi)}).$$

We substitute the series coefficients $f_{mn}(x)$ into (2.174), which are expressed in terms of $f(x,y,z)$ as

$$f_{mn}(x) = \frac{4}{bh}\int_0^h \int_0^b f(x,\eta,\zeta)\sin\mu\eta \sin\nu\zeta d\eta d\zeta$$

in (2.177), and then substitute $u_{mn}(x)$ into (2.173). This reduces the solution of the boundary-value problem described by (2.169)-(2.171) to

$$u(x,y,z) = \int_0^h \int_0^b \int_0^\infty \frac{2}{bh} \sum_{m,n=1}^{\infty} \frac{1}{\lambda}(\mathrm{e}^{-\lambda|x-\xi|} - \mathrm{e}^{-\lambda(x+\xi)})$$

$$\times \sin\mu y \sin\mu\eta \sin\nu z \sin\nu\zeta f(\xi,\eta,\zeta) dD(\xi,\eta,\zeta). \tag{2.178}$$

Hence, in light of (2.172), we conclude that the kernel of the integral in (2.178)

$$G(x,y,z;\xi,\eta,\zeta) = \frac{2}{bh}\sum_{m,n=1}^{\infty} \frac{1}{\lambda}\left(\mathrm{e}^{-\lambda|x-\xi|} - \mathrm{e}^{-\lambda(x+\xi)}\right) \sin\mu y \sin\mu\eta \sin\nu z \sin\nu\zeta \tag{2.179}$$

represents the Green's function for the homogeneous boundary-value problem corresponding to (2.169)–(2.171).

2.5 Chapter Exercises

1. Derive the Green's function shown in (2.13) for the Dirichlet problem for the Laplace equation on the infinite circular sector of $\pi/2$.
2. Derive the Green's function shown in (2.14) for the Dirichlet problem corresponding to the Laplace equation on the infinite circular sector of $\pi/6$.
3. Show that the method of images fails in the construction of the Green's function for the Dirichlet problem on the infinite circular sector of $2\pi/5$.
4. Show that the method of images fails in the construction of the Green's function for the Dirichlet problem on the infinite circular sector of $2\pi/7$.
5. Prove that the method of images is effective for the construction of the Green's function of the Dirichlet problem on the infinite circular sector of π/k, where k is an integer.
6. Use the method of eigenfunction expansion to construct the Green's function for the Laplace equation for the boundary-value problem
$$u(x,0) = \frac{\partial u(x,b)}{\partial y} = 0$$
on the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$.
7. Construct the Green's function for the Laplace equation for the boundary-value problem
$$u(0,y) = u(x,0) = \frac{\partial u(x,b)}{\partial y} = 0$$
on the semi-infinite strip $\Omega = \{0 < x < \infty, 0 < y < b\}$.
8. Construct the Green's function for the Laplace equation for the boundary-value problem
$$\frac{\partial u(0,y)}{\partial x} = u(x,0) = \frac{\partial u(x,b)}{\partial y} = 0$$
on the semi-infinite strip $\Omega = \{0 < x < \infty, 0 < y < b\}$.
9. Use the method of eigenfunction expansion to construct the Green's function for the Laplace equation for the mixed boundary-value problem
$$u(0,y) = u(x,0) = u(x,b) = \frac{\partial u(a,y)}{\partial x} + \beta u(a,y) = 0,$$
where $\beta \geq 0$, on the rectangle $\Omega = \{0 < x < a, 0 < y < b\}$.

10. Construct the Green's function for the Laplace equation for the "Dirichlet-Mixed" problem

$$u(a,\varphi)=0, \quad \frac{\partial u(b,\varphi)}{\partial r}+\beta u(b,\varphi)=0, \quad \beta \geq 0,$$

on the annular region $\Omega=\{a<r<b, 0\leq\varphi<2\pi\}$.

11. Construct the Green's function for the Laplace equation for the "Mixed-Dirichlet" problem

$$u(b,\varphi)=0, \quad \frac{\partial u(a,\varphi)}{\partial r}-\beta u(a,\varphi)=0, \quad \beta \geq 0,$$

on the annular region $\Omega=\{a<r<b, 0\leq\varphi<2\pi\}$.

12. Use the method of eigenfunction expansion to construct the Green's function for the Laplace equation for the "Neumann-Mixed" problem

$$\frac{\partial u(a,\varphi)}{\partial r}=0, \quad \frac{\partial u(b,\varphi)}{\partial r}+\beta u(b,\varphi)=0, \quad \beta > 0,$$

on the annular region $\Omega=\{a<r<b, 0\leq\varphi<2\pi\}$. Explain why for $\beta=0$ the problem is ill-posed.

13. Use the method of eigenfunction expansion to construct the Green's function for the Laplace equation for the mixed boundary-value problem

$$\frac{\partial u(a,\varphi)}{\partial r}-\beta u(a,\varphi)=0, \quad \frac{\partial u(b,\varphi)}{\partial r}=0$$

on the annular region $\Omega=\{a<r<b, 0\leq\varphi<2\pi\}$.

14. Use the method of eigenfunction expansion to construct the Green's function for the Laplace equation for the mixed boundary-value problem

$$\frac{\partial u(a,\varphi)}{\partial r}-\beta_1 u(a,\varphi)=0, \quad \frac{\partial u(b,\varphi)}{\partial r}+\beta_2 u(b,\varphi)=0, \quad \beta_1,\beta_2 > 0,$$

on the annular region $\Omega=\{a<r<b, 0\leq\varphi<2\pi\}$. Show why the problem is ill-posed when both β_1 and β_2 are equal to zero.

15. Use the method of images to construct the Green's function for the three-dimensional Laplace equation for the mixed boundary-value problem

$$u(x,y,0)=0, \quad \frac{\partial u(x,y,h)}{\partial z}=0$$

in the infinite layer $D=\{0<z<h\}$ of width h.

16. Use the method of eigenfunction expansion to construct the Green's function for the three-dimensional Laplace equation for the Dirichlet problem stated on the parallelepiped $D=\{0<x<a, 0<y<b, 0<z<h\}$.

Chapter 3
The Static Klein–Gordon Equation

We will now apply the techniques to construct Green's functions, that have successfully been used earlier in this book, to a number of boundary-value problems stated for the Klein–Gordon equation in two and three dimensions. This is another elliptic type partial differential equation which represents a natural domain for the extension of several of the approaches that appear productive in the case of Laplace equation. It is evident that the direct implementation of the conformal mapping method is problematic for the Klein–Gordon equation. On the other hand, the perspective of the other two standard methods (images and eigenfunction expansion) looks attractive and promising.

In our earlier book [45] and in a recent paper [51], several problem settings for the static Klein–Gordon equation have been reviewed and their Green's functions constructed. In this chapter, we summarize the experience gained in working with the Laplace equation. This will allow us to obtain Green's functions for an extensive list of boundary-value problems for the Klein–Gordon equation. The computability of their formulas is a critical issue, for example for potential users of the boundary element method. Hence, we deal with this topic in detail in this chapter.

The material in this chapter is organized in such a way that the definition of the Green's function for the two-dimensional Klein–Gordon equation is introduced in Section 3.1 in a fashion similar to that used in Chapter 2 for the Laplace equation. We then focus on the image method which, is covered in Section 3.2, whilst the method of eigenfunction expansion is discussed in detail in Section 3.3. Finally, Section 3.4 deals with several three-dimensional problems.

3.1 Definition of Green's Function

A variety of boundary-value problems will be considered in this chapter for the elliptic type equation:

$$\nabla^2 u(P) - k^2 u(P) = 0 \tag{3.1}$$

with ∇^2, the two-dimensional Laplace operator written in the coordinates of the point P, and the parameter k is a real constant.

Note that, in customary terminology, equation (3.1) is called the *static Klein–Gordon* equation. For brevity, we will simply call it the *Klein–Gordon* equation within the scope of the present book.

The Klein–Gordon and Laplace equations are similar in nature, and share many common properties: setting $k = 0$, reduces (3.1) to the Laplace formula. In fact, later in this chapter, the reader will find that once a Green's function for the Klein–Gordon equation is constructed, setting $k = 0$ reduces it the corresponding Green's function for the Laplace equation.

For more evidence of the likeness of the two equations, we turn to their applications in physics. Both the two-dimensional Laplace and Klein–Gordon equation can be used [13] to simulate the steady-state heat conduction process in a thin plate, made of homogeneous isotropic conductive material. The only difference we have to take into account is that the lateral surfaces of the plate are assumed to be insulated in the case of Laplace equation, whereas in the case of the Klein–Gordon equation, the lateral surfaces are not assumed to be perfectly insulated, but allow some heat flow through, with the flow intensity directly proportional to the parameter k.

Using the methodology developed and tested in Chapter 2, we will introduce the Green's function $G(P, Q)$ for the Klein–Gordon equation by setting up the homogeneous boundary-value problem

$$M_i u(P) \equiv \alpha_i(P)\frac{\partial u(P)}{\partial n_i} + \beta_i(P)u(P) = 0, \quad P \in \Gamma_i, \tag{3.2}$$

for the inhomogeneous Klein–Gordon equation

$$\nabla^2 u(P) - k^2 u(P) = -f(P), \quad P \in \Omega, \tag{3.3}$$

where Ω represents a simply connected region in two-dimensional Euclidean space, $\Gamma = \bigcup_{i=1}^{m} \Gamma_i$ denotes a piecewise smooth contour of Ω ,with $\alpha_i(P)$ and $\beta_i(P)$ given functions defined on Γ such that at least one of them is nonzero for every piece Γ_i of Γ, and n_i the direction normal to Γ_i at point P. Note that, within our approach, we can also consider multiply-connected regions. The right-hand side function $f(P)$ in (3.3) is assumed to be integrable on Ω.

Assume that the boundary-value problem described by (3.2) and (3.3) is well-posed. This means that it has a unique solution or, in other words, the corresponding homogeneous problem, with $f(P) \equiv 0$, has only the trivial $u(P) \equiv 0$ solution. With this in mind, we will now define the Green's function for the Klein–Gordon equation in the fashion used in Chapter 2 for the Laplace equation.

If for any right-hand side term $f(P)$ described by (3.3), integrable on Ω, the solution of the boundary-value problem described by (3.2) and (3.3) is found in the form

$$u(P) = \iint_{\Omega} G(P, Q) f(Q) d\Omega(Q), \tag{3.4}$$

then the kernel $G(P, Q)$ of the above formula is said to be the *Green's function* for the homogeneous problem corresponding to (3.2) and (3.3).

We will apply terminology similar to the case of Laplace equation, according to which P and Q in (3.4) will be referred to as the field (observation) point and the source point, respectively.

For any location of the source point $Q \in \Omega$, the Green's function $G(P, Q)$, as a function of the coordinates of the observation point P, has the following properties (being referred to as the *defining properties*):

1. At any point $P \in \Omega$, with the exception of $P = Q$, $G(P, Q)$ satisfies the homogeneous Klein–Gordon equation, that is
$$(\nabla^2 - k^2)G(P, Q) = 0, \quad P \neq Q.$$
2. For $P \to Q$, $G(P, Q)$ approaches infinity like the modified cylindrical Bessel (or Macdonald) function $K_0(k|P - Q|)$ of the second kind of order zero.
3. $G(P, Q)$ satisfies the boundary conditions in (3.2), that is
$$M_i G(P, Q) = 0, \quad P \in \Gamma_i, \quad i = \overline{1, m}.$$

Due to the properties of the Macdonald function (see, for example, [1, 27, 37]), the Green's function $G(P, Q)$ of the Klein–Gordon equation contains the same type of logarithmic singularity as the Green's function of the Laplace equation. To aid in comprehending the nature of the Macdonald function $K_0(x)$ and to deliver a practical tool for its computation, we present its standard series expansion (see [73, 74], for example)

$$K_0(x) = -\left(C + \ln\frac{x}{2}\right)\sum_{j=0}^{\infty}\frac{x^{2j}}{2^{2j}(j!)^2} + \sum_{m=1}^{\infty}\frac{x^{2m}}{2^{2m}(m!)^2}\left(\sum_{n=1}^{m}\frac{1}{n}\right) \tag{3.5}$$

with $C \approx 0.5772157$ known as the *Euler's constant*. There are several formulas for C, one of which [37] is

$$C = \lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{1}{k} - \ln m\right).$$

The expansion in (3.5) might raise several concerns about its computability: its appearance might be considered somewhat cumbersome. However, upon closer analysis it is revealed, that the opposite is the case. The formula in (3.5) turns out to be computer-friendly, with two points supporting this assertion. First, the infinite series components in (3.5) converge uniformly for any value of x; and, second, their rate of convergence is fairly high.

From the definition we just introduced, it follows that the Green's function of the homogeneous boundary-value problem corresponding to (3.2) and (3.3) can be expressed as

$$G(P, Q) = \frac{1}{2\pi}K_0(k|P - Q|) + R(P, Q), \tag{3.6}$$

where $R(P, Q)$, as a function of P, satisfies the homogeneous Klein–Gordon equation everywhere on Ω, regardless of the locations of P and Q.

The formula in (3.6) merits a special comment, as to the use of the terms *regular component* and *singular component* of a Green's function: upon explicitly expressing the first additive term in (3.5) as

$$-\ln\frac{x}{2}\sum_{j=0}^{\infty}\frac{x^{2j}}{2^{2j}(j!)^2} = -\ln\frac{x}{2}\left(1+\frac{x^2}{4}+\frac{x^4}{64}+\cdots\right)$$

we realize that the singularity in (3.6) only applies to the $j = 0$ term of the above series component in $K_0(x)$; the terms directly proportional to $x^2\ln x, x^4\ln x$ and so on, as well as the second series term in (3.5), being regular as x approaches zero, add no singularity to $G(P, Q)$.

Keeping in mind the above comment and following the pattern treated in Chapter 2, we will by convention refer to the additive terms $\frac{1}{2\pi}K_0(k|P - Q|)$ and $R(P, Q)$ of $G(P, Q)$ in (3.6) as the *singular component* and *regular component*, respectively. This should aid proceeding through the presentation that follows.

3.2 Method of Images

We turn our attention to the construction of Green's functions for the Klein–Gordon equation in two dimensions. In this section, the technique based on the method of images is directly applied to a variety of boundary-value problems.

The singular component $\frac{1}{2\pi}K_0(k|P - Q|)$ of $G(P, Q)$ is, similar to the case of Laplace equation, interpreted as the field generated at a field (observation) point P by a unit source placed at an arbitrary point Q. With this in mind, the method of images intends to express the regular component $R(P, Q)$ of $G(P, Q)$ as a sum of a number of unit sources and sinks placed at points $Q_1^*, Q_2^*, \ldots, Q_m^*$ outside of Ω, the region under consideration. Hence, the regular component of $G(P, Q)$ becomes

$$R(P, Q) = \sum_{j=1}^{m} \pm\frac{1}{2\pi}K_0(k|P - Q_j^*|),$$

a function satisfying the homogeneous Klein–Gordon equation at any point P in Ω (since all the source points Q_j^* are outside Ω).

In the following, we will explore the algorithm for the method of images in detail through a series of examples. The experience we gained in dealing with this method, earlier in Chapter 2 will be especially helpful to us.

Example 3.1. We consider the trivial case of the Dirichlet problem for the Klein–Gordon equation on the upper half-plane $\Omega(x, y) = \{-\infty < x < \infty, y > 0\}$, and will construct its Green's function.

The field generated by a unit source at a point $Q(\xi, \eta) \in \Omega$ represents the singular component of the Green's function

$$\frac{1}{2\pi} K_0(k\sqrt{(x-\xi)^2 + (y-\eta)^2})$$

The above can be canceled out by a single unit sink placed at $Q^*(\xi, -\eta)$ located at the lower half-plane and symmetric with respect to $Q(\xi, \eta)$ about the boundary $y = 0$. With the field generated by this sink

$$-\frac{1}{2\pi} K_0(k\sqrt{(x-\xi)^2 + (y+\eta)^2}),$$

the Green's function for the Dirichlet problem on the upper half-plane is found to be

$$G(x, y; \xi, \eta) = \frac{1}{2\pi}[K_0(k|z-\zeta|) - K_0(k|z-\overline{\zeta}|)], \tag{3.7}$$

where the complex variable notation

$$z = x + iy \quad \text{and} \quad \zeta = \xi + i\eta$$

is used for compactness, to denote the observation and the source point, respectively.

Note that we will refer to the relevant illustrations from Chapter 2 in the derivation of Green's functions in this section.

Example 3.2. For the next example, we consider another trivial case of the Dirichlet problem for the Klein–Gordon equation on the infinite circular sector $\Omega(r, \varphi) = \{(r, \varphi)|0 < r < \infty, 0 < \varphi < \pi/2\}$ with the angle $\pi/2$, which represents the quarter-plane.

Since the distance between two points (r_1, φ_1) and (r_2, φ_2) is defined in polar coordinates as

$$\sqrt{r_1^2 - 2r_1 r_2 \cos(\varphi_1 - \varphi_2) + r_2^2}$$

the singular component of the sought-after Green's function $G(r, \varphi; \varrho, \psi)$ reads

$$\frac{1}{2\pi} K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - \psi) + \varrho^2}) \tag{3.8}$$

representing the field strength at an observation point $M(r, \varphi) \in \Omega$ generated by the unit source acting at (ϱ, ψ). The latter is depicted in Figure 2.1 of Chapter 2 with the plus sign symbol.

In order to compensate the trace of the function in (3.8) (in other words, to satisfy the Dirichlet condition) on the boundary segment $y = 0$, we place the unit sink (labeled with the minus sign in Figure 2.1) at $D(\varrho, 2\pi - \psi)$. The field generated by this sink is given by

$$-\frac{1}{2\pi}K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - (2\pi - \psi)) + \varrho^2}). \tag{3.9}$$

Similarly, we compensate the trace of (3.8) on the boundary segment $x = 0$ with the unit sink at $B(\varrho, \pi - \psi)$, which generates the field

$$-\frac{1}{2\pi}K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - (\pi - \psi)) + \varrho^2}), \tag{3.10}$$

whilst to compensate the traces of the functions in (3.9) and (3.10) on $x = 0$ and $y = 0$, respectively, we require a unit source at $C(\varrho, \pi + \psi)$, with the field

$$\frac{1}{2\pi}K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - (\pi + \psi)) + \varrho^2}). \tag{3.11}$$

Hence, the Green's function for the Dirichlet problem on the infinite circular sector with angle $\pi/2$ represents the sum of the components shown in (3.8), (3.9), (3.10), and (3.11), which converts, after a trivial transformation, to the compact formula

$$G(r, \varphi; \varrho, \psi) = \frac{1}{2\pi}\sum_{n=1}^{2}[K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - ((n-1)\pi + \psi)) + \varrho^2}) \\ - K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - (n\pi - \psi)) + \varrho^2})]. \tag{3.12}$$

The method of images enables us to construct the Green's function for a mixed boundary-value problem on the infinite circular sector $\Omega(r, \varphi) = \{(r, \varphi) | 0 < r < \infty, 0 < \varphi < \pi/2\}$. We refer the reader to the following example for more details.

Example 3.3. Consider a mixed boundary-value problem for the infinite circular sector with angle $\pi/2$, with Dirichlet and Neumann boundary conditions imposed on the boundary segments $y = 0$ and $x = 0$, respectively.

The boundary conditions can be satisfied by placing an appropriate set of sources and sinks to compensate traces of the fundamental solution

$$\frac{1}{2\pi}K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - \psi) + \varrho^2}) \tag{3.13}$$

of the Klein–Gordon equation. We now follow the scheme shown in Figure 2.2 of Chapter 2: in order to support the Dirichlet condition on the boundary segment $y = 0$, we compensate the trace of the function in (3.13) with the unit sink (labeled with the minus sign in Figure 2.2) at $D(\varrho, 2\pi - \psi)$. The field generated by this sink is given by

$$-\frac{1}{2\pi}K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - (2\pi - \psi)) + \varrho^2}). \tag{3.14}$$

The Neumann condition on the boundary segment $x = 0$ will be satisfied if the unit source is placed at $B(\varrho, \pi - \psi)$, generating a field defined as

$$\frac{1}{2\pi} K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - (\pi - \psi)) + \varrho^2}), \tag{3.15}$$

whilst to satisfy both of the boundary conditions imposed on $x = 0$ and $y = 0$, a unit sink is required at $C(\varrho, \pi + \psi)$, generating the field

$$-\frac{1}{2\pi} K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - (\pi + \psi)) + \varrho^2}). \tag{3.16}$$

After summing the components in (3.13) through (3.16), we obtain the Green's function for the mixed problem under consideration, on the infinite circular sector $\{0 < r < \infty, 0 < \varphi < \pi/2\}$. After a trivial transformation, this converts to a compact formula as

$$G(r, \varphi; \varrho, \psi) = \frac{1}{2\pi} \sum_{n=1}^{2} [(-1)^n K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - (n\pi - \psi)) + \varrho^2})$$
$$+ (-1)^n K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - ((n-1)\pi + \psi)) + \varrho^2})]. \tag{3.17}$$

Analogous to the Laplace equation, the method of images turns out to be productive for several boundary-value problems for the Klein–Gordon equation on infinite circular sectors. For several others, it is, however, not productive. In a series of examples that follow, we will find both successful as well as not so successful cases of application of the method.

Example 3.4. Consider the Dirichlet problem for the Klein–Gordon equation on the infinite circular sector $\Omega(r, \varphi) = \{(r, \varphi) | 0 < r < \infty, 0 < \varphi < \pi/3\}$ with angle $\pi/3$.

To construct the Green's function, we refer the reader to follow our procedure by studying and utilizing the scheme depicted in Figure 2.3 of Chapter 2. In order to cancel out the influence of the singular component of the Green's function

$$\frac{1}{2\pi} K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - \psi) + \varrho^2})$$

on the boundary fragment $\varphi = 0$, we place a compensatory unit sink at $F(\varrho, 2\pi - \psi)$, whilst another unit sink is required at $B(\varrho, 2\pi/3 - \psi)$ to satisfy the Dirichlet condition on $\varphi = \pi/3$. To compensate the trace of the latter sink on the boundary fragment $\varphi = 0$, a unit source is required at $E(\varrho, 4\pi/3 + \psi)$. The trace of the latter source is compensated on $\varphi = \pi/3$ with a unit sink at $D(\varrho, 4\pi/3 - \psi)$, while the trace of this sink is compensated on $\varphi = 0$ with a unit source placed at $C(\varrho, 2\pi/3 + \psi)$.

Thus, the regular component $R(r,\varphi;\varrho,\psi)$ of the sought-after Green's function can be formed as the aggregate of the five compensatory sources and sinks located outside Ω, in the manner exposed in Figure 2.3. This implies that the Green's function $G(r,\varphi;\varrho,\psi)$ itself is obtained by adding the singular component to $R(r,\varphi;\varrho,\psi)$, which yields

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\sum_{n=1}^{3}\left[K_0\left(k\sqrt{r^2 - 2r\varrho\cos\left(\varphi - \left(2(n-1)\frac{\pi}{3} + \psi\right)\right) + \varrho^2}\right)\right.$$
$$\left. - K_0\left(k\sqrt{r^2 - 2r\varrho\cos\left(\varphi - \left(2n\frac{\pi}{3} - \psi\right)\right) + \varrho^2}\right)\right]. \tag{3.18}$$

Recall Example 3.3, where we successfully constructed the Green's function for the mixed (Dirichlet–Neumann) boundary-value problem on the infinite circular sector with angle $\pi/2$. in contrast to that problem, the method of images fails for the same type of mixed problem on another infinite circular sector. We will justify this assertion in the next example.

Example 3.5. When trying to apply the method of images, we surprisingly find it fails in the case of the Dirichlet–Neumann problem for the infinite circular sector with angle $\pi/3$. To follow our procedure in detail, we refer the reader to the scheme depicted in Figure 2.4.

Clearly, the Dirichlet condition on $\varphi = 0$ is satisfied by a unit sink placed at $F(\varrho, 2\pi-\psi)$. To allow this sink to satisfy the Neumann condition on $\varphi = \pi/3$, a unit sink is also required at $C(\varrho, 2\pi/3+\psi)$. As to the Neumann condition on $\varphi = \pi/3$, the unit source at $A(\varrho,\psi)$ must be augmented by the unit source at $B(\varrho, 2\pi/3-\psi)$, which, in turn, must be augmented by the unit sink placed at $E(\varrho, 4\pi/3+\psi)$. The latter sink has to be paired with the unit sink at $D(\varrho, 4\pi/3-\psi)$ to satisfy the Neumann condition on $\varphi = \pi/3$. Inspecting the two sinks in $C(\varrho, 2\pi/3+\psi)$ and $D(\varrho, 4\pi/3-\psi)$, we note that they don't support the Dirichlet condition on the boundary fragment $\varphi = 0$. This is what indicates the method's failure for the mixed problem under consideration.

Although, as Example 3.5 illustrates, the method of images fails for several problem settings on an infinite circular sector, the range of problems for which the method proves to be efficient is not limited to the ones we have considered so far. To support this point, we provide the following examples.

Example 3.6. Consider the Dirichlet problem on the infinite circular sector $\Omega(r,\varphi) = \{(r,\varphi)|0 < r < \infty, 0 < \varphi < \pi/4\}$ with angle $\pi/4$.

Our experience from the previous examples enables the reader to obtain the Green's function for our problem, by following the procedure in detail. With the aid of the scheme depicted in Figure 2.5 of Chapter 2, we can express $G(r, \varphi; \varrho, \psi)$ as

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\sum_{n=1}^{4}\Bigg[K_0\Big(k\sqrt{r^2 - 2r\varrho\cos\Big(\varphi - \Big((n-1)\frac{\pi}{2} + \psi\Big)\Big) + \varrho^2}\Big) - K_0\Big(k\sqrt{r^2 - 2r\varrho\cos\Big(\varphi - \Big(n\frac{\pi}{2} - \psi\Big)\Big) + \varrho^2}\Big)\Bigg]. \tag{3.19}$$

Green's functions for other mixed boundary-values problem can also be constructed within the scope of the method of images.

Example 3.7. Consider the Dirichlet–Neumann problem for the infinite circular sector with angle $\pi/4$, with the Dirichlet and Neumann conditions imposed on the boundary segments $\varphi = 0$ and $\varphi = \pi/4$, respectively.

In order to outline the application of the method of images, we examine the scheme depicted in Figure 2.6. Combining the field generated by a total number of eight sources and sinks, we come up with the sought-after Green's function, expressed in the compact form

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\sum_{n=1}^{2}\Bigg[K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - ((n-1)\pi + \psi)) + \varrho^2}) + K_0\Big(k\sqrt{r^2 - 2r\varrho\cos\Big(\varphi - \Big((2n-1)\frac{\pi}{2} - \psi\Big)\Big) + \varrho^2}\Big) - K_0\Big(k\sqrt{r^2 - 2r\varrho\cos\Big(\varphi - \Big((2n-1)\frac{\pi}{2} + \psi\Big)\Big) + \varrho^2}\Big) - K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - (n\pi - \psi)) + \varrho^2})\Bigg]. \tag{3.20}$$

Example 3.8. As to the Dirichlet problem on the infinite circular sector with angle $\pi/6$, we proceed, following the method of images, and obtain the Green's function as an aggregate of the total number of twelve appropriately placed unit sources and unit sinks. Omitting the details, we present the ultimate form of the Green's function

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\sum_{n=1}^{6}\Bigg[K_0\Big(k\sqrt{r^2 - 2r\varrho\cos\Big(\varphi - \Big((n-1)\frac{\pi}{3} + \psi\Big)\Big) + \varrho^2}\Big) - K_0\Big(k\sqrt{r^2 - 2r\varrho\cos\Big(\varphi - \Big(n\frac{\pi}{3} - \psi\Big)\Big) + \varrho^2}\Big)\Bigg]. \tag{3.21}$$

We can make a generalization from the analysis of the forms of the Green's functions for the boundary-value problems on an infinite circular sector, derived thus far. The following two examples present compact expressions of Green's functions for Dirichlet problems on sets of infinite circular sectors.

Example 3.9. Observing the expression derived earlier for the Green's function of the Dirichlet problem on the half-plane, on a circular sector with angle $\pi/4$, and the one obtained for the circular sector with angle $\pi/2$, we arrive at the following generalization

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\sum_{n=1}^{2^m}\Big[K_0\Big(k\sqrt{r^2 - 2r\varrho\cos\Big(\varphi - \Big(2(n-1)\frac{\pi}{2^m} + \psi\Big)\Big) + \varrho^2}\Big) \\ - K_0\Big(k\sqrt{r^2 - 2r\varrho\cos\Big(\varphi - \Big(2n\frac{\pi}{2^m} - \psi\Big)\Big) + \varrho^2}\Big)\Big] \tag{3.22}$$

which represents the Green's function of the Dirichlet problem on the set of infinite circular sectors with angle $\pi/2^m$, where $m = 0, 1, 2, \ldots$.

Note that for $m = 0$, corresponding to the circular sector with angle π, in other words, the upper half-plane $y > 0$, we read from (3.22)

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}[K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2}) \\ - K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi+\psi) + \varrho^2})] \tag{3.23}$$

representing the Green's function derived in (3.7) and expressed here in polar coordinates. Clearly, the cases of the circular sectors with angle $\pi/2$, exhibited in (3.12) and with angle $\pi/4$ (see (3.19)) also follow from (3.22).

Example 3.10. We can make another significant generalization from the examples treated so far: upon analyzing the expressions in (3.18) and (3.21), representing the Klein–Gordon equation's Green's functions for Dirichlet problems on infinite circular sectors with angles $\pi/3$ and $\pi/6$, we derive the Green's function for the Dirichlet problem on the set of circular sectors with angle $\pi/(3\cdot 2^m)$, where $m = 0, 1, 2, \ldots,$ in the form

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\sum_{n=1}^{3\cdot 2^m}\Big[K_0\Big(k\sqrt{r^2 - 2r\varrho\cos\Big(\varphi - \Big(2(n-1)\frac{\pi}{3\cdot 2^m} + \psi\Big)\Big) + \varrho^2}\Big) \\ - K_0\Big(k\sqrt{r^2 - 2r\varrho\cos\Big(\varphi - \Big(\frac{2n\pi}{3\cdot 2^m} - \psi\Big)\Big) + \varrho^2}\Big)\Big], \tag{3.24}$$

where $m = 0$ represents the sector with angle $\pi/3$, while the $m = 1$ represents the sector with angle $\pi/6$.

In discussing the success or failure of the method of images in constructing Green's functions for the Klein–Gordon equation on infinite circular sectors, recall that earlier in Example 3.5 we have shown that the method fails for a mixed (Dirichlet–Neumann) boundary-value problem on a sector with angle $\pi/3$. However, it is worth noting that the method is not necessarily successful for Dirichlet problems either and could fail for some of them. We illustrate this point in the following example.

Example 3.11. Consider the Dirichlet problem on the infinite circular sector $\Omega(r,\varphi) = \{(r,\varphi)|0 < r < \infty, 0 < \varphi < 2\pi/3\}$ and attempt to construct its Green's function.

Analogously to the same problem for the Laplace equation, the failure of the method can be understood with the aid of the scheme shown in Figure 2.7 of Chapter 2. Let the unit source (producing the singular component of the Green's function) be located at $A(\varrho,\psi) \in \Omega$. To compensate its trace on the segment $\varphi = 0$ of the boundary Ω, we place the compensatory sink at $D(\varrho, 2\pi - \psi) \notin \Omega$. The trace of the latter on the boundary segment $\varphi = 2\pi/3$ is in turn compensated by a unit source at $C(\varrho, 4\pi/3 + \psi) \notin \Omega$, the trace of which on $\varphi = 0$ must be compensated by a unit sink at $B(\varrho, 2\pi/3 - \psi)$, which is located *inside* Ω. This is what leads to the failure of the method.

We have just illustrated the fact that the method of images could potentially fail in constructing the Green's function for the Dirichlet problem on an infinite circular sector allowing use of cyclic symmetry. To examine other cases of the failure of the method, the reader is recommended to refer to the Chapter Exercises, and apply the method to Dirichlet problems stated on other infinite circular sectors (of $2\pi/5$ or $2\pi/7$, for example) also allowing the cyclic symmetry.

Note that all the Green's functions of the Klein–Gordon equation, constructed in this section so far, are expressed in terms of a finite sum of the Macdonald functions. In the following examples, we focus on several other problem settings for which the method of images is applicable, but where we obtain Green's functions expressed in terms of infinite series of the Macdonald functions.

Example 3.12. We begin with one of the classical settings, namely, the Dirichlet problem for the Klein–Gordon equation on the infinite strip $\Omega(x,y) = \{(x,y)|-\infty < x < \infty, 0 < y < b\}$.

We illustrate the application of the method of images with the aid of the scheme depicted in Figure 2.9, where we place a unit source ξ_0^+ at an arbitrary point $A(\xi,\eta)$ inside Ω. The value of the field generated by ξ_0^+ at an observation point $M(x,y)$ represents the fundamental solution

$$G_0^+(x,y;\xi,\eta) = \frac{1}{2\pi} K_0(\sqrt{(x-\xi)^2 + (y-\eta)^2})$$

of the Klein–Gordon equation.

It is evident that $G_0^+(x, y; \xi, \eta)$ does not satisfy the Dirichlet conditions on the boundary fragments $y = 0$ and $y = b$ (it does not vanish on those lines). To compensate the traces of $G_0^+(x, y; \xi, \eta)$ on $y = 0$ and $y = b$, we place two unit sinks $\xi_{1,0}^-$ and $\xi_{1,b}^-$ at the points $B(\xi, -\eta)$ and $C(\xi, 2b - \eta)$, representing the images of (ξ, η) about the lines $y = 0$ and $y = b$, respectively. The field strengths of these sinks, in (x, y) evidently are

$$G_{1,0}^-(x, y; \xi, -\eta) = -\frac{1}{2\pi} K_0(\sqrt{(x-\xi)^2 + (y+\eta)^2})$$

and

$$G_{1,b}^-(x, y; \xi, 2b - \eta) = -\frac{1}{2\pi} K_0(\sqrt{(x-\xi)^2 + (y-(2b-\eta))^2}).$$

The functions $G_{1,0}^-(x, y; \xi, -\eta)$ and $G_{1,b}^-(x, y; \xi, 2b - \eta)$ do not vanish on the boundary lines $y = 0$ and $y = b$; and their traces can, in turn, be compensated with the unit sources $\xi_{2,0}^+$ and $\xi_{2,b}^+$ located in $D(\xi, -2b + \eta)$ and $E(\xi, 2b + \eta)$. Their field strengths at (x, y) are given as

$$G_{2,0}^+(x, y; \xi, -2b + \eta) = \frac{1}{2\pi} K_0(\sqrt{(x-\xi)^2 + (y-(-2b+\eta))^2})$$

and

$$G_{2,b}^+(x, y; \xi, 2b + \eta) = \frac{1}{2\pi} K_0(\sqrt{(x-\xi)^2 + (y-(2b+\eta))^2}).$$

To properly compensate the traces of the functions $G_{2,0}^+(x, y; \xi, -2b + \eta)$ and $G_{2,b}^+(x, y; \xi, 2b + \eta)$ on $y = 0$ and $y = b$, we place unit sinks $\xi_{3,0}^-$ and $\xi_{3,b}^-$ at $F(\xi, -2b - \eta)$ and $H(\xi, 4b - \eta)$, respectively.

Upon following the described procedure of placing appropriate compensatory unit sources alternating with compensatory unit sinks, we express the sought-after Green's function $G = G(x, y; \xi, \eta)$ in the formula

$$G = G_0^+ + \sum_{i=1}^{\infty} (G_{2i-1,0}^- + G_{2i-1,b}^-) + \sum_{i=1}^{\infty} (G_{2i,0}^+ + G_{2i,b}^+)$$

which transforms ultimately into the single infinite series of the Macdonald functions

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} [K_0(\sqrt{(x-\xi)^2 + (y-\eta+2nb)^2}) - K_0(\sqrt{(x-\xi)^2 + (y+\eta-2nb)^2})]. \quad (3.25)$$

Since this Green's function is obtained in series form, the convergence of the latter has to be specifically addressed, in order to ensure its applicability for practical computations. First, note that the singularity of $G(x, y; \xi, \eta)$ in (3.25) is provided by the component $K_0(\sqrt{(x-\xi)^2 + (y-\eta)^2})$, which is part of a single term ($n = 0$), and cannot, in any way, affect the convergence of the series.

From close analysis we learn that the series representation in (3.25) converges at a high rate and is therefore suitable for immediate computer implementations. This assertion can be supported by noting that

$$\lim_{n\to\infty} \frac{\sqrt{(x-\xi)^2 + (y \pm \eta \pm 2nb)^2}}{2nb} = 1$$

which allows us to assert that the arguments of the Macdonald functions in (3.25) are asymptotically close to $2nb$. This implies that, when n increases, the terms of the series in (3.25) converge to zero at the same rate as terms of the sequence $\{K_0(2nb)\}$. It is well known [1, 27, 37] that the Macdonald function $K_0(x)$ is continuous, positive, decreasing (at a very high rate) and bounded by zero. For a strip of unit width ($b = 1$), for example, approximate values of the first few terms of the sequence $\{K_0(2nb)\}$ are

$$K_0(2) = 0.11389, \quad K_0(4) = 0.01116, \quad K_0(6) = 0.00124, \quad K_0(8) = 0.00015$$

which approaches zero at a rate close to that of a geometric sequence with a ratio of the order of 10^{-1} and which therefore rapidly converges to zero.

One might call into question the rigor of the brief analysis that we just completed. However, it is hard to deny that our conclusions are sufficient to assert that the series in (3.25) converges at a high rate. With this, it is safe to use this formula for computer implementations.

Example 3.13. We now consider a mixed problem for the Klein–Gordon equation on the infinite strip $\Omega = \{(x, y) | -\infty < x < \infty, 0 < y < b\}$, with the Dirichlet condition imposed on $y = 0$ and the Neumann condition imposed on $y = b$.

Similar to the treatment of the problem for the Laplace equation, considered in Chapter 2, the scheme depicted in Figure 2.10 helps us to grasp the procedure of the method of images when applied to the present problem.

The trace of the fundamental solution $G_0^+(x, y; \xi, \eta)$ on the boundary line $y = 0$, can be compensated with a unit sink $\xi_{1,0}^-$ placed at $B(\xi, -\eta)$, with at $M(x, y)$ generates a field given by

$$G_{1,0}^-(x, y; \xi, -\eta) = -\frac{1}{2\pi} K_0(\sqrt{(x-\xi)^2 + (y+\eta)^2}).$$

The Neumann condition imposed on $y = b$, can be satisfied by placing a unit source $\xi_{1,b}^+$ at the point $C(\xi, 2b - \eta)$, yielding

$$G_{1,b}^+(x, y; \xi, 2b - \eta) = \frac{1}{2\pi} K_0(\sqrt{(x-\xi)^2 + (y-(2b-\eta))^2}).$$

To cancel out the trace of the function $G^{+}_{1,b}(x, y; \xi, 2b - \eta)$ on the boundary line $y = 0$ a unit sink $\xi^{-}_{2,0}$ is placed at $D(\xi, -2b + \eta)$, which, in (x, y)

$$G^{-}_{2,0}(x, y; \xi, -2b + \eta) = -\frac{1}{2\pi} K_0(\sqrt{(x - \xi)^2 + (y + (2b - \eta))^2})$$

whilst the Neumann condition on $y = b$ can be satisfied by placing a unit sink $\xi^{-}_{2,b}$ at $E(\xi, 2b + \eta)$, which at (x, y) generates

$$G^{-}_{2,b}(x, y; \xi, 2b + \eta) = -\frac{1}{2\pi} K_0(\sqrt{(x - \xi)^2 + (y - (2b + \eta))^2}).$$

In turn, the trace of the function $G^{-}_{2,b}(x, y; \xi, 2b+\eta)$ on $y = 0$ can be compensated by a unit source $\xi^{+}_{3,0}$ placed at $F(\xi, -2b - \eta)$, which generates the field

$$G^{+}_{3,0}(x, y; \xi, -2b - \eta) = \frac{1}{2\pi} K_0(\sqrt{(x - \xi)^2 + (y + (2b + \eta))^2})$$

whilst the Neumann condition on $y = b$ can be satisfied by placing a unit sink $\xi^{-}_{3,b}$ at $H(\xi, 4b - \eta)$, with field strength at (x, y) given as

$$G^{-}_{3,b}(x, y; \xi, 4b - \eta) = -\frac{1}{2\pi} K_0(\sqrt{(x - \xi)^2 + (y - (4b - \eta))^2}).$$

Further following the approach described in Example 3.12, the sought-after Green's function is obtained in the following series form

$$\begin{aligned} G(x, y; \xi, \eta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \big[& K_0(\sqrt{(x - \xi)^2 + (y - \eta + 4nb)^2}) \\ & - K_0(\sqrt{(x - \xi)^2 + (y + \eta + 4nb)^2}) \\ & + K_0(\sqrt{(x - \xi)^2 + (y + \eta + 2(2n + 1)b)^2}) \\ & - K_0(\sqrt{(x - \xi)^2 + (y - \eta + 2(2n + 1)b)^2}) \big] \end{aligned} \quad (3.26)$$

which has a high rate of convergence, comparable to that of the series in (3.25). This makes equation (3.26) suitable for immediate computer implementations.

We now turn to several other boundary-value problems for the Klein–Gordon equation and apply the the method of images to obtain their Green's functions.

Example 3.14. Consider first the Dirichlet problem on the semi-infinite strip $\Omega = \{(x, y)|0 < x < \infty, 0 < y < b\}$.

We here follow the derivation scheme depicted in Figure 2.11 of Chapter 2 in sketching our procedure, similar to the one described earlier in detail in Examples 3.12 and 3.13.

The fundamental solution of the Klein–Gordon equation, which represents the field generated by the unit source acting at an arbitrary point $A(\xi,\eta)$ in Ω, can be compensated on the edges $y=0$ and $y=b$ by a set of unit sources and sinks placed at the regular set of points $B(\xi,-\eta)$, $C(\xi,2b-\eta)$, $D(\xi,-2b+\eta)$, $E(\xi,2b+\eta)$, $F(\xi,-2b-\eta)$, $H(\xi,4b-\eta),\ldots$ located outside of Ω. In other words, these sources and sinks allow us to satisfy the homogeneous Dirichlet boundary conditions imposed on the edges $y=0$ and $y=b$ of Ω.

To satisfy the boundary condition imposed on the edge $x=0$, the field generated by the sources and sinks acting at $A,B,C,D,E,F,H,\ldots$ can, in turn, be compensated with unit sources and sinks on the boundary line, placed in another regular set of points $K(-\xi,\eta)$, $L(-\xi,-\eta)$, $N(-\xi,2b-\eta)$, $P(-\xi,-2b+\eta)$, $R(-\xi,2b+\eta)$, $S(-\xi,-2b-\eta)$, $T(-\xi,4b-\eta),\ldots$ located out of Ω. It is evident that the latter sources and sinks do not conflict with the boundary conditions on $y=0$ and $y=b$.

Upon combining the field strength of all the compensatory sources and sinks in the scheme shown in Figure 2.11, we arrive at the series form

$$G(x,y;\xi,\eta)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\Big[K_0\big(\sqrt{(x-\xi)^2+(y-\eta+2nb)^2}\big)\\ -K_0\big(\sqrt{(x-\xi)^2+(y+\eta-2nb)^2}\big)\\ +K_0\big(\sqrt{(x+\xi)^2+(y+\eta-2nb)^2}\big)\\ -K_0\big(\sqrt{(x+\xi)^2+(y-\eta+2nb)^2}\big)\Big] \tag{3.27}$$

of the Green's function for the Dirichlet problem for the Klein–Gordon equation on the semi-infinite strip $\Omega=\{(x,y)|0<x<\infty, 0<y<b\}$. It is evident that the rate of convergence of the series in the above representation is comparable with that the formulas in (3.25) and (3.26).

Example 3.15. In another example on the semi-infinite strip $\Omega=\{(x,y)|0<x<\infty, 0<y<b\}$, we consider a mixed boundary-value problem, namely, that of Dirichlet conditions imposed on the boundary fragments $y=0$ and $y=b$, and the Neumann condition imposed on $x=0$.

The Green's function for this problem can be derived with the aid of the scheme depicted in Figure 2.12 of Chapter 2.

Similarly to the Dirichlet problem that we considered in the previous example, the traces of the fundamental solution of the Klein–Gordon equation (the field generated by the unit source acting at an arbitrary point $A(\xi,\eta)$ in Ω) on the edges $y=0$ and $y=b$ are compensated with unit sources and sinks placed at the set of points $B(\xi,-\eta)$, $C(\xi,2b-\eta)$, $D(\xi,-2b+\eta)$, $E(\xi,2b+\eta)$, $F(\xi,-2b-\eta)$, $H(\xi,4b-\eta),\ldots,$ exterior to Ω.

The Neumann condition that is imposed on the boundary fragment $x = 0$ can be satisfied with the aggregate influence of the sources and sinks acting at $A, B, C, D, E, F, H, \ldots$. The latter can be compensated, similar to the Dirichlet problem, with unit sources and sinks placed at the set of points $K(-\xi, \eta)$, $L(-\xi, -\eta)$, $N(-\xi, 2b - \eta)$, $P(-\xi, -2b + \eta)$, $R(-\xi, 2b + \eta)$, $S(-\xi, -2b - \eta)$, $T(-\xi, 4b - \eta), \ldots$, exterior to Ω. However, in this case the order of sources and sinks is different from the one suggested earlier for the Dirichlet problem.

Continuing our procedure, after combining the field strength of all sources and sinks, we arrive at the rapidly converging series

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \Big[K_0\Big(\sqrt{(x-\xi)^2 + (y - \eta + 2nb)^2}\Big) \\ - K_0\Big(\sqrt{(x-\xi)^2 + (y + \eta - 2nb)^2}\Big) \\ + K_0\Big(\sqrt{(x+\xi)^2 + (y - \eta + 2nb)^2}\Big) \\ - K_0\Big(\sqrt{(x+\xi)^2 + (y + \eta - 2nb)^2}\Big) \Big] \tag{3.28}$$

for the Green's function for this mixed problem.

3.3 Method of Eigenfunction Expansion

Reviewing the boundary-value problems discussed in the previous section, the reader might notice that the method of images turns out to only be successful for settings with the Dirichlet and Neumann conditions imposed. With this in mind, we now turn our attention to another standard method which, similar to the Laplace equation, also turns out to be productive for the static Klein–Gordon equation in two dimensions, and whose application range is wider than that for the method of images. In this section, we will construct a number of Green's using the method of eigenfunction expansion.

Example 3.16. We start with one of the trivial problems already reviewed in Section 3.2, and will obtain an alternative formula for the Green's function for the Dirichlet problem

$$(\nabla^2 - k^2) u(x, y) = -f(x, y), (x, y) \in \Omega, \tag{3.29}$$

$$u(x, 0) = u(x, b) = 0 \tag{3.30}$$

on the infinite strip $\Omega = \{(x, y) | -\infty < x < \infty, 0 < y < b\}$ with width b. In addition, we require the solution of the problem defined by (3.29) and (3.30), to remain bounded as $x \to \pm\infty$. Further, assume the right-hand side term $f(x, y)$ in

(3.29) to be a sufficiently well-behaved function over Ω in the sense that

$$\left| \iint_{\Omega} f(x, y) d\Omega(x, y) \right| < \infty.$$

Expand the solution $u(x, y)$ of the problem in (3.29) and (3.30), and the right-hand side function $f(x, y)$ of (3.29) into the following Fourier sine-series

$$u(x, y) = \sum_{n=1}^{\infty} u_n(x) \sin \nu y, \quad \nu = \frac{n\pi}{b}, \tag{3.31}$$

and

$$f(x, y) = \sum_{n=1}^{\infty} f_n(x) \sin \nu y. \tag{3.32}$$

This yields the set of self-adjoint boundary-value problems

$$\frac{d^2 u_n(x)}{dx^2} - (\nu^2 + k^2) u_n(x) = -f_n(x), \quad n = 1, 2, 3, \ldots, \tag{3.33}$$

$$|u_n(-\infty)| < \infty, \quad |u_n(\infty)| < \infty \tag{3.34}$$

for the coefficients $u_n(x)$ of the series in (3.31).

Clearly, a fundamental set of solutions of (3.33) can be represented by the functions

$$\exp(\sqrt{\nu^2 + k^2} x) \quad \text{and} \quad \exp(-\sqrt{\nu^2 + k^2} x)$$

and the Green's function $g_n(x, \xi)$ of the homogeneous problem corresponding to (3.33) and (3.34) can be constructed by using either the method based on its defining properties (see Section 1.1) or the method of variation of parameters (see Section 1.3). This yields

$$g_n(x, \xi) = \frac{1}{2\sqrt{\nu^2 + k^2}} \begin{cases} \exp(\sqrt{\nu^2 + k^2}(x - \xi)), & \text{for } x \le \xi, \\ \exp(\sqrt{\nu^2 + k^2}(\xi - x)), & \text{for } \xi \le x, \end{cases} \tag{3.35}$$

and the series representation

$$G(x, y; \xi, \eta) = \frac{2}{b} \sum_{n=1}^{\infty} g_n(x, \xi) \sin \nu y \sin \nu \eta \tag{3.36}$$

of the Green's function to the homogeneous boundary-value problem corresponding to (3.29) and (3.30) can be obtained by following the scheme developed in Chapter 2 for the Laplace equation.

It is evident that the non-uniform convergence of the series in (3.36) is preconditioned by the logarithmic singularity of the Green's function. This significantly constrains the effectiveness of direct computational implementations of (3.36). However, the situation can be radically improved by rewriting the branch of the function $g_n(x,\xi)$, valid for $x \le \xi$ (the other branch can also be used instead), in the form

$$g_n(x,\xi) = \left[\frac{\exp(h(x-\xi))}{2h} - \frac{\exp(\nu(x-\xi))}{2\nu}\right] + \frac{\exp(\nu(x-\xi))}{2\nu},$$

where we introduced $h = \sqrt{\nu^2 + k^2}$ for compactness. With this, the expression in (3.36) transforms into

$$G(x,y;\xi,\eta) = \frac{1}{b}\sum_{n=1}^{\infty}\left[\frac{\exp(h(x-\xi))}{h} - \frac{\exp(\nu(x-\xi))}{\nu}\right]\sin\nu y \sin\nu\eta$$
$$+ \frac{1}{b}\sum_{n=1}^{\infty}\frac{\exp(\nu(x-\xi))}{\nu}\sin\nu y\sin\nu\eta \tag{3.37}$$

with the first series uniformly convergent and its numerical implementations posing no problems. The second series in (3.37) does not converge uniformly and is, in fact, responsible for the logarithmic singularity of the Green's function. The good news is that it allows us a complete summation, possible with the aid of the standard summation formula which can be found in (2.62) of Chapter 2. After some elementary algebra, we get

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\frac{E(z-\overline{\zeta})}{E(z-\zeta)} - \frac{1}{b}\sum_{n=1}^{\infty} H_n(x,\xi)\sin\nu y\sin\nu\eta \tag{3.38}$$

with z and ζ being the observation and the source point, respectively; the real-valued function $E(w)$ of a complex variable w is introduced as

$$E(w) = \left|1 - \exp\left(\frac{\pi}{b}w\right)\right|$$

with $H_n(x,\xi)$ in (3.38) given by

$$H_n(x,\xi) = \frac{1}{\nu h}[h\exp(\nu(x-\xi)) - \nu\exp(h(x-\xi))] \quad \text{for } x \le \xi.$$

Note that the expression for $H_n(x,\xi)$ for $x \ge \xi$ can be obtained from the above by exchanging x and ξ. Also note, that $H_n(x,\xi)$ vanishes for $k = 0$ and (3.38) reduces to the well-known closed form

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\frac{E(z-\overline{\zeta})}{E(z-\zeta)}$$

of the Green's function for the Dirichlet problem for the Laplace equation on the infinite strip (see, for example, [45, 64]).

We now turn to the trigonometric series in (3.38). To explore its convergence, we analyze the modulus of its remainder $R_N(x, y; \xi, \eta)$

$$\Big| \sum_{n=N+1}^{\infty} \frac{1}{\nu h}[h \exp(\nu(x-\xi)) - \nu \exp(h(x-\xi))] \sin(\nu y) \sin(\nu \eta)\Big|.$$

Upon ignoring the trigonometric factors, we obtain

$$|R_N(x, y; \xi, \eta)| \leq \sum_{n=N+1}^{\infty} \left| \frac{1}{\nu h}[h \exp(\nu(x-\xi)) - \nu \exp(h(x-\xi))] \right|.$$

Clearly, since $k^2 > 0$ and $x \leq \xi$, the parameter h exceeds ν and, consequently, the first term in the brackets always exceeds the second, which is why we can omit the modulus sign on the right-hand side of the above inequality, yielding

$$|R_N(x, y; \xi, \eta)| \leq \sum_{n=N+1}^{\infty} \frac{1}{\nu h}[h \exp(\nu(x-\xi)) - \nu \exp(h(x-\xi))].$$

Taking into account that $\nu < h$, from the above estimate follows:

$$|R_N(x, y; \xi, \eta)| < \sum_{n=N+1}^{\infty} \frac{1}{\nu^2}[h \exp(\nu(x-\xi)) - \nu \exp(h(x-\xi))]. \tag{3.39}$$

We can develop this further by using the relation

$$h = \sqrt{\nu^2 + k^2} < \nu + k$$

which may be regarded as the *triangle inequality* for h, ν, and k. With this, we enforce the inequality in (3.39) by increasing the first term in the brackets (based on the fact that $\nu + k > h$) while decreasing the second term (because $x \leq \xi$ and $\nu + k > h$). That is

$$|R_N(x, y; \xi, \eta)| < \sum_{n=N+1}^{\infty} \frac{1}{\nu^2}[(\nu + k) \exp(\nu(x-\xi)) - \nu \exp((\nu + k)(x-\xi))].$$

Some elementary algebra yields

$$\begin{aligned}
&|R_N(x, y; \xi, \eta)| \\
&\quad < \sum_{n=N+1}^{\infty} \frac{1}{\nu^2}\{[1 - \exp(k(x-\xi))]\nu \exp(\nu(x-\xi)) + k \exp(\nu(x-\xi))\} \\
&\quad = [1 - \exp(k(x-\xi))] \sum_{n=N+1}^{\infty} \frac{1}{\nu} \exp(\nu(x-\xi)) + k \sum_{n=N+1}^{\infty} \frac{1}{\nu^2} \exp(\nu(x-\xi)) \\
&\quad \leq [1 - \exp(k(x-\xi))] \sum_{n=N+1}^{\infty} \frac{1}{\nu} \exp(\nu(x-\xi)) + k \sum_{n=N+1}^{\infty} \frac{1}{\nu^2}.
\end{aligned}$$

Now, after recalling the value of ν expressed in terms of n and rewriting the remainders of the above two series in an explicit form, the above inequality transforms as

$$|R_N(x,y;\xi,\eta)| < \frac{b}{\pi}[1-\exp(k(x-\xi))] \times \left[\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(\frac{n\pi}{b}(x-\xi)\right) - \sum_{n=1}^{N}\frac{1}{n}\exp\left(\frac{n\pi}{b}(x-\xi)\right)\right] + k\frac{b^2}{\pi^2}\left[\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{N}\frac{1}{n^2}\right]. \quad (3.40)$$

With regard to the two infinite series in (3.40), we note that the second one represents the p-series the sum of which is [1, 27, 37]

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

The sum of the first series in (3.40) can be found by taking a convergent geometric series and integrate it term-by-term: consider the series

$$\sum_{n=1}^{\infty}\exp np = \frac{\exp p}{1-\exp p}, \quad p<0,$$

and integrate it, along with its sum, yielding

$$\sum_{n=1}^{\infty}\frac{1}{n}\exp np = -\ln(1-\exp p)$$

which translates, in our terms, into

$$\sum_{n=1}^{\infty}\frac{1}{n}\exp n\left(\frac{\pi}{b}(x-\xi)\right) = -\ln\left(1-\exp\left(\frac{\pi}{b}(x-\xi)\right)\right).$$

This finally yields

$$|R_N(x,y;\xi,\eta)| < \frac{b}{\pi}\left\{[1-\exp(k(x-\xi))]\left[\ln\left(1-\exp\left(\frac{\pi}{b}(x-\xi)\right)\right) - \sum_{n=1}^{N}\frac{1}{n}\exp\left(\frac{n\pi}{b}(x-\xi)\right)\right] + k\frac{b}{\pi}\left[\frac{\pi^2}{6} - \sum_{n=1}^{N}\frac{1}{n^2}\right]\right\} \quad (3.41)$$

for the estimate in (3.40).

Hence, the series in (3.38) converges uniformly and we can accomplish an accurate assessment by direct truncation. Based on the estimate derived in (3.41), we can compute the value of the truncating parameter N required to attain the desired level of accuracy.

Example 3.17. We now set up the mixed boundary-value problem

$$u(x,0) = \frac{\partial u(x,b)}{\partial y} = 0 \tag{3.42}$$

for the inhomogeneous Klein–Gordon equation in (3.29) on the infinite strip $\Omega = \{(x,y) | -\infty < x < \infty, 0 < y < b\}$.

In this case, the solution $u(x,y)$ as well as the right-hand side $f(x,y)$ in (3.29) must be expressed by the following Fourier sine-series

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x) \sin \nu y, \quad \nu = \frac{(2n-1)\pi}{2b} \tag{3.43}$$

and

$$f(x,y) = \sum_{n=1}^{\infty} f_n(x) \sin \nu y.$$

It is evident that such a formula for $u(x,y)$ satisfies the boundary conditions in (3.42).

Continuing with the algorithm used in the previous example, we now arrive at the boundary-value problem in (3.33) and (3.34) for the coefficient $u_n(x)$ of the series in (3.43), whose Green's function was presented earlier, in (3.35). This leads to the Green's function for (3.29) and (3.42), also obtained in series form in (3.36), which in the current setting converts into the quite compact and computer-friendly form

$$\begin{aligned} G(x,y;\xi,\eta) = {} & \frac{1}{2\pi} \ln \frac{E_1(z-\zeta)E_2(z-\overline{\zeta})}{E_2(z-\zeta)E_1(z-\overline{\zeta})} \\ & - \frac{1}{b} \sum_{n=1}^{\infty} \left[\frac{\exp(\nu(x-\xi))}{\nu} - \frac{\exp(h(x-\xi))}{h} \right] \sin \nu y \sin \nu \eta, \end{aligned} \tag{3.44}$$

where the real-valued functions $E_1(w)$ and $E_2(w)$ of a complex variable w are defined as

$$E_1(w) = \left| \exp\left(\frac{\pi w}{2b}\right) + 1 \right|, \quad E_2(w) = \left| \exp\left(\frac{\pi w}{2b}\right) - 1 \right|.$$

Note that the coefficient

$$\frac{\exp(\nu(x-\xi))}{\nu} - \frac{\exp(h(x-\xi))}{h}$$

of the series in (3.44) is valid for $x \leq \xi$, whereas its expression for $x \geq \xi$ can be obtained from the above by exchanging x and ξ.

The series in (3.44) converges uniformly on Ω. This can be verified in the exact same fashion as for the Dirichlet problem that we discussed earlier, and the estimate of the series remainder can be readily obtained.

In following examples, we consider two different boundary-value problems for the Klein–Gordon equation on a semi-infinite strip.

Example 3.18. Omitting the details of the lengthy but straightforward procedure we present, in this example, an expression for the Green's function of the Dirichlet problem

$$u(0, y) = u(x, 0) = u(x, b) = 0$$

on the semi-infinite strip $\Omega = \{(x, y)|0 < x < \infty, 0 < y < b\}$, found to be

$$\begin{aligned} G(x, y; \xi, \eta) = & \frac{1}{2\pi} \ln \frac{E(z - \overline{\zeta})E(z + \overline{\zeta})}{E(z - \zeta)E(z + \zeta)} \\ & - \frac{2}{b} \sum_{n=1}^{\infty} \left(\frac{\sinh \nu x}{\nu \exp \nu\xi} - \frac{\sinh hx}{h \exp h\xi} \right) \sin \nu y \sin \nu\eta. \end{aligned} \tag{3.45}$$

The coefficient of the above series is valid for $x \leq \xi$, whereas its expression for $x \geq \xi$ can be obtained from the above by exchanging x and ξ. The series has a high rate of uniform convergence, and its remainder can be readily estimated.

Example 3.19. For our second example for the semi-infinite strip $\Omega = \{(x, y)|0 < x < \infty, 0 < y < b\}$, we set up the mixed boundary-value problem

$$u(x, 0) = \frac{\partial u(x, b)}{\partial y} = \frac{\partial u(0, y)}{\partial x} - \beta u(0, y) = 0, \quad \beta \geq 0. \tag{3.46}$$

The remarkable feature is that three different kinds of boundary conditions are imposed on different fragments of the contour of Ω.

Following along with our approach, the Green's function for (3.46) is ultimately obtained in the form

$$\begin{aligned} G(x, y; \xi, \eta) = & \frac{1}{2\pi} \ln \frac{E_1(z - \zeta)E_2(z + \zeta)E_1(z + \overline{\zeta})E_2(z - \overline{\zeta})}{E_1(z + \zeta)E_2(z - \zeta)E_1(z - \overline{\zeta})E_2(z + \overline{\zeta})} \\ & - \frac{2}{b} \sum_{n=1}^{\infty} \left(\frac{\cosh \nu x}{\nu \exp \nu\xi} - \frac{\cosh hx}{h \exp h\xi} \right) \sin \nu y \sin \nu\eta \\ & - \frac{2\beta}{b} \sum_{n=1}^{\infty} \frac{\exp(-h(x + \xi))}{h(h + \beta)} \sin \nu y \sin \nu\eta, \quad x \leq \xi, \end{aligned} \tag{3.47}$$

where

$$h = \sqrt{\nu^2 + k^2}, \quad \nu = (2n-1)\frac{\pi}{2b}. \tag{3.48}$$

The coefficient of the first of the two series in (3.47) is valid for $x \leq \xi$, while for $x \geq \xi$ it can be obtained by exchanging x and ξ. Clearly, the coefficient of the second series in (3.47) is invariant under exchanging the variables.

Notice that both series in (3.47) are uniformly convergent, and one can readily estimate the remainder of the first one by using the approach already implemented when we analyzed the series (3.38). To estimate the remainder of the second series in (3.47), $R_N(x, y; \xi, \eta)$, we proceed as follows

$$\begin{aligned} |R_N(x, y; \xi, \eta)| &= \left| \sum_{n=N+1}^{\infty} \frac{\exp(-h(x+\xi))}{h(h+\beta)} \sin \nu y \sin \nu \eta \right| \\ &\leq \sum_{n=N+1}^{\infty} \left| \frac{\exp(-h(x+\xi))}{h(h+\beta)} \right| = \sum_{n=N+1}^{\infty} \frac{\exp(-h(x+\xi))}{h(h+\beta)} \\ &< \sum_{n=N+1}^{\infty} \frac{\exp(-\nu(x+\xi))}{\nu(\nu+\beta)}. \end{aligned} \tag{3.49}$$

In order to justify the last step in this estimation, note from (3.48) that h is greater than or equal to ν. Hence, by replacing h with ν in (3.49), we increase the numerator and, at the same time, decrease the denominator of the fraction. Hence, the inequality is satisfied.

Recalling the expression for ν in terms of n from (3.48), we rewrite the estimate in (3.49) in the explicit form

$$|R_N(x, y; \xi, \eta)| < \frac{4b^2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{\exp(-\frac{\pi(2n-1)}{2b}(x+\xi))}{(2n-1)[(2n-1)+\beta_0]},$$

where $\beta_0 = 2b\beta/\pi$. The above can be rewritten as

$$\begin{aligned} |R_N(x, y; \xi, \eta)| &< \frac{4b^2}{\pi^2} \sum_{n=N+1}^{\infty} \frac{[\exp(-\frac{\pi(x+\xi)}{2b})]^{2n-1}}{(2n-1)[(2n-1)+\beta_0]} \\ &= \frac{4b^2}{\pi^2 p} \sum_{n=N+1}^{\infty} \frac{p^{2n}}{(2n-1)2n}, \end{aligned} \tag{3.50}$$

where, for notational convenience, we introduce

$$p = \exp\left(-\frac{\pi}{2b}(x+\xi)\right).$$

Proceeding from estimate obtained in (3.50), we will derive the summation formula for the series

$$\sum_{n=1}^{\infty} \frac{p^{2n}}{(2n-1)2n}$$

whose Nth remainder appears in (3.50). In doing so, we take advantage of the standard relation [1, 27, 37]

$$\sum_{n=1}^{\infty} \frac{p^{2n-1}}{2n-1} = \frac{1}{2} \ln\left(\frac{1+p}{1-p}\right) \tag{3.51}$$

valid for $p^2 < 1$. Since the above series converges uniformly for $p \in (0, 1)$, we may integrate it term-by-term, yielding

$$\int_0^p \sum_{n=1}^{\infty} \frac{\eta^{2n-1}}{2n-1} d\eta = \sum_{n=1}^{\infty} \frac{1}{2n-1} \int_0^p \eta^{2n-1} d\eta = \sum_{n=1}^{\infty} \frac{p^{2n}}{(2n-1)2n}.$$

On the other hand, after integrating the right-hand side of the relation in (3.51), we obtain

$$\int_0^p \frac{1}{2} \ln\left(\frac{1+\eta}{1-\eta}\right) d\eta = \frac{1}{2}[(1+p)\ln(1+p) + (1-p)\ln(1-p)].$$

Hence, we arrive at the summation formula

$$\sum_{n=1}^{\infty} \frac{p^{2n}}{(2n-1)2n} = \frac{1}{2}[(1+p)\ln(1+p) + (1-p)\ln(1-p)] \tag{3.52}$$

which is important to the estimation we are in the process of deriving.

We now return to the inequality (3.50) and rewrite it as

$$\begin{aligned} |R_N(x, y; \xi, \eta)| &< \frac{4b^2}{\pi^2 p} \sum_{n=N+1}^{\infty} \frac{p^{2n}}{(2n-1)2n} \\ &= \frac{4b^2}{\pi^2 p} \left(\sum_{n=1}^{\infty} \frac{p^{2n}}{(2n-1)2n} - \sum_{n=1}^{N} \frac{p^{2n}}{(2n-1)2n} \right) \end{aligned}$$

which can be completed, in view of the relation derived in (3.52). That is,

$$\begin{aligned} |R_N(x, y; \xi, \eta)| < \frac{4b^2}{\pi^2 p} \bigg\{ &\frac{1}{2}[(1+p)\ln(1+p) + (1-p)\ln(1-p)] \\ &- \sum_{n=1}^{N} \frac{p^{2n}}{(2n-1)2n} \bigg\}, \end{aligned}$$

where we introduced the parameter p right after the relation in (3.50).

Thus, from the estimate we just obtained, it follows that the formula for the Green's function presented in (3.47), can be accurately computed by direct truncation of its series components.

It is evident that the estimate we just derived is not uniform: it depends upon the positions of the field and source point to which we apply the estimate. Hence, this makes it possible to use different truncations of the series on different sub-zones of the region Ω in order to keep a certain fixed level accuracy for the entire region.

The suggested technique provides a productive tool for constructing Green's functions for boundary-value problems for the Klein–Gordon equation on regions of differing configurations.

Example 3.20. We now focus on the Dirichlet problem on the rectangle $\Omega = \{(x, y) | 0 < x < a, 0 < y < b\}$. Following again the technique that has been so productive before, we obtains the series representation of the Green's function

$$G(x, y; \xi, \eta) = \frac{2}{b} \sum_{n=1}^{\infty} g_n(x, \xi) \sin \nu y \sin \nu \eta, \quad \nu = \frac{n\pi}{b}. \tag{3.53}$$

For $x \le \xi$, the coefficient $g_n(x, \xi)$ appears as

$$g_n(x, \xi) = \frac{1}{4h \sinh(ha)} [\exp(h(x - \xi - a)) - \exp(h(x + \xi - a)) + \exp(-h(x - \xi - a)) - \exp(-h(x + \xi - a))],$$

where we again introduce the parameter $h = \sqrt{\nu^2 + k^2}$.

After some algebra, the above expression reduces to

$$g_n(x, \xi) = -\frac{\sinh hx \sinh h\xi}{h \exp ha \sinh ha} + \frac{1}{2h}[\exp(-h(x - \xi)) - \exp(-h(x + \xi))].$$

Upon adding and subtracting the function

$$\frac{1}{2\nu}[\exp(\nu(x + \xi)) - \exp(-\nu(x - \xi))]$$

to the bracketed term in $g_n(x, \xi)$, conducting a partial summation of the series in (3.53), and performing some trivial algebra, we obtain the following form

$$\begin{aligned} G(x, y; \xi, \eta) = {} & \frac{1}{2\pi} \ln \frac{E(z - \overline{\zeta}) E(z + \overline{\zeta})}{E(z - \zeta) E(z + \zeta)} \\ & - \frac{2}{b} \sum_{n=1}^{\infty} \left(\frac{\sinh \nu x}{\nu \exp \nu\xi} - \frac{\sinh hx}{h \exp h\xi} \right) \sin \nu y \sin \nu \eta \\ & - \frac{2}{b} \sum_{n=1}^{\infty} \frac{\sinh hx \sinh h\xi}{h \exp ha \sinh ha} \sin \nu y \sin \nu \eta, \quad x \le \xi, \end{aligned} \tag{3.54}$$

of the Green's function for the Dirichlet problem for the Klein–Gordon equation on the rectangle Ω.

Note that $G(x, y; \xi, \eta)$ valid for $x \geq \xi$, follows from (3.54) upon exchanging the variables x and ξ in just the first of the two series, since the second is invariant to this operation.

Also note that the formula for the Green's function for the semi-strip, which was earlier derived in (3.45), follows immediately from the one in (3.54), if a is taken to infinity. This assertion is supported by the following features: (i) the logarithmic term as well as the first series in (3.54) do not depend on a, and (ii) whereas the limit of the second series term is zero, for a going to infinity,

$$\lim_{a\to\infty}\sum_{n=1}^{\infty}\frac{\sinh hx\sinh h\xi}{h\exp ha\sinh ha}\sin\nu y\sin\nu\eta = 0.$$

Speaking of computational implementations based on the use of the equation (3.54), note that, as we mentioned before, the first series in it is quite computer-friendly and can be accurately assessed by suitable truncation.

In contrast, the second series in (3.54) is not uniformly convergent. It has a logarithmic singularity, which shows up if both the observation and the source point are approaching the edge $x = a$ of the rectangle. This singularity can be separated from the regular part and expressed then in a closed analytical form by following the approach we used earlier in Chapter 2, where we treated the Dirichlet problem for the Laplace equation on a rectangle.

To further extend the list of boundary-value problems for the static Klein–Gordon equation, for which the method of eigenfunction expansion proves to be productive, we now turn to a several other problem settings written in polar coordinates.

Example 3.21. Consider first the Dirichlet problem

$$u(r, 0) = u(r, \alpha) = 0 \tag{3.55}$$

for the inhomogeneous Klein–Gordon equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u(r,\varphi)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u(r,\varphi)}{\partial\varphi^2}-k^2u(r,\varphi)=-f(r,\varphi) \tag{3.56}$$

on the infinite circular sector $\Omega(r, \varphi) = \{(r, \varphi)|0 < r < \infty, 0 < \varphi < \alpha\}$ with angle α constrained to be between 0 and 2π. In addition, we require that the solution $u(r, \varphi)$ remains bounded when r approaches zero or infinity. Also, assume that the right-hand side $f(r, \varphi)$ in (3.56) is integrable on Ω, that is

$$\left|\iint_{\Omega} f(r,\varphi)d\Omega(r,\varphi)\right| < \infty.$$

We have learned in Section 3.2 that the method of images fails for such a setting. But, as the current example shows, the method of eigenfunction expansion turns out to be efficient.

To obtain the solution to the problem in (3.55) and (3.56) in the integral form

$$u(r,\varphi) = \int_0^{\alpha}\int_0^{\infty} G(r,\varphi;\varrho,\psi)f(\varrho,\psi)\varrho d\varrho d\psi, \tag{3.57}$$

where the kernel $G(r,\varphi;\varrho,\psi)$ is the Green's function that needs to be found, we express $u(r,\varphi)$ and $f(r,\varphi)$ as the following Fourier sine-series

$$u(r,\varphi) = \sum_{n=1}^{\infty} u_n(r)\sin\nu\varphi, \quad \nu = \frac{n\pi}{\alpha}, \tag{3.58}$$

and

$$f(r,\varphi) = \sum_{n=1}^{\infty} f_n(r)\sin\nu\varphi. \tag{3.59}$$

This yields the set of boundary-value problems

$$\frac{d^2u_n(r)}{dr^2} + \frac{1}{r}\frac{du_n(r)}{\partial r} - \left(k^2 + \frac{\nu^2}{r^2}\right)u_n(r) = -f_n(r), \quad n = 1,2,3,\ldots, \tag{3.60}$$

$$\lim_{r\to 0}|u_n(r)| < \infty, \quad \lim_{r\to\infty}|u_n(r)| < \infty \tag{3.61}$$

for the coefficients $u_n(r)$ of the series in (3.58). The equation in (3.60) is [33] a modified Bessel equation whose fundamental set of solutions can be represented by the modified Bessel functions of the first and the second kind, respectively, $I_\nu(kr)$and $K_\nu(kr)$ of order ν [37, 73, 74].

The Green's function $g_n(r,\varrho)$ for the homogeneous problem corresponding to (3.60) and (3.61) can be constructed by using either the method based on the defining properties (see Section 1.1 of Chapter 1) or the method of variation of parameters (see Section 1.3), yielding

$$g_n(r,\varrho) = \begin{cases} I_\nu(kr)K_\nu(k\varrho), & \text{for } r \le \varrho, \\ I_\nu(k\varrho)K_\nu(kr), & \text{for } \varrho \le r. \end{cases} \tag{3.62}$$

Omitting the details of the construction procedure, which are left to the Chapter Exercises, we present the final series representation

$$G(r,\varphi;\varrho,\psi) = \frac{2}{\alpha}\sum_{n=1}^{\infty} g_n(r,\varrho)\sin\nu\varphi\sin\nu\psi \tag{3.63}$$

of the Green's function for the problem in (3.55) and (3.56).

One particular case of the expansion in (3.63) leads us to a surprising observation: assuming $\alpha = \pi$, we find the series expansion

$$G(r,\varphi;\varrho,\psi) = \frac{2}{\pi}\sum_{n=1}^{\infty} I_n(kr)K_n(k\varrho)\sin n\varphi \sin n\psi \quad \text{for } r \leq \varrho \tag{3.64}$$

of the Green's function for the Dirichlet problem for the Klein–Gordon equation on the half-plane $\Omega = \{(r,\varphi)|0 < r < \infty, 0 < \varphi < \pi\}$. Note that, in compliance with (3.62), the variables r and ϱ in (3.64) must be exchanged for $\varrho \leq r$.

At this point, it is worthwhile to recall that in Section 3.2 we have already obtained another alternative form of the Green's function for the Dirichlet on the half-plane, by making use of the the method of images. This formula is shown in (3.7) as

$$\begin{aligned} G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\big[&K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2}) \\ &- K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi+\psi) + \varrho^2})\big], \end{aligned} \tag{3.65}$$

where the moduli of the expressions $z - \zeta$ and $z - \overline{\zeta}$ are expressed, for convenience, in polar coordinates.

And this is where the actual surprise occurs: upon equating the two equivalent formulas of (3.64) and (3.65), one arrives at the following relation for $r \leq \varrho$,

$$\begin{aligned} \sum_{n=1}^{\infty} I_n(kr)K_n(k\varrho)\sin n\varphi \sin n\psi = \frac{1}{4}\big[&K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2}) \\ &- K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi+\psi) + \varrho^2})\big] \end{aligned} \tag{3.66}$$

which is, in fact, a multi-variable identity for the modified Bessel functions.

To the authors' best knowledge, the identity (3.66) is not available in the classical texts on special functions (see, for example, [3, 73, 74]) and it cannot be found in the existing handbooks like [1] or [27] either.

It is evident that several other particular cases of the expansion in (3.63) might also generate interesting identities for the modified Bessel functions. Assuming, for example, $\alpha = \pi/2$ yields

$$G(r,\varphi;\varrho,\psi) = \frac{4}{\pi}\sum_{n=1}^{\infty} I_{2n}(kr)K_{2n}(k\varrho)\sin 2n\varphi \sin 2n\psi \quad \text{for } r \leq \varrho \tag{3.67}$$

representing the Green's function for the Dirichlet problem for the Klein–Gordon equation on the quarter-plane $\Omega = \{(r,\varphi)|0 < r < \infty, 0 < \varphi < \pi/2\}$. Upon equating (3.67) with its equivalent obtained earlier in Section 3.2 by the method of images as displayed in (3.12), we obtain another nontrivial identity

$$\sum_{n=1}^{\infty} I_{2n}(kr)K_{2n}(k\varrho)\sin 2n\varphi \sin 2n\psi$$
$$= \frac{1}{8}\sum_{n=1}^{2}\big[K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - ((n-1)\pi + \psi)) + \varrho^2})$$
$$- K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - (n\pi - \psi)) + \varrho^2})\big] \qquad (3.68)$$

for $r \le \varrho$, where for $\varrho \le r$, on the left-hand side, the variables r and ϱ must be exchanged.

We encourage the reader to derive several similar identities, by working the Chapter Exercises.

Example 3.22. Consider the Dirichlet problem

$$u(R,\varphi) = u(r,0) = u(r,\pi) = 0$$

for the Klein–Gordon equation on the half-disk $\Omega = \{(r,\varphi)|0 < r < R, 0 < \varphi < \pi\}$ with radius R. In the Chapter Exercises, we present the reader with the challenge of going through the derivation procedure in detail. Here, we just present an expression for the Green's function to this problem, which is obtained in the form

$$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\big[K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi - \psi) + \varrho^2})$$
$$- K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi + \psi) + \varrho^2})\big]$$
$$- \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{I_n(kr)I_n(k\varrho)K_n(kR)}{I_n(kR)}\sin n\varphi \sin n\psi. \qquad (3.69)$$

Clearly, the Green's function for the Dirichlet problem on the half-plane, displayed in (3.65), follows from the one on the half-disk shown in (3.69), for R approaching infinity. Indeed, the series term in (3.69) vanishes for $R \to \infty$, because of the evident observation [27, 37]

$$\lim_{R\to\infty}\frac{K_n(kR)}{I_n(kR)} = 0$$

which directly follows from the asymptotic behavior of the modified Bessel functions (for $R \to \infty$)

$$\lim_{R\to\infty} K_n(kR) = 0 \quad \text{and} \quad \lim_{R\to\infty} I_n(kR) = \infty.$$

Example 3.23. For our final example, consider the following mixed boundary-value problem

$$\frac{\partial u(R,\varphi)}{\partial r} + \beta u(R,\varphi) = u(r,0) = u(r,\pi) = 0, \quad \beta \geq 0, \tag{3.70}$$

for the Klein–Gordon equation on the half-disk $\Omega = \{(r,\varphi)|0 < r < R, 0 < \varphi < \pi\}$ with radius R. The Green's function for this problem can also be found using our technique. After splitting off the singular component, we express it as

$$\begin{aligned} G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\big[&K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2}) \\ &- K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi+\psi) + \varrho^2})\big] \\ &- \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{I_n(kr)I_n(k\varrho)[K_n'(kR) + \beta K_n(kR)]}{I_n'(kR) + \beta I_n(kR)} \sin n\varphi \sin n\psi. \end{aligned} \tag{3.71}$$

The Green's functions for two other boundary-value problems can be obtained as particular cases of this equation. First, the formula in (3.71) reduces to the (3.69) (the Dirichlet problem for the half-disk), if the parameter β in the boundary condition of (3.70) approaches infinity. Second, for β going to zero, equation (3.71) reduces to the Green's function

$$\begin{aligned} G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\big[&K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi-\psi) + \varrho^2}) \\ &- K_0(k\sqrt{r^2 - 2r\varrho\cos(\varphi+\psi) + \varrho^2})] \\ &- \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{I_n(kr)I_n(k\varrho)K_n'(kR)}{I_n'(kR)} \sin n\varphi \sin n\psi \end{aligned} \tag{3.72}$$

for another mixed boundary-value problem on the semi-circle

$$\frac{\partial u(R,\varphi)}{\partial r} = u(r,0) = u(r,\pi) = 0.$$

The range of boundary-value problems for the static Klein–Gordon equations, for which Green's functions can potentially be constructed with the aid of the method of eigenfunction expansion, is not limited to the cases considered in this section. The reader can readily use the experience gained in this chapter to obtain compact formulas for other Green's functions.

3.4 Three-Dimensional Problems

As we have already outlined, this book is primarily intended to deal with partial differential equations in two dimensions. However, in a number of places in this book, we also bring forward the construction of Green's functions for three-dimensional problems. In the present section, we intend to provide the reader with a few sample settings for the static Klein–Gordon equation in three dimensions. It will be shown that the methods of images and eigenfunction expansion, which appeared efficient in two dimensions, have potential for application to three-dimensional problems as well.

Taking into account the form

$$\frac{\exp(-k|P-Q|)}{4\pi|P-Q|} \tag{3.73}$$

of the fundamental solution of the three-dimensional static Klein–Gordon equation [3, 37], its Green's function for the well-posed boundary-value problem

$$\nabla^2 u(P) - k^2 u(P) = 0, \quad P \in D, \tag{3.74}$$

$$T[u(P)] = 0, \quad P \in S, \tag{3.75}$$

in a simply-connected region D bounded with a piecewise smooth surface S can be written as

$$G(P,Q) = \frac{\exp(-k|P-Q|)}{4\pi|P-Q|} + R(P,Q), \quad P, Q \in D, \tag{3.76}$$

with $R(P,Q)$ representing the regular component of $G(P,Q)$.

Hence, similar to two-dimensional problems, finding a three-dimensional Green's function for the Klein–Gordon equation is a matter of finding the regular component $R(P,Q)$. A number of 3-D Green's functions can be constructed with the aid of the method of images. We illustrated this assertion with the following examples, starting with the simplest of them.

Example 3.24. Construct the Green's function for the Dirichlet problem for the Klein–Gordon equation in the half-space $D = \{z > 0\}$.

Placing a unit source, generating the field defined by (3.73), at an arbitrary point $Q(\xi, \eta, \zeta)$ in D, we compensate its trace on the boundary plane $z = 0$ with the unit sink

$$-\frac{\exp(-k\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2})}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}} \tag{3.77}$$

located at $Q^*(\xi, \eta, -\zeta)$ symmetric to $Q(\xi, \eta, \zeta)$ with respect to the plane $z = 0$. Clearly, the function in (3.77) is a solution of the Klein–Gordon equation everywhere in D, and the sum of (3.73) and (3.77)

$$G(x,y,z;\xi,\eta,\zeta)=\frac{1}{4\pi}\left(\frac{\exp(-k\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2})}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\right.$$
$$\left.-\frac{\exp(-k\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2})}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}}\right), \quad (3.78)$$

as a function of the coordinates of the field point P, satisfies the Klein–Gordon equation everywhere in D except for $(x,y,z)=(\xi,\eta,\zeta)$. Additionally, it contains the fundamental solution singularity, and vanishes on the boundary $z=0$ of D. In other words, it does, indeed, represent the sought-after Green's function.

Example 3.25. Construct the Green's function for the Dirichlet problem for the three-dimensional Klein–Gordon equation in a sphere of radius a.

To derive the required Green's function, we follow procedure used in Chapter 2 for the case of Laplace equation. Similar to that case, for any location of the source point inside the sphere, there exists a proper location for a compensatory sink outside of the sphere so that the face of the sphere represents a surface of zero potential for the field generated by both the source and the sink.

The only thing different in the current case, compared to the case of Laplace equation, is the form of the fundamental solution in (3.73). Hence, with the field point and the source point, in spherical coordinates, denoted by $P(r,\vartheta,\varphi)$ and $Q(\rho,\chi,\psi)$, respectively, the expression for the Green's function for the Dirichlet problem for the Klein–Gordon equation stated in a sphere of radius a is found as

$$G(P,Q)=\frac{1}{4\pi}\left(\frac{\exp(-k|P-Q|)}{|P-Q|}-\frac{\exp(-k|P-Q^*|)}{|P-Q^*|}\right) \quad (3.79)$$

with the distance between P and Q defined in spherical coordinates as

$$|P-Q|=\sqrt{r^2-2r\rho\cos\gamma+\rho^2}$$

and with γ representing the angle between the vectors $\overrightarrow{P}$ and $\overrightarrow{Q}$. We define an expression for $\cos\gamma$ in terms of the spherical angle coordinates of P and Q as

$$\sin\vartheta\sin\chi\cos(\varphi-\psi)+\cos\vartheta\cos\chi.$$

The term $|P-Q^*|$ in (3.79) represents the distance between the field point P and the compensatory sink point Q^*, located outside the sphere and is defined as

$$|P-Q^*|=\frac{1}{a}\sqrt{r^2\rho^2-2a^2r\rho\cos\gamma+a^4}.$$

Example 3.26. Construct the Green's function for the Dirichlet problem for the 3-D static Klein–Gordon equation in the infinite layer $D = \{0 < z < h\}$ of width h.

Following the method of images as applied to the corresponding problem for the Laplace equation, in Chapter 2, we set out to write a series representation of the required Green's function. That is, if we place the unit source

$$G_0(x, y, z; \xi, \eta, \zeta) = \frac{\exp(-k\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2})}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \tag{3.80}$$

at an arbitrary point (ξ, η, ζ) inside D, then the traces of (3.80) on the boundary planes $z = 0$ and $z = h$ can be canceled out by the unit sinks

$$G_0(x, y, z; \xi, \eta, -\zeta) = -\frac{\exp(-k\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2})}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta)^2}}$$

and

$$G_0(x, y, z; \xi, \eta, 2h-\zeta) = -\frac{\exp(-k\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-2h)^2})}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-2h)^2}}$$

placed at $(\xi, \eta, -\zeta)$ and $(\xi, \eta, 2h-\zeta)$, respectively.

Clearly, $G_0(x, y, z; \xi, \eta, -\zeta)$ and $G_0(x, y, z; \xi, \eta, 2h-\zeta)$ leave non-zero traces on the boundary planes $z = 0$ and $z = h$. To compensate those, we place the unit sources

$$G_0(x, y, z; \xi, \eta, -2h+\zeta) = \frac{\exp(-k\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta+2h)^2})}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta+2h)^2}}$$

and

$$G_0(x, y, z; \xi, \eta, 2h+\zeta) = \frac{\exp(-k\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta-2h)^2})}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta-2h)^2}}$$

at the points $(\xi, \eta, -2h+\zeta)$ and $(\xi, \eta, 2h+\zeta)$, respectively.

Traces of $G_0(x, y, z; \xi, \eta, -2h+\zeta)$ and $G_0(x, y, z; \xi, \eta, 2h+\zeta)$ on the boundary planes can then be compensated with the unit sinks

$$G_0(x, y, z; \xi, \eta, -2h-\zeta) = -\frac{\exp(-k\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta+2h)^2})}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta+2h)^2}}$$

and

$$G_0(x, y, z; \xi, \eta, 4h-\zeta) = -\frac{\exp(-k\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-4h)^2})}{4\pi\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z+\zeta-4h)^2}}$$

placed at $(\xi, \eta, -2h-\zeta)$ and $(\xi, \eta, 4h-\zeta)$, respectively.

Following the established pattern, we place N appropriate pairs of compensatory sources and sinks and sum the field they generate:

$$\frac{1}{4\pi} \sum_{n=-N}^{N} [G_0(x,y,z;\xi,\eta,\zeta-2nh) - G_0(x,y,z;\xi,\eta,2nh-\zeta)] .$$

Taking the limit for N approaching infinity, we obtain the infinite series

$$G(x,y,z;\xi,\eta,\zeta) = \frac{1}{4\pi} \sum_{n=-\infty}^{\infty} \left(\frac{\exp(-k\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta+2nh)^2})}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta+2nh)^2}} \right.$$
$$\left. - \frac{\exp(-k\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta-2h)^2})}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta-2nh)^2}} \right) \tag{3.81}$$

representing, the sought-after Green's function.

It is evident that the first additive component in the $n = 0$ term of the series in (3.81)

$$\frac{\exp(-k\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2})}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}$$

represents the fundamental solution to the three-dimensional static Klein–Gordon equation. This implies that if the $n = 0$ term is omitted then the series in (3.81) must converge uniformly in D.

To illustrate the potential of the eigenfunction expansion method in the construction of Green's functions for the three-dimensional Klein–Gordon equation, we present the following sample problem.

Example 3.27. Construct the Green's function for the homogeneous boundary-value problem corresponding to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - k^2 u = -f(x,y,z), \quad (x,y,z) \in D, \tag{3.82}$$

$$u(x,y,0) = u(x,y,h) = u(x,0,z) = u(x,b,z) = 0, \tag{3.83}$$

$$\frac{\partial u(0,y,z)}{\partial x} - \beta u(0,y,z) = 0, \quad \lim_{x\to\infty} |u(x,y,z)| < \infty \tag{3.84}$$

in the semi-infinite bar $D = \{0 < x < \infty, 0 < y < b, 0 < z < h\}$ with rectangular cross-section. Note also that the parameter β in (3.84) is supposed to be non-negative.

Recall that once the solution of the problem in (3.82)–(3.84) is found in the form

$$u(x,y,z) = \int_0^h \int_0^b \int_0^\infty K(x,y,z;\xi,\eta,\zeta) f(\xi,\eta,\zeta) dD(\xi,\eta,\zeta) \tag{3.85}$$

the kernel $K(x, y, z; \xi, \eta, \zeta)$ in the above integral is immediately recognized to be the Green's function to the corresponding homogeneous problem.

Taking into account the boundary conditions in (3.83), we expand the functions $u = u(x, y, z)$ and $f(x, y, z)$ into the double Fourier series

$$u(x, y, z) = \sum_{m,n=1}^{\infty} u_{mn}(x) \sin \mu y \sin \nu z \tag{3.86}$$

and

$$f(x, y, z) = \sum_{m,n=1}^{\infty} f_{mn}(x) \sin \mu y \sin \nu z \tag{3.87}$$

with μ and ν defined in terms of the summation indexes of the series as $\mu = m\pi/b$ and $\nu = n\pi/h$.

Upon substituting (3.86) and (3.87) into (3.82), we arrive at the boundary-value problem

$$\frac{d^2 u_{mn}(x)}{dx^2} - \lambda^2 u_{mn}(x) = -f_{mn}(x), \quad \lambda^2 = \mu^2 + \nu^2 + k^2, \tag{3.88}$$

$$\frac{du_{mn}(0)}{dx} - \beta u_{mn}(0) = 0, \quad \lim_{x\to\infty} |u_{mn}(x)| < \infty \tag{3.89}$$

for the coefficients $u_{mn}(x)$ of the series in (3.86).

If $g_{mn}(x, \xi)$ represents the Green's function to the homogeneous problem corresponding to (3.88) and (3.89), then the solution to the problem itself can be written as

$$u_{mn}(x) = \int_0^{\infty} g_{mn}(x, \xi) f_{mn}(\xi) d\xi. \tag{3.90}$$

Earlier in Chapter 1, we constructed the Green's function $g_{mn}(x, \xi)$ (see Example 1.15). Using our current notation, it reads as

$$g_{mn}(x, \xi) = \frac{1}{2\lambda}\left(\mathrm{e}^{-\lambda|x-\xi|} + \frac{\lambda - \beta}{\lambda + \beta}\mathrm{e}^{-\lambda(x+\xi)}\right).$$

We substitute the series coefficients $f_{mn}(x)$, expressed in terms of $f(x, y, z)$ as

$$f_{mn}(x) = \frac{4}{bh}\int_0^h \int_0^b f(x, \eta, \zeta) \sin \mu\eta \sin \nu\zeta d\eta d\zeta$$

into (3.87), and then substitute then $u_{mn}(x)$ into (3.86), reducing the solution of the boundary-value problem in (3.82)–(3.84) to

$$u(x, y, z) = \int_0^h \int_0^b \int_0^{\infty} \frac{2}{bh} \sum_{m,n=1}^{\infty} \frac{1}{\lambda}\left(\mathrm{e}^{-\lambda|x-\xi|} + \frac{\lambda - \beta}{\lambda + \beta}\mathrm{e}^{-\lambda(x+\xi)}\right)$$
$$\times \sin \mu y \sin \mu\eta \sin \nu z \sin \nu\zeta f(\xi, \eta, \zeta) dD(\xi, \eta, \zeta). \tag{3.91}$$

Hence, in light of the relation in (3.85), we conclude that the kernel of the integral in (3.91)

$$G(x,y,z;\xi,\eta,\zeta) = \frac{2}{bh}\sum_{m,n=1}^{\infty}\frac{1}{\lambda}\left(e^{-\lambda|x-\xi|} + \frac{\lambda-\beta}{\lambda+\beta}e^{-\lambda(x+\xi)}\right) \times \sin\mu y \sin\mu\eta \sin\nu z \sin\nu\zeta \quad (3.92)$$

represents the Green's function for the homogeneous boundary-value problem corresponding to (3.82)–(3.84).

3.5 Chapter Exercises

1. Construct the Green's functions for the static Klein–Gordon equation on the semi-infinite strip $\Omega = \{(x,y)|0 < x < \infty, 0 < y < b\}$ for the following boundary-value problems:

   (a) $\partial u(0,y)/\partial x = u(x,0) = u(x,b) = 0$;

   (b) $\partial u(0,y)/\partial x - \beta u(0,y) = u(x,0) = u(x,b) = 0$.

2. Construct the Green's function for the static Klein–Gordon equation on the infinite circular sector $\Omega = \{(r,\varphi)|0 < r < \infty, 0 < \varphi < \alpha\}$ for the boundary-value problem

   $$u(r,0) = 0, \quad \partial u(r,\alpha)/\partial\varphi = 0$$

   and obtain, as a particular case (for $\alpha = \pi$), the Green's function for the corresponding mixed boundary-value problem on a half-plane.

3. Derive an expression for the Green's function for the static Klein–Gordon equation on an infinite circular sector shown in (3.63).

4. Construct the Green's functions for the static Klein–Gordon equation on the half–disk $\Omega = \{(r,\varphi)|0 < r < R, 0 < \varphi < \pi\}$ with radius R for the following boundary-value problems:

   (a) $u(r,0) = \partial u(r,\pi)/\partial\varphi = u(R,\varphi) = 0$;

   (b) $u(r,0) = \partial u(r,\pi)/\partial\varphi = \partial u(R,\varphi)/\partial r = 0$;

   (c) $u(r,0) = \partial u(r,\pi)/\partial\varphi = \partial u(R,\varphi)/\partial r + \beta u(R,\varphi) = 0, \beta \geq 0$.

5. Construct the Green's functions for the static Klein–Gordon equation on the disk $\Omega = \{(r,\varphi)|0 < r < R, 0 \leq \varphi < 2\pi\}$ for the following boundary-value problems:

   (a) $u(R,\varphi) = 0$;

   (b) $\partial u(R,\varphi)/\partial r + \beta u(R,\varphi) = 0, \beta > 0$.

6. Derive an identity for the modified Bessel functions by assuming $\alpha = \pi/3$ in (3.63).
7. Derive an identity for the modified Bessel functions by assuming $\alpha = \pi/4$ in (3.63).
8. Construct the Green's function shown in (3.69).
9. Construct the Green's function shown in (3.71).
10. Construct the Green's function shown in (3.72).
11. Use the method of images to construct the Green's function for the mixed problem

$$u(x, y, 0) = 0, \quad \frac{\partial u(x, y, h)}{\partial z} = 0$$

for the three-dimensional static Klein–Gordon equation in the infinite layer $D = \{0 < z < h\}$ of width h.

12. Use the method of eigenfunction expansion to construct the Green's function for the three-dimensional Klein–Gordon equation for the Dirichlet problem in the parallelepiped $D = \{0 < x < a, 0 < y < b, 0 < z < h\}$.

Chapter 4
Higher Order Equations

So far in this book, we have been involved with second order elliptic equations. In Chapters 2 and 3, we developed efficient approaches to the construction of Green's functions for the two-dimensional Laplace and the static Klein–Gordon equation. In this chapter, we concentrate on applied higher order elliptic equations and systems. We will consider particularly the biharmonic equation, which finds many important applications in engineering and science, and its Green's functions are very rare in the literature. This equation is traditionally linked to a number of physical phenomena and processes. It is associated, for example, with the plane problem in the theory of elasticity [7, 19, 26, 31, 38, 45, 47, 71]. Another application of the biharmonic equation in mechanics is tied to the bending of thin plates made of isotropic homogeneous elastic materials and considered within the scope of the so-called Poisson–Kirchhoff model [32, 47, 60, 72].

This chapter strives to make our technique, as developed earlier for second order elliptic equations, also workable for higher order equations. It is our objective to develop, for example, a working methodology that aims at the construction of Green's functions for boundary-value problems for the biharmonic equation on regions of standard shape. We will then extend our elaborations to other higher-order elliptic equations and systems (also applicable to several problem settings in the Poisson–Kirchhoff plate and shell theory), for which the technique that we developed for the biharmonic equation, also turns out to be productive.

This chapter is organized similar to the cases of the Laplace and static Klein–Gordon equation treated in Chapters 2 and 3. In a short opening section we plot the course for a specific approach that was productive for second order equations, and show how it will be used for the construction of Green's functions for the biharmonic equation. Our approach is based on an integral representation of the solution of a well-posed problem for the inhomogeneous equation subject to homogeneous boundary conditions.

Section 4.2 is devoted to the construction of Green's functions for the biharmonic equation for problems formulated in Cartesian coordinates on which we will impose various boundary conditions. In Section 4.3, we present several results for a number of problems in a circular region. Note that whilst formulating particular boundary-value problems, we stay focused on specific settings that occur in mechanics. Section 4.4 directs our interest to another fourth order elliptic equation, which arises in the analysis of the stress-strain state of thin elastic plates, resting on an elastic foundation. Section 4.5 touches upon an eighth order elliptic system modeling several shell problems.

4.1 Definition of Green's Function

We start this section by providing a backdrop for our involvement with the biharmonic equation. In doing so, we will sketch an approach to the construction of its Green's functions. This approach will later be applied to a number of boundary-value problems.

Let Ω represent a simply connected region in two-dimensional Euclidean space, bounded by a piecewise smooth contour L.

To introduce the basic idea of our approach, consider the well-posed boundary-value problem

$$\nabla^2\nabla^2 w(P) = -f(P), \quad P \in \Omega, \tag{4.1}$$

$$B_1[w(P)] = 0, \quad B_2[w(P)] = 0, \quad P \in L, \tag{4.2}$$

where ∇^2 represents the Laplace operator written in terms of the coordinates of the field point P and the right-hand side function $f(P)$ is assumed to be integrable on Ω. B_1 and B_2 are linear differential operators specifying the boundary conditions imposed on L.

From the qualitative theory of partial differential equations [3, 18, 22, 25, 39, 53, 54, 57, 61, 66, 67, 77], it follows that the solution $w(P)$ to the problem in (4.1) and (4.2) can be expressed in compact integral form

$$w(P) = \iint_\Omega G(P,Q)f(Q)d\Omega(Q), \quad P \in \Omega, \tag{4.3}$$

in terms of the Green's function $G(P,Q)$ for the homogeneous (with $f(P) = 0$) boundary-value problem corresponding to (4.1) and (4.2).

Hence, once a computer-friendly form of the Green's function $G(P,Q)$ has been found, the relation in (4.3) provides an explicit expression for the solution to the problem in (4.1) and (4.2), for which $G(P,Q)$ turns out to be a key part of the resolving operator.

On the other hand, the relation in (4.3) provides us with a hint as to a possible strategy for the construction of the Green's function: instead of looking for a direct method for obtaining $G(P,Q)$ by using, say, its defining properties, we might choose an indirect path. That is, we want to develop a procedure that expresses the solution to (4.1) and (4.2) in the integral form of (4.3). Indeed, if such a procedure is developed and the solution $w(P)$ is found as the integral in (4.3), then the sought-after Green's function $G(P,Q)$ appears explicitly as the kernel of that integral representation.

Before continuing our actual presentation and in order to maintain a flexible terminological background for our developments, it is convenient to turn to a physical interpretation of the notion of the Green's function $G(P,Q)$ for the homogeneous problem corresponding to (4.1) and (4.2). This can be attained by recalling a process or phenomenon which is mathematically modeled by (4.1) and (4.2). In doing so, we assume that the middle plane of a thin elastic plate, undergoing a lateral load directly proportional to the function $f(P)$, occupies the region Ω.

Within the scope of the above interpretation, the setting in (4.1) and (4.2) can be thought of as a mathematical model for the bending of the plate, with $w(P)$ representing the plate's lateral deflection. In mechanics, the Green's function $G(P, Q)$ is called the influence function of a unit concentrated force [45, 47, 72] and is interpreted as the deflection of the middle plane of the plate at P due to a lateral unit point force applied at Q.

Similar to the cases of Laplace and static Klein–Gordon equation, the Green's function $G(P, Q)$ of the two-dimensional biharmonic equation can be split into a singular and a regular component. The singular component of $G(P, Q)$ represents the fundamental solution

$$S(Q, P) = \frac{1}{8\pi}|P - Q|^2 \ln \frac{1}{|P - Q|} \tag{4.4}$$

of the homogeneous biharmonic equation [3, 39, 45, 47]. With $S(P, Q)$ available in a closed form, the construction of $G(P, Q)$ focuses on its regular component $R(P, Q)$. Note, however, that the procedure for obtaining Green's functions advocated in this book does not target the regular component alone. Instead, our procedure allows us to at once obtain both the singular and the regular components. It is also important to note that the word "singular" must be put in quotes because it can only conditionally be applied to the term in (4.4). The point is that, as can easily be justified by L'Hôpital's rule, the limit of $S(P, Q)$ since $Q \to P$ exists, it is equal to zero. That is,

$$\lim_{Q \to P} |P - Q|^2 \ln \frac{1}{|P - Q|} = 0.$$

Hence, the component $S(P, Q)$ itself has no singularity, and the word "singular" refers to the logarithmic singularity of the second order derivatives of $S(P, Q)$ with respect to the coordinates of the observation point P.

In contrast to the case of the Laplace equation, only a limited number of Green's functions for the biharmonic equation are available in the current literature. We will review them and propose a derivation procedure which proves efficient for a number of boundary-value problems. The procedure is based on the method of eigenfunction expansion [29, 66].

4.2 Rectangular-Shaped Regions

In this section, we will consider number of boundary-value problems for the biharmonic equation on standard rectangular-shaped regions (infinite strip, semi-infinite strip and rectangle). These problems represent just examples, illustrating the potential of this procedure for the construction of Green's functions. As to the boundary conditions imposed for all the problems considered herein, we keep in mind the plate theory applications and try to allow for interpretation of our problems, in a physically meaningful way in mechanics.

Example 4.1. To illustrate the basics of the method, we begin with a simple problem. Consider the inhomogeneous biharmonic equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w(x,y)}{\partial x^2}+\frac{\partial^2 w(x,y)}{\partial y^2}\right)=-f(x,y), \quad (x,y)\in\Omega, \tag{4.5}$$

set up on the rectangular region $\Omega = \{(x,y)|0 < x < a, 0 < y < b\}$. Assume equation in (4.5) to be subject to the boundary conditions

$$w=\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0,b}=0, \quad w=\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0,a}=0. \tag{4.6}$$

The above is one of the classical settings in the so-called Poisson–Kirchhoff plate theory, where $w(x,y)$ in (4.5) and (4.6) is interpreted as the lateral deflection of a rectangular plate whose edges are *simply supported*, whilst the plate undergoes an exterior lateral load directly proportional to the function $f(x,y)$.

A specific form of the boundary conditions in (4.6) makes it possible to expand the solution $w(x,y)$ of the boundary-value problem in (4.5) and (4.6) into the double Fourier sine-series

$$w(x,y)=\sum_{m,n=1}^{\infty} w_{mn}\sin\mu x\sin\nu y, \quad \mu=\frac{m\pi}{a},\ \nu=\frac{n\pi}{b}. \tag{4.7}$$

Clearly, the above formula is the expansion of $w(x,y)$ in terms of its eigenfunctions, which makes it satisfy all the boundary conditions imposed in (4.6).

Continuing the method of eigenfunction expansion, we now represent the right-hand side function $f(x,y)$ of the equation in (4.5) in the identical double sine-series form

$$f(x,y)=\sum_{m,n=1}^{\infty} f_{mn}\sin\mu x\sin\nu y. \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.5) and combining like terms in the left-hand side yields

$$\sum_{m,n=1}^{\infty}\left(\mu^4+2\mu^2\nu^2+\nu^4\right)w_{mn}\sin\mu x\sin\nu y=-\sum_{m,n=1}^{\infty} f_{mn}\sin\mu x\sin\nu y$$

from which, after equating the corresponding coefficients of the trigonometric series and performing some trivial algebra, we obtain

$$w_{mn} = -\frac{f_{mn}}{\left(\mu^2 + \nu^2\right)^2}.$$

Now, substitution of w_{mn} into (4.7) yields

$$w(x, y) = -\sum_{m,n=1}^{\infty} \frac{f_{mn}}{\left(\mu^2 + \nu^2\right)^2} \sin \mu x \sin \nu y. \tag{4.9}$$

Upon recalling the Euler–Fourier formula [16, 35] and adapting it to the double-series situation, we express the Fourier coefficients f_{mn} of the right-hand side function $f(x, y)$ as

$$f_{mn} = \frac{4}{ab} \int_0^a \int_0^b f(\xi, \eta) \sin \mu\xi \sin \nu\eta d\xi d\eta.$$

Substituting f_{mn} into (4.9), we obtain the solution to the boundary-value problem in (4.5) and (4.6) in the form

$$w(x, y) = -\frac{4}{ab} \sum_{m,n=1}^{\infty} \left(\int_0^a \int_0^b f(\xi, \eta) \sin \mu\xi \sin \nu\eta d\xi d\eta \right) \frac{\sin \mu x \sin \nu y}{\left(\mu^2 + \nu^2\right)^2}$$

which reduces to a more compact form by exchanging the order of summation and integration. That is,

$$w(x, y) = -\frac{4}{ab} \int_0^a \int_0^b \sum_{m,n=1}^{\infty} \frac{\sin \mu x \sin \mu\xi \sin \nu y \sin \nu\eta}{\left(\mu^2 + \nu^2\right)^2} f(\xi, \eta) d\xi d\eta. \tag{4.10}$$

This exchange, is well justified by the uniform convergence of the double-series involved in this problem.

Hence, once the solution to the problem in (4.5) and (4.6) is obtained in the integral form of (4.3), the kernel

$$G(x, y; \xi, \eta) = -\frac{4}{ab} \sum_{m,n=1}^{\infty} \frac{\sin \mu x \sin \mu\xi \sin \nu y \sin \nu\eta}{\left(\mu^2 + \nu^2\right)^2} \tag{4.11}$$

of (4.10) represents the Green's function for the homogeneous (with $f(x, y) = 0$) problem corresponding to (4.5) and (4.6).

Note that the series in (4.11) converges at a relatively fast rate. It is also evident that the convergence is uniform on Ω, implying that it is not affected by the locations of the field point (x, y) and the force application point (ξ, η). These observations justify concluding that the (4.11) is convenient for direct computer implementations.

Example 4.2. Consider the boundary-value problem stated as

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w(x,y)}{\partial x^2}+\frac{\partial^2 w(x,y)}{\partial y^2}\right)=-f(x,y), \quad (x,y)\in\Omega, \tag{4.12}$$

and

$$w=\left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0,b}=0,\ w=\left.\frac{\partial w}{\partial x}\right|_{x=0}=0 \tag{4.13}$$

on the semi-infinite strip-shaped region $\Omega=\{(x,y)|0<x<\infty,0<y<b\}$.

In addition to the boundary conditions of (4.13), the solution of the problem $w(x,y)$ is required to be bounded as x approaches infinity. This requirement is important to ensure finding a unique solution for the problem.

To obtain the solution of the boundary-value problem in (4.12) and (4.13), we express it in the Fourier sine-series form

$$w(x,y)=\sum_{n=1}^{\infty} w_n(x)\sin\nu y, \quad \nu=\frac{n\pi}{b}, \tag{4.14}$$

and expand the right-hand side function $f(x,y)$ of (4.12) in the identical sine-series form

$$f(x,y)=\sum_{n=1}^{\infty} f_n(x)\sin\nu y. \tag{4.15}$$

Representation (4.14) satisfies the two conditions in (4.13), which are imposed on the boundary fragments $y=0$ and $y=b$ of Ω. Similar to the development in the previous example, we now substitute the expansions from (4.14) and (4.15) into (4.12) and (4.13), and equate the corresponding coefficients of the two Fourier sine-series that arise on the right-hand side and the left-hand side of (4.12). This results in the following set ($n=1,2,3,\dots$) of boundary-value problems

$$\frac{d^4 w_n(x)}{dx^4}-2\nu^2\frac{d^2 w_n(x)}{dx^2}+\nu^4 w_n(x)=-f_n(x), \quad x\in(0,\infty), \tag{4.16}$$

$$w_n(0)=\frac{dw_n(0)}{dx}=0 \tag{4.17}$$

and

$$\lim_{x\to\infty}|w_n(x)|<\infty, \quad \lim_{x\to\infty}\left|\frac{dw_n(x)}{dx}\right|<\infty \tag{4.18}$$

for the coefficients $w_n(x)$ of the series in (4.14).

The Green's function of the homogeneous problem ($f_n(x) \equiv 0$) corresponding to (4.16), (4.17) and (4.18), $g_n(x,\xi)$, has been derived earlier (see (1.118) in Example 1.17 of Chapter 1). Using our current notation, we reproduce $g_n(x,\xi)$ here as

$$g_n(x,\xi) = \frac{1}{4\nu^3}[(1+\nu(x+\xi)+2\nu^2 x\xi)e^{-\nu(x+\xi)} - (1+\nu|x-\xi|)e^{-\nu|x-\xi|}]. \tag{4.19}$$

Notice that this expression for $g_n(x,\xi)$ is written, in contrast to (1.118), in a compact single-piece formula, valid for any location of the variables x and ξ. This is possible due to the symmetry of $g_n(x,\xi)$, which provides $g_n(x,\xi) = g_n(\xi,x)$, and by using the absolute value function $|x-\xi|$.

Theorem 1.4 of Chapter 1 suggests that the solution $w_n(x)$ to the boundary-value problem in (4.16)–(4.18) can be written in terms of $g_n(x,\xi)$ as the improper integral

$$w_n(x) = \int_0^\infty g_n(x,\xi) f_n(\xi) d\xi$$

the convergence of which is ensured by two features of its integrand: first, the Green's function $g_n(x,\xi)$ is integrable on $(0,\infty)$ and second, the right-hand side function $f_n(\xi)$ of (4.16) is also integrable on $(0,\infty)$. Clearly, the second assertion is a direct consequence of the integrability of $f(x,y)$ on Ω.

Using the Euler–Fourier formula

$$f_n(\xi) = \frac{2}{b}\int_0^b f(\xi,\eta)\sin\nu\eta d\eta$$

to expressing the Fourier coefficients $f_n(\xi)$ of $f(x,y)$ on the right-hand side of (4.12), we rewrite the $w_n(x)$ as

$$w_n(x) = \frac{2}{b}\int_0^b \int_0^\infty g_n(x,\xi)\sin\nu\eta f(\xi,\eta) d\xi d\eta.$$

After substituting the above into (4.14) and exchanging summation and the integration, the solution of the boundary-value problem in (4.12) and (4.13) is ultimately found to be the double-integral

$$w(x,y) = \int_0^b \int_0^\infty \left(\frac{2}{b}\sum_{n=1}^\infty g_n(x,\xi)\sin\nu y\sin\nu\eta\right) f(\xi,\eta) d\xi d\eta$$

the kernel of which

$$G(x,y;\xi,\eta) = \frac{2}{b}\sum_{n=1}^\infty g_n(x,\xi)\sin\nu y\sin\nu\eta \tag{4.20}$$

is the Green's function of the homogeneous problem corresponding to (4.12) and (4.13), with (x, y) and (ξ, η) representing the observation and the force application point, respectively.

Note that, because its convergence rate is too low, the series form in (4.20) is not efficient for computer implementation. This notably restricts its practical value and hinders its direct numerical use. We suggest the following strategy to eliminate the deficiency of the series in (4.20): finding its slowy converging component, we isolate it, and sum it analytically, resulting in a part analytic-part series form of the Green's function, where the series component converges rapidly.

To accomplish this, we substitute $g_n(x, \xi)$ from (4.19) into (4.20) and split up the first additive exponential term in $g_n(x, \xi)$. This transforms $G(x, y; \xi, \eta)$ into

$$G(x, y; \xi, \eta) = \frac{1}{b} \sum_{n=1}^{\infty} \left(\frac{x\xi}{\nu} \mathrm{e}^{-\nu(x+\xi)} + \widetilde{g}_n(x, \xi) \right) \sin \nu y \sin \nu \eta, \tag{4.21}$$

where

$$\widetilde{g}_n(x, \xi) = \frac{1 + \nu(x + \xi)}{2\nu^3} \mathrm{e}^{-\nu(x+\xi)} - \frac{1 - \nu|x - \xi}{2\nu^3} \mathrm{e}^{-\nu|x-\xi|}. \tag{4.22}$$

Of the two series in (4.21), the second one (with the coefficient $\widetilde{g}_n(x, \xi)$) converges at the rate $1/n^2$, whereas the convergence rate of the first series in (4.21) is of the order $1/n$. So, the first step in improving the practicality of (4.20) is a success. Indeed, its slowly converging component is already isolated. To complete the job, we must sum the first series in (4.21). In doing so, we take out the factor $x\xi$ and transform it by applying the standard trigonometric identity

$$\sin \alpha \sin \beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

yielding

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\nu(x+\xi)}}{\nu} \sin \nu y \sin \nu \eta \\ &\quad = \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\nu(x+\xi)}}{\nu} \cos \nu(y - \eta) - \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\nu(x+\xi)}}{\nu} \cos \nu(y + \eta) \right) \\ &\quad = \frac{1}{2p} \left(\sum_{n=1}^{\infty} \frac{(\mathrm{e}^{-p(x+\xi)})^n}{n} \cos np(y - \eta) - \sum_{n=1}^{\infty} \frac{(\mathrm{e}^{-p(x+\xi)})^n}{n} \cos np(y + \eta) \right), \end{aligned} \tag{4.23}$$

where $p = \pi/b$.

The two series in (4.23) are completely summable. This can be shown with the aid of the classical summation formula

$$\sum_{n=1}^{\infty} \frac{r^n}{n} \cos n\vartheta = -\ln \sqrt{1 - 2r \cos \vartheta + r^2} \tag{4.24}$$

which we have repeatedly referred to earlier in our book.

The parameters r and ϑ for the series in (4.23) are defined as

$$r = \mathrm{e}^{-p(x+\xi)} \quad \text{and} \quad \vartheta = p(y \pm \eta)$$

and it is evident that they meet the constraints

$$r^2 < 1 \quad \text{and} \quad 0 \le \vartheta < 2\pi$$

required for the summation formula in (4.24).

Hence, the series in (4.23) sums to

$$\begin{aligned}
&\frac{1}{2p}\left(\sum_{n=1}^{\infty} \frac{(\mathrm{e}^{-p(x+\xi)})^n}{n} \cos np(y-\eta) - \sum_{n=1}^{\infty} \frac{(\mathrm{e}^{-p(x+\xi)})^n}{n} \cos np(y+\eta)\right) \\
&\quad = \frac{1}{2p}\Big(\ln \sqrt{1 - 2\mathrm{e}^{-p(x+\xi)} \cos p(y+\eta) + \mathrm{e}^{-2p(x+\xi)}} \\
&\qquad\quad - \ln \sqrt{1 - 2\mathrm{e}^{-p(x+\xi)} \cos p(y-\eta) + \mathrm{e}^{-2p(x+\xi)}}\Big) \\
&\quad = \frac{1}{2p} \ln \sqrt{\frac{1 - 2\mathrm{e}^{-p(x+\xi)} \cos p(y+\eta) + \mathrm{e}^{-2p(x+\xi)}}{1 - 2\mathrm{e}^{-p(x+\xi)} \cos p(y-\eta) + \mathrm{e}^{-2p(x+\xi)}}}.
\end{aligned}$$

Multiplying the numerator and the denominator of the radicand by $\mathrm{e}^{2p(x+\xi)}$, we can reduce the above expression to

$$\frac{1}{2p} \ln \sqrt{\frac{1 - 2\mathrm{e}^{p(x+\xi)} \cos p(y+\eta) + \mathrm{e}^{2p(x+\xi)}}{1 - 2\mathrm{e}^{p(x+\xi)} \cos p(y-\eta) + \mathrm{e}^{2p(x+\xi)}}}$$

which can be written in a more compact form by introducing the complex variables $z = x + iy$ and $\zeta = \xi + i\eta$ for the observation and the force application point, respectively: it reads as the real-valued function of z and ζ

$$\frac{1}{2p} \ln \frac{|1 - \mathrm{e}^{p(z+\zeta)}|}{|1 - \mathrm{e}^{p(z+\overline{\zeta})}|}, \tag{4.25}$$

where the bar on ζ denotes the complex conjugate of ζ, that is $\overline{\zeta} = \xi - i\eta$.

Hence, the function in (4.25) is the sum of the first series in (4.21), so that the entire expression for the sought-after Green's function is ultimately found as

$$G(x, y; \xi, \eta) = \frac{x\xi}{2\pi} \ln \frac{|1 - \mathrm{e}^{p(z+\zeta)}|}{|1 - \mathrm{e}^{p(z+\overline{\zeta})}|} + \frac{1}{b} \sum_{n=1}^{\infty} \widetilde{g}_n(x, \xi) \sin \nu y \sin \nu\eta. \tag{4.26}$$

It is evident that, compared to (4.20), this formula has greater practical merit, because the series in (4.26) converges at a faster rate $1/n^2$. Later in this section, we will examine the convergence issue in detail. This makes it possible to accurately compute values of $G(x, y; \xi, \eta)$ in (4.26) by truncating its series component appropriately.

Later in this section, we will examine series similar to the one in (4.20) and address the issue of their convergence in more detail. We will provide an analysis of differential properties, which are of great importance in mechanics in order to obtain the so-called stress-related components of the stress-strain state of a plate undergoing transverse loads.

Example 4.3. The technique based on the eigenfunction expansion can be used successfully in the construction of Green's functions for the semi-infinite strip-shaped region with other types of boundary conditions imposed: consider the case with the boundary conditions

$$w = \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0,b} = 0 \tag{4.27}$$

imposed on the boundary segments $y = 0$ and $y = b$, whilst the conditions

$$\left.\left(\frac{\partial^2 w}{\partial x^2} + \sigma\frac{\partial^2 w}{\partial y^2}\right)\right|_{x=0} = 0, \tag{4.28}$$

$$\left.\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + (2-\sigma)\frac{\partial^2 w}{\partial y^2}\right)\right|_{x=0} = 0 \tag{4.29}$$

are imposed on the segment $x = 0$. In mechanics, the conditions in (4.28) and (4.29) model the so-called free edge, with σ representing the Poisson ratio of the isotropic elastic material of which the plate is made.

We derive the Green's function for the problem defined by (4.27)–(4.29) in the form of the series expansion shown in (4.20). The coefficient of that expansion $g_n(x,\xi)$ represents, in this case, the Green's function of the homogeneous boundary-value problem

$$\frac{d^4 w_n(x)}{dx^4} - 2\nu^2\frac{d^2 w_n(x)}{dx^2} + \nu^4 w_n(x) = 0, \quad x \in (0,\infty), \tag{4.30}$$

$$\frac{d^2 w(x)}{dx^2} - \sigma\nu^2 w_n(0) = \frac{d^3 w_n(0)}{dx^3} - (2-\sigma)\nu^2\frac{dw_n(0)}{dx} = 0, \tag{4.31}$$

$$\lim_{x\to\infty}|w_n(x)| < \infty, \quad \lim_{x\to\infty}\left|\frac{dw_n(x)}{dx}\right| < \infty \tag{4.32}$$

with ν defined as $n\pi/b$.

Leaving details of the derivation procedure as an exercise for the reader, we present an expression for the Green's function of the problem in (4.30)–(4.32)

$$g_n(x,\xi) = -\frac{1+\nu|x-\xi|}{4\nu^3}e^{-\nu|x-\xi|} - \frac{(4+(1+\sigma)^2)+\nu(1-\sigma)^2(x+\xi+2\nu x\xi)}{4\nu^3(1-\sigma)(3+\sigma)}e^{-\nu(x+\xi)}.$$

In this case, we can improve the convergence of the series in (4.20) in a manner similar to what we used in Example 4.2. We recommend the reader to explore the issue in detail, in order to gain a deeper understanding.

The Green's function of the biharmonic equation stated on the semi-infinite strip with boundary conditions

$$w = \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0,b} = 0, \quad w = \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = 0 \tag{4.33}$$

is represented by the expansion shown in equation (4.20), with the coefficient $g_n(x, s)$

$$g_n(x,\xi) = \frac{1+\nu(x+\xi)}{4\nu^3}\mathrm{e}^{-\nu(x+\xi)} - \frac{1+\nu|x-\xi|}{4\nu^3}\mathrm{e}^{-\nu|x-\xi|}. \tag{4.34}$$

Derivation of the above expression is a more or less routine procedure, given the experience that we gained in our earlier work, and we believe that it can be left as an exercise for the reader.

At this point in our presentation, we will revisit the rectangular region and retrace the procedure based on the eigenfunction expansion, as applied to a problem for the biharmonic equation with boundary conditions different from those of Example 4.1.

Example 4.4. Consider the boundary-value problem

$$w = \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0,b} = 0, \quad w = \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = 0, \quad w = \left.\frac{\partial w}{\partial x}\right|_{x=a} = 0 \tag{4.35}$$

for the biharmonic equation on the rectangular region $\Omega = \{(x, y)|0 < x < a, 0 < y < b\}$.

The Green's function for this problem reduces to the series in equation (4.20), whose coefficients $g_n(x, \xi)$ represent the Green's function for the following boundary-value problem

$$\frac{d^4 w_n(x)}{dx^4} - 2\nu^2\frac{d^2 w_n(x)}{dx^2} + \nu^4 w_n(x) = 0, x \in (0, a), \tag{4.36}$$

$$w_n(0) = \frac{d^2 w_n(0)}{dx^2} = 0, \quad w_n(a) = \frac{d w_n(a)}{dx} = 0, \tag{4.37}$$

where $\nu = n\pi/b$.

Following our routine but quite cumbersome procedure, we derive the Green's function for the problem in (4.36) and (4.37) as

$$\begin{aligned} g_n(x,\xi) = \frac{1}{2\nu^3\Delta^*}\{&\nu x\cosh\nu x[2\nu(\xi-a)\cosh\nu\xi - \sinh\nu(\xi-2a) - \sinh\nu\xi] \\ &- \sinh\nu x[\nu\xi\cosh\nu(\xi-2a) - \sinh\nu(\xi-2a) \\ &+ \nu(\xi-2a)\cosh\nu\xi + (2\nu^2 a(\xi-a)-1)\sinh\nu\xi]\}, \\ &x \le \xi, \end{aligned} \tag{4.38}$$

where $\Delta^* = \sinh 2\nu a - 2\nu a$.

Note that due to the problem in (4.36) and (4.37) being self-adjoint, the expression for $g_n(x, \xi)$, valid for $x \geq \xi$, can be obtained from the above by exchanging x and ξ.

It follows from what we considered so far, that Green's functions for a number of particular problems for the biharmonic equation can be expressed in the series form of (4.20). Hence, the convergence of that series represents a critical issue in developing Green's function-based computational procedures.

Going to the exploration of the convergence of the series in (4.20), we turn to plate theory applications and focus on plates undergoing a transverse point concentrated force, in which case the corresponding Green's function $G(x, y; \xi, \eta)$ itself represents the deflection of the plate, whereas in order to obtain the components of the stress-strain state, caused by the point force, derivatives of the Green's function are required. So, differential properties of the series in (4.20) become critically important for users of the Green's function method.

To be more specific, we choose the problem in (4.33) as modeling the semi-infinite strip-shaped plate with all the edges simply-supported. The Green's function in (4.20) in this case has the coefficient $g_n(x, \xi)$ as in (4.34). For any location of the observation point (x, y) and the force application point (ξ, η), the expansion in (4.20) converges uniformly, and its rate of convergence is of order $1/n^2$. Hence, computational implementations requiring just the Green's function $G(x, y; \xi, \eta)$ itself can be carried out accurately, by appropriate truncation of the series. If, however, high-order partial derivatives of $G(x, y; \xi, \eta)$ are required (which is the case for components of the stress-strain state), the situation becomes less trivial and needs more attention.

In the following, we will show that, as an example, the second partial derivatives of the expansion in (4.20) represent non-uniformly convergent series that diverge logarithmically when the observation point (x, y) approaches the force application point (ξ, η). This agrees with the Poisson–Kirchhoff plate theory, which asserts that the bending moments M_x and M_y in the plate undergoing a transverse unit force concentrated at a point (ξ, η), are expressed as

$$M_x(x, y) = -D\left(\frac{\partial^2 G(x, y; \xi, \eta)}{\partial x^2} + \sigma \frac{\partial^2 G(x, y; \xi, \eta)}{\partial y^2}\right)$$

and

$$M_y(x, y) = -D\left(\frac{\partial^2 G(x, y; \xi, \eta)}{\partial y^2} + \sigma \frac{\partial^2 G(x, y; \xi, \eta)}{\partial x^2}\right),$$

where D is referred to as the plate's flexural rigidity, defined in terms of the thickness of the plate and the properties of the homogeneous elastic isotropic material of which the plate is made. The parameter σ is, as mentioned before in Example 4.3, the Poisson ratio of the material.

After substituting $G(x, y; \xi, \eta)$ from (4.20), with coefficients $g_n(x, \xi)$ from (4.34), into the above equations, the latter convert to

$$M_x(x, y) = -\frac{D}{2b}\sum_{n=1}^{\infty}\left\{\left[\frac{1+\sigma}{\nu} - (1-\sigma)|x-\xi|\right]\mathrm{e}^{-\nu|x-\xi|}\right.$$
$$\left. + \left[(1-\sigma)(x+\xi) - \frac{1+\sigma}{\nu}\right]\mathrm{e}^{-\nu(x+\xi)}\right\}\sin\nu y\sin\nu\eta$$

and

$$M_y(x, y) = -\frac{D}{2b}\sum_{n=1}^{\infty}\left\{\left[\frac{1+\sigma}{\nu} + (1-\sigma)|x-\xi|\right]\mathrm{e}^{-\nu|x-\xi|}\right.$$
$$\left. - \left[(1-\sigma)(x+\xi) + \frac{1+\sigma}{\nu}\right]\mathrm{e}^{-\nu(x+\xi)}\right\}\sin\nu y\sin\nu\eta.$$

These series representations can be summed readily with the aid of the classical summation formula exhibited earlier in (4.24) and another standard formula (see [1, 27], for example)

$$\sum_{n=1}^{\infty} r^n \cos n\vartheta = \frac{1 - r\cos\vartheta}{1 - 2r\cos\vartheta + r^2}, \tag{4.39}$$

where

$$r^2 < 1 \quad \text{and} \quad 0 \le \vartheta < 2\pi.$$

This finally yields

$$M_x(x, y) = -\frac{D}{4b}\left\{\frac{b(1+\sigma)}{\pi}\ln\frac{E(z-\overline{\zeta})E(z+\overline{\zeta})}{E(z-\zeta)E(z+\zeta)}\right.$$
$$+ (1-\sigma)\left[|x-\xi|\left(\frac{R(z-\overline{\zeta})}{E^2(z-\overline{\zeta})} - \frac{R(z-\zeta)}{E^2(z-\zeta)}\right)\right.$$
$$\left.\left. - (x+\xi)\left(\frac{R(-(z+\overline{\zeta}))}{E^2(-(z+\overline{\zeta}))} - \frac{R(-(z+\zeta))}{E^2(-(z+\zeta))}\right)\right]\right\} \tag{4.40}$$

and

$$M_y(x,y) = -\frac{D}{4b}\left\{\frac{b(1+\sigma)}{\pi}\ln\frac{E(z-\overline{\zeta})E(z+\overline{\zeta})}{E(z-\zeta)E(z+\zeta)}\right.$$
$$-(1-\sigma)\left[|x-\xi|\left(\frac{R(z-\overline{\zeta})}{E^2(z-\overline{\zeta})}-\frac{R(z-\zeta)}{E^2(z-\zeta)}\right)\right.$$
$$\left.\left.-(x+\xi)\left(\frac{R(-(z+\overline{\zeta}))}{E^2(-(z+\overline{\zeta}))}-\frac{R(-(z+\zeta))}{E^2(-(z+\zeta))}\right)\right]\right\} \tag{4.41}$$

with $z = x + iy$ and $\zeta = \xi + i\eta$ denoting the observation and the force application point, respectively. The real-valued functions $E(s)$ and $R(s)$ of a complex variable s are defined as

$$E(s) = \left|1 - \exp\left(\frac{\pi s}{b}\right)\right|$$

and

$$R(s) = \text{Re}\left(1 - \exp\left(\frac{\pi s}{b}\right)\right).$$

Thus, using the series form of the Green's function in (4.20), the bending moments M_x and M_y, caused in the plate by a force concentrated in a point, are obtained in a closed readily computable form.

It is evident that the formulas for the bending moments we derived in (4.40) and (4.41) contain a logarithmic singularity if the observation point z coincides with the force application point ζ. The singularity is caused only by the logarithmic term $\ln(E(z-\zeta))$. There are no other singularities: due to the presence of the factor $|x-\xi|$, the non-logarithmic components containing the term $E(z-\zeta)$ in the denominators of (4.40) and (4.41) represent a removable singularity.

4.3 Circular-Shaped Regions

The series of examples that we tackled successfully in the previous section raises confidence in our approach to constructing Green's functions for the biharmonic equation for problems on rectangular-shaped regions. In this section, our intention is to show that the approach also turns out to be productive when considering circular-shaped regions with various boundary conditions. In carrying out our intention, we first turn to a classical problem the Green's function of which is available in the literature, and demonstrate that we can also find it following our recommended approach.

Example 4.5. Consider the inhomogeneous biharmonic equation

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)^2 w(r,\varphi)=-f(r,\varphi),\quad (r,\varphi)\in\Omega, \tag{4.42}$$

on the circular region $\Omega=\{(r,\varphi)|0<r<a, 0\leq\varphi<2\pi\}$ and impose the boundary conditions

$$w(a,\varphi)=0,\quad \frac{\partial w(a,\varphi)}{\partial r}=0 \tag{4.43}$$

at the contour $r=a$ of Ω. Since $r=0$ is a singular point in (4.42), we complete specifying the problem by adding the boundedness conditions

$$\lim_{r\to 0}|w(r,\varphi)|<\infty,\quad \lim_{r\to 0}\left|\frac{\partial^2 w(r,\varphi)}{\partial r^2}\right|<\infty \tag{4.44}$$

to (4.42) and (4.43).

As to a possible physical interpretation of the above setting, we might consider a case where the problem in (4.42)–(4.44) models the deflection $w(r,\varphi)$ of a thin circular plate (considered within the scope of the Poisson–Kirchhoff theory), made from an elastic isotropic homogeneous material and undergoing a transverse distributed load proportional to $f(r,\varphi)$. The plate's edge $r=a$ is assumed to be clamped, according to the conditions in (4.43).

The Green's function

$$G(z,\zeta)=\frac{1}{8\pi}\left[\frac{1}{2a^2}\left(a^2-|z|^2\right)\left(a^2-|\zeta|^2\right)-|z-\zeta|^2\ln\frac{|a^2-z\bar{\zeta}|}{a|z-\zeta|}\right] \tag{4.45}$$

of the homogeneous problem corresponding to (4.42)–(4.44) has been known for many decades (see, for instance, [39, 59]). For compactness, we introduce complex variable notation in (4.45), for $z=r(\cos\varphi+i\sin\varphi)$ and $\zeta=\rho(\cos\psi+i\sin\psi)$ denoting the observation and the force application point, respectively.

It is worth noting that (4.45) represents the only closed analytical form of Green's function for the biharmonic equation available in the literature. In what follows, we show that the technique, developed earlier in this section, provides an alternative way for deriving $G(z,\zeta)$ shown in (4.45).

Since the problem setting in (4.42)–(4.44) is 2π-periodic with respect to the variable φ, we assume the following trigonometric Fourier expansions for the functions $w(r,\varphi)$ and $f(r,\varphi)$

$$w(r,\varphi)=\frac{1}{2}w_0(r)+\sum_{n=1}^{\infty}\left(w_n^c(r)\cos n\varphi+w_n^s(r)\sin n\varphi\right) \tag{4.46}$$

and

$$f(r,\varphi) = \frac{1}{2} f_0(r) + \sum_{n=1}^{\infty} \left(f_n^c(r)\cos n\varphi + f_n^s(r)\sin n\varphi \right). \tag{4.47}$$

After substituting these equations into (4.42)–(4.44), we obtain the following set ($n = 0, 1, 2, \ldots$) of boundary-value problems for the coefficients $w_n(r)$ in (4.46)

$$\left(\frac{d^4}{dr^4} + \frac{2}{r}\frac{d^3}{dr^3} - \frac{1+2n^2}{r^2}\frac{d^2}{dr^2} + \frac{1+2n^2}{r^3}\frac{d}{dr} + \frac{n^2(n^2-4)}{r^4} \right) w_n(r) = -f_n(r), \tag{4.48}$$

$$\lim_{r\to 0} |w_n(r)| < \infty, \quad \lim_{r\to 0} \left| \frac{d^2 w_n(r)}{dr^2} \right| < \infty, \quad w_n(a) = \frac{dw_n(a)}{dr} = 0, \tag{4.49}$$

where we omit, for notational convenience, the superscripts on $w_n(r)$ and $f_n(r)$, since both the cosine- and the sine-mode in (4.46) and (4.47) will be treated similarly until a certain stage in the development.

It turns out that the problem in (4.48) and (4.49) is not in self-adjoint form. This implies that if its Green's function is constructed, it would not be symmetric. However, the property of symmetry can be restored by reducing (4.48) and (4.49) to self-adjoint form. This can be achieved by introducing the integrating factor r^2. Now, the boundary-value problem in (4.49) for the homogeneous equation

$$\left(r^2\frac{d^4}{dr^4} + 2r\frac{d^3}{dr^3} - (1+2n^2)\frac{d^2}{dr^2} + \frac{1+2n^2}{r}\frac{d}{dr} + \frac{n^2(n^2-4)}{r^2} \right) w_n(r) = 0 \tag{4.50}$$

is in a form for which the Green's function must be symmetric.

Before we get down to the construction of Green's function to the boundary-value problem in (4.49) and (4.50), we must make important comment: it is impossible to obtain a single fundamental set of solutions (the set valid for the entire range of the parameter $n = 0, 1, 2, \ldots$) for the governing equation in (4.50). Indeed, three individual instances ($n = 0$, $n = 1$, and $n \geq 2$) of (4.50) must be distinguished and treated separately, because their fundamental sets of solutions are different.

First, consider $n = 0$, for which the problem in (4.49) and (4.50) reads

$$\left(r^2\frac{d^4}{dr^4} + 2r\frac{d^3}{dr^3} - \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \right) w_0(r) = 0, \tag{4.51}$$

$$\lim_{r\to 0} |w_0(r)| < \infty, \quad \lim_{r\to 0} \left| \frac{d^2 w_0(r)}{dr^2} \right| < \infty, \quad w_0(a) = \frac{dw_0(a)}{dr} = 0. \tag{4.52}$$

Clearly, equation (4.51) represents the well-known type of Cauchy–Euler [20, 33, 37]. Therefore, as its solution, we can try $w_0(r) = r^k$, where k are to be determined by substituting this formula into (4.51). This yields the fourth order algebraic auxiliary equation

$$k(k-1)(k-2)(k-3)+2k(k-1)(k-2)-k(k-1)+k=0 \tag{4.53}$$

for k. It can be readily shown that the above equation allows an analytic solution. In order to obtain this, we rewrite its last two additive terms, thus reducing the equation to

$$k(k-1)(k-2)(k-3)+2k(k-1)(k-2)-k(k-2)=0$$

Factoring out the term of $k(k-2)$, we have

$$k(k-2)[(k-1)(k-3)+2(k-1)-1]=0$$

finally reducing (4.53) to the compact, directly solvable form

$$k^2(k-2)^2=0$$

implying that $k=0$ and $k=2$, with each representing a double-root of (4.53). Hence, taking into account the multiplicity of roots of the auxiliary equation (see Chapter 1), a fundamental set of solutions of (4.51),

$$1, \quad \ln r, \quad r^2, \quad \text{and} \quad r^2 \ln r,$$

can be compiled which allows us to derive the Green's function $g_0(r,\rho)$ for the boundary-value problem in (4.51) and (4.52). For this, we can use the procedure based on either the defining properties of the Green's function (see Section 1.1) or on the method of variation of parameters (see Section 1.3). We omit the details of the derivation procedure and display only the final expression for $g_0(r,\rho)$:

$$g_0(r,\rho)=\frac{1}{8}\left[\frac{1}{a^2}(a^2-\rho^2)(a^2+r^2)+2(r^2+\rho^2)\ln\frac{\rho}{a}\right], \quad r\le\rho. \tag{4.54}$$

In the second of our three individual cases, for $n=1$, (4.49) and (4.50) read as

$$\left(r^2\frac{d^4}{dr^4}+2r\frac{d^3}{dr^3}-3\frac{d^2}{dr^2}+\frac{3}{r}\frac{d}{dr}-\frac{3}{r^2}\right)w_1(r)=0, \tag{4.55}$$

$$\lim_{r\to 0}|w_1(r)|<\infty, \quad \lim_{r\to 0}\left|\frac{d^2w_1(r)}{dr^2}\right|<\infty, \quad w_1(a)=\frac{dw_1(a)}{dr}=0 \tag{4.56}$$

which also represents a boundary-value problem for the Cauchy–Euler type equation, the auxiliary equation of which is different from that in (4.53), appearing as

$$k(k-1)(k-2)(k-3)+2k(k-1)(k-2)-3k(k-1)+3k-3=0.$$

After some trivial algebra, similar to what we applied to (4.53), the left-hand side in the above equation reduces to a product of two linear factors and a quadratic factor. That is,

$$(k-1)(k-3)(k^2-1)=0$$

revealing the four real roots

$$k=1 \text{ (double-root)}, \quad k=-1 \quad \text{and} \quad k=3.$$

This leads us to the fundamental set of solutions of (4.55), compiled with the functions

$$r^{-1}, \quad r, \quad r^3, \quad \text{and} \quad r\ln r$$

bringing us to the Green's function $g_1(r,\rho)$ for (4.55) and (4.56), which, for $r\le\rho$, is found as

$$g_1(r,\rho)=\frac{r(\rho^2-a^2)}{16a^4\rho}\left[r^2(a^2-\rho^2)+2a^2\rho^2\right]-\frac{1}{4}r\rho\ln\frac{\rho}{a}. \tag{4.57}$$

Our final case, where $n\ge 2$, is the most complex of the three. It generates an auxiliary equation for (4.50)

$$\begin{aligned}k(k-1)(k-2)(k-3)&+2k(k-1)(k-2)\\ &-(1+2n^2)k(k-1)+(1+2n^2)k+n^2(n^2-4)=0.\end{aligned} \tag{4.58}$$

Finding the roots of this equation is not as trivial as it has been in the cases of $n=0$ and $n=1$. It requires a specific approach, and we will describe its procedure in more detail. In doing so, we take the first two additive terms in (4.58) and factor their sum as shown:

$$k(k-1)(k-2)(k-3)+2k(k-1)(k-2)=k(k-1)^2(k-2).$$

Now, the sum of the third and the fourth terms in (4.58) simplifies to

$$-(1+2n^2)k(k-1)+(1+2n^2)k=-(1+2n^2)k(k-2)$$

converting (4.58) to a quite compact form

$$\begin{aligned}k(k-1)^2(k-2)&-(1+2n^2)k(k-2)+n^2(n^2-4)\\ &=k(k-2)\left[(k-1)^2-(1+2n^2)\right]+n^2(n^2-4)=0\end{aligned}$$

which, after the two additive terms in the brackets are reordered, reduces to

$$k(k-2)\left[k(k-2)-2n^2\right]+n^2(n^2-4)=0$$

or, on removing the brackets

$$\underline{k^2(k-2)^2 - 2n^2k(k-2) + n^4} - 4n^2 = 0. \tag{4.59}$$

At this stage in the development, it is useful to note that the sum of the three underlined terms in the above equation forms a complete square. With this in mind, we transform (4.59) into

$$\left[k(k-2) - n^2\right]^2 - 4n^2 = 0.$$

Interpreting the left-hand side of the above equation as a difference of squares, we factor it as

$$\left[k(k-2) - n^2 - 2n\right]\left[k(k-2) - n^2 + 2n\right] = 0.$$

Hence, with a series of elegant transformations, we have managed to simplify the auxiliary equation in (4.58), reducing it to two trivial quadratic equations in k:

$$k^2 - 2k - n(n+2) = 0$$

with the roots

$$k = -n \quad \text{and} \quad k = 2 + n,$$

and

$$k^2 - 2k - n(n-2) = 0,$$

whose roots are

$$k = n \quad \text{and} \quad k = 2 - n.$$

This gives us four distinct real roots for (4.58) as

$$k = n, \quad k = -n, \quad k = n+2, \quad \text{and} \quad k = 2 - n.$$

Hence, a fundamental set of solutions for the equation in (4.50) can be compiled using the functions

$$r^n, \quad r^{-n}, \quad r^{n+2}, \quad \text{and} \quad r^{2-n}.$$

Taking this fundamental set of solutions, and using one of the standard techniques described in Chapter 1, we derive the following expression

$$g_n(r,\rho) = -\frac{1}{8}\left\{\frac{r\rho}{n-1}\left[\left(\frac{r\rho}{a^2}\right)^{n-1} - \left(\frac{r}{\rho}\right)^{n-1}\right] + \frac{r^2+\rho^2}{n}\left[\left(\frac{r}{\rho}\right)^{n} - \left(\frac{r\rho}{a^2}\right)^{n}\right]\right.$$
$$\left. + \frac{r\rho}{n+1}\left[\left(\frac{r\rho}{a^2}\right)^{n+1} - \left(\frac{r}{\rho}\right)^{n+1}\right]\right\}, \quad r \le \rho, \tag{4.60}$$

for the Green's function for the problem in (4.49) and (4.50), for the general case of $n \geq 2$.

Note that the Green's functions that we derived in (4.54), (4.57), and (4.60) are valid for $r \leq \rho$. However, due to the self-adjointness of the problem in (4.49) and (4.50), we can immediately obtain expressions for $g_0(r, \rho)$, $g_1(r, \rho)$, and $g_n(r, \rho)$, valid for $r \geq \rho$, from the corresponding ones in (4.54), (4.57), and (4.60) by exchanging of r and ρ.

The functions $g_0(r, \rho)$, $g_1(r, \rho)$, and $g_n(r, \rho)$, can be used to get the Green's function $G(r, \varphi; \rho, \psi)$ for the homogeneous boundary-value problem, corresponding to (4.42)–(4.44). This can be achieved by following the method of eigenfunction expansion as developed earlier in Chapter 2, when we considered the Dirichlet problem for the Laplace equation on a disk (see Section 2.3.3). In this case the Green's function, we are looking for, is found as

$$G(r, \varphi; \rho, \psi) = \frac{1}{2\pi}\left[g_0(r, \rho) + 2\sum_{n=1}^{\infty} g_n(r, \rho)\cos n(\varphi - \psi)\right]. \tag{4.61}$$

The expression in (4.60) reveals nonuniform convergence of the series in (4.61). In the following, we will show that the series can be summed completely. Notice that it does not matter which of the two branches of its coefficients $g_1(r, \rho)$ and $g_n(r, \rho)$ are taken for the summation procedure. We can use either the ones valid for $r \leq \rho$ or the other ones. In our procedure, we use the branches displayed in (4.54), (4.57), and (4.60).

To sum the series in (4.61), we regroup its terms somewhat. Since the coefficient of the first term in the series $g_1(r, \rho)$ is obtained in a form different from the rest of the coefficients $g_n(r, \rho)$, obtained for $n = 2, 3, \ldots$, we isolate the term

$$g_1(r, \rho)\cos(\varphi - \psi)$$

and rewrite (4.61) as

$$G(r, \varphi; \rho, \psi) = \frac{1}{2\pi}\left[g_0(r, \rho) + 2g_1(r, \rho)\cos(\varphi - \psi) + 2\sum_{n=2}^{\infty} g_n(r, \rho)\cos n(\varphi - \psi)\right]. \tag{4.62}$$

After substituting $g_0(r, \rho)$ and $g_1(r, \rho)$ from (4.54) and (4.57) into this expansion, the first two terms in the brackets read

$$\begin{aligned} &g_0(r, \rho) + 2g_1(r, \rho)\cos(\varphi - \psi) \\ &\quad = \frac{1}{8}\left[\frac{1}{a^2}(a^2 - \rho^2)(a^2 + r^2) + 2(r^2 + \rho^2)\ln\frac{\rho}{a}\right] \\ &\quad\quad - \left[\frac{r(a^2 - \rho^2)}{8a^4\rho}\left(r^2(a^2 - \rho^2) + 2a^2\rho^2\right) + \frac{1}{2}r\rho\ln\frac{\rho}{a}\right]\cos(\varphi - \psi). \end{aligned}$$

Combining the logarithmic terms, we rewrite it as

$$
\begin{aligned}
&g_0(r,\rho) + 2g_1(r,\rho)\cos(\varphi-\psi) \\
&\quad = \frac{1}{8}\left\{\underline{\frac{1}{a^2}(a^2-\rho^2)(a^2+r^2)} + 2\left[r^2 - 2r\rho\cos(\varphi-\psi) + \rho^2\right]\ln\frac{\rho}{a}\right. \\
&\qquad \left.\underline{\underline{-\frac{r(a^2-\rho^2)}{a^4\rho}\left[r^2(a^2-\rho^2) + 2a^2\rho^2\right]\cos(\varphi-\psi)}}\right\}. \qquad (4.63)
\end{aligned}
$$

We will revisit this expression at a later time, when the series in (4.62) is ready for a final summation. Prior to that we examine the remaining series term in (4.62) and write it in an explicit form, after recalling the expression for $g_n(r,\rho)$ from (4.60). This yields

$$
\begin{aligned}
&2\sum_{n=2}^{\infty} g_n(r,\rho)\cos n(\varphi-\psi) \\
&\quad = -\frac{1}{4}\left\{r\rho\sum_{n=2}^{\infty}\frac{1}{n-1}\left[\left(\frac{r\rho}{a^2}\right)^{n-1} - \left(\frac{r}{\rho}\right)^{n-1}\right]\cos n(\varphi-\psi)\right. \\
&\qquad + (r^2+\rho^2)\sum_{n=2}^{\infty}\frac{1}{n}\left[\left(\frac{r}{\rho}\right)^{n} - \left(\frac{r\rho}{a^2}\right)^{n}\right]\cos n(\varphi-\psi) \\
&\qquad \left. + r\rho\sum_{n=2}^{\infty}\frac{1}{n+1}\left[\left(\frac{r\rho}{a^2}\right)^{n+1} - \left(\frac{r}{\rho}\right)^{n+1}\right]\cos n(\varphi-\psi)\right\}. \qquad (4.64)
\end{aligned}
$$

We need an individual approach for treating each of the three series in (4.64). To partially sum the first one, we change its summation index n by making the substitution $k = n-1$. This yields

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\frac{1}{n-1}\left[\left(\frac{r\rho}{a^2}\right)^{n-1} - \left(\frac{r}{\rho}\right)^{n-1}\right]\cos n(\varphi-\psi) \\
&\quad = \sum_{k=1}^{\infty}\frac{1}{k}\left[\left(\frac{r\rho}{a^2}\right)^{k} - \left(\frac{r}{\rho}\right)^{k}\right]\cos(k+1)(\varphi-\psi) \\
&\quad = \sum_{k=1}^{\infty}\frac{1}{k}\left[\left(\frac{r\rho}{a^2}\right)^{k} - \left(\frac{r}{\rho}\right)^{k}\right] \\
&\qquad\qquad \times\left[\cos k(\varphi-\psi)\cos(\varphi-\psi) - \sin k(\varphi-\psi)\sin(\varphi-\psi)\right] \\
&\quad = \cos(\varphi-\psi)\sum_{k=1}^{\infty}\frac{1}{k}\left[\left(\frac{r\rho}{a^2}\right)^{k} - \left(\frac{r}{\rho}\right)^{k}\right]\cos k(\varphi-\psi) \\
&\qquad - \sin(\varphi-\psi)\sum_{k=1}^{\infty}\frac{1}{k}\left[\left(\frac{r\rho}{a^2}\right)^{k} - \left(\frac{r}{\rho}\right)^{k}\right]\sin k(\varphi-\psi). \qquad (4.65)
\end{aligned}
$$

Of the two above series, the cosine series can be summed by implementing the standard summation formula

$$\sum_{k=1}^{\infty} \frac{p^k}{k} \cos k\vartheta = -\frac{1}{2} \ln(1 - 2p\cos\vartheta + p^2) \tag{4.66}$$

which we repeatedly used in Chapters 2 and 3, as well as earlier in this chapter (see (4.24)). This yields, for the entire first term in (4.64),

$$\begin{aligned} r\rho \sum_{n=2}^{\infty} \frac{1}{n-1} \left[\left(\frac{r\rho}{a^2}\right)^{n-1} - \left(\frac{r}{\rho}\right)^{n-1} \right] \cos n(\varphi - \psi) \\ = r\rho \cos(\varphi - \psi) \left[\frac{1}{2} \ln\left(1 - 2\frac{r}{\rho}\cos(\varphi - \psi) + \left(\frac{r}{\rho}\right)^2\right) \right. \\ \left. - \frac{1}{2} \ln\left(1 - 2\frac{r\rho}{a^2}\cos(\varphi - \psi) + \left(\frac{r\rho}{a^2}\right)^2\right) \right] \\ - r\rho \sin(\varphi - \psi) \sum_{k=1}^{\infty} \frac{1}{k} \left[\left(\frac{r\rho}{a^2}\right)^k - \left(\frac{r}{\rho}\right)^k \right] \sin k(\varphi - \psi). \end{aligned} \tag{4.67}$$

The above sine-series is also summable. Its summation may be accomplished with the aid of another standard summation formula [1, 27]

$$\sum_{k=1}^{\infty} \frac{p^k}{k} \sin k\vartheta = \arctan \frac{p\sin\vartheta}{1 - p\cos\vartheta}. \tag{4.68}$$

Note that the relations in (4.66) and (4.68) apply if p and ϑ satisfy the following constraints

$$p < 1 \quad \text{and} \quad 0 \le \vartheta < 2\pi.$$

As to the sine-series of (4.65), for the sake of our further development, we leave it in its current form.

In order to sum the series in the second term of (4.64), we rewrite it as

$$\begin{aligned} (r^2 + \rho^2) \sum_{n=2}^{\infty} \frac{1}{n} \left[\left(\frac{r}{\rho}\right)^n - \left(\frac{r\rho}{a^2}\right)^n \right] \cos n(\varphi - \psi) \\ = (r^2 + \rho^2) \sum_{n=1}^{\infty} \frac{1}{n} \left[\left(\frac{r}{\rho}\right)^n - \left(\frac{r\rho}{a^2}\right)^n \right] \cos n(\varphi - \psi) \\ + (r^2 + \rho^2) \left[\left(\frac{r\rho}{a^2}\right) - \left(\frac{r}{\rho}\right) \right] \cos(\varphi - \psi) \end{aligned}$$

and implement the summation formula (4.66), allowing us to obtain

$$(r^2+\rho^2)\sum_{n=2}^{\infty}\frac{1}{n}\left[\left(\frac{r}{\rho}\right)^n-\left(\frac{r\rho}{a^2}\right)^n\right]\cos n(\varphi-\psi)$$

$$=-(r^2+\rho^2)\left[\frac{1}{2}\ln\left(1-2\frac{r}{\rho}\cos(\varphi-\psi)+\left(\frac{r}{\rho}\right)^2\right)\right.$$

$$\left.-\frac{1}{2}\ln\left(1-2\frac{r\rho}{a^2}\cos(\varphi-\psi)+\left(\frac{r\rho}{a^2}\right)^2\right)\right]$$

$$+\underline{\underline{(r^2+\rho^2)\left[\left(\frac{r\rho}{a^2}\right)-\left(\frac{r}{\rho}\right)\right]\cos(\varphi-\psi).}} \tag{4.69}$$

For a partial summation of the series of the last term in (4.64), we change its summation index n by introducing $k=n+1$, yielding

$$r\rho\sum_{n=2}^{\infty}\frac{1}{n+1}\left[\left(\frac{r\rho}{a^2}\right)^{n+1}-\left(\frac{r}{\rho}\right)^{n+1}\right]\cos n(\varphi-\psi)$$

$$=r\rho\sum_{k=3}^{\infty}\frac{1}{k}\left[\left(\frac{r\rho}{a^2}\right)^k-\left(\frac{r}{\rho}\right)^k\right]\cos(k-1)(\varphi-\psi)$$

$$=r\rho\sum_{k=1}^{\infty}\frac{1}{k}\left[\left(\frac{r\rho}{a^2}\right)^k-\left(\frac{r}{\rho}\right)^k\right]\cos(k-1)(\varphi-\psi)$$

$$-r\rho\left[\left(\frac{r\rho}{a^2}\right)-\left(\frac{r}{\rho}\right)\right]-\frac{r\rho}{2}\left[\left(\frac{r\rho}{a^2}\right)^2-\left(\frac{r}{\rho}\right)^2\right]\cos(\varphi-\psi)$$

$$=r\rho\cos(\varphi-\psi)\sum_{k=1}^{\infty}\frac{1}{k}\left[\left(\frac{r\rho}{a^2}\right)^k-\left(\frac{r}{\rho}\right)^k\right]\cos k(\varphi-\psi)$$

$$+r\rho\sin(\varphi-\psi)\sum_{k=1}^{\infty}\frac{1}{k}\left[\left(\frac{r\rho}{a^2}\right)^k-\left(\frac{r}{\rho}\right)^k\right]\sin k(\varphi-\psi)$$

$$-r\rho\left[\left(\frac{r\rho}{a^2}\right)-\left(\frac{r}{\rho}\right)\right]-\frac{r\rho}{2}\left[\left(\frac{r\rho}{a^2}\right)^2-\left(\frac{r}{\rho}\right)^2\right]\cos(\varphi-\psi).$$

Summing the cosine-series in the above expression, and leaving the sine-series in its current form, the last term in (4.64) is finally expressed as

$$
\begin{aligned}
r\rho \sum_{n=2}^{\infty} \frac{1}{n+1}\left[\left(\frac{r\rho}{a^2}\right)^{n+1} - \left(\frac{r}{\rho}\right)^{n+1}\right] \cos n(\varphi - \psi) \\
= r\rho \cos(\varphi - \psi)\left[\frac{1}{2}\ln\left(1 - 2\frac{r}{\rho}\cos(\varphi - \psi) + \left(\frac{r}{\rho}\right)^2\right)\right. \\
\left. - \frac{1}{2}\ln\left(1 - 2\frac{r\rho}{a^2}\cos(\varphi - \psi) + \left(\frac{r\rho}{a^2}\right)^2\right)\right] \\
+ r\rho \sin(\varphi - \psi)\sum_{k=1}^{\infty}\frac{1}{k}\left[\left(\frac{r\rho}{a^2}\right)^k - \left(\frac{r}{\rho}\right)^k\right]\sin k(\varphi - \psi) \\
- \underline{r\rho\left[\left(\frac{r\rho}{a^2}\right) - \left(\frac{r}{\rho}\right)\right]} - \underline{\underline{\frac{r\rho}{2}\left[\left(\frac{r\rho}{a^2}\right)^2 - \left(\frac{r}{\rho}\right)^2\right]\cos(\varphi - \psi)}}. \qquad (4.70)
\end{aligned}
$$

At this point in our development, we substitute equations (4.67), (4.69), and (4.70) into (4.64). Now, the two sine-series (one from (4.67) and another from (4.70)) cancel out. All the logarithmic terms, as well as the two double-underlined terms are combined accordingly yielding, for the series term of equation (4.62)

$$
\begin{aligned}
2\sum_{n=2}^{\infty} g_n(r,\rho)\cos n(\varphi - \psi) \\
= -\left[r^2 - 2r\rho\cos(\varphi - \psi) + \rho^2\right]\ln\frac{\rho^2\left[a^4 - 2a^2 r\rho\cos(\varphi - \psi) + r^2\rho^2\right]}{a^4\left[r^2 - 2r\rho\cos(\varphi - \psi) + \rho^2\right]} \\
+ \underline{r\rho\left[\left(\frac{r\rho}{a^2}\right) - \left(\frac{r}{\rho}\right)\right]} + \underline{\underline{\frac{r(a^2 - \rho^2)}{a^4\rho}\left[r^2(a^2 - \rho^2) + 2a^2\rho^2\right]\cos(\varphi - \psi)}}.
\end{aligned}
$$

Upon substituting this expression, along with (4.63), into (4.62), the two double-underlined terms cancel out, while the two logarithmic and the two once-underlined terms are combined accordingly, yielding the following representation for the Green's function $G(r, \varphi; \rho, \psi)$ of the homogeneous problem corresponding to (4.42)–(4.44).

$$
\begin{aligned}
G(r, \varphi; \rho, \psi) = \frac{1}{16\pi}\left\{\frac{1}{a^2}(a^2 - \rho^2)(a^2 - r^2)\right. \\
\left. - \left[r^2 - 2r\rho\cos(\varphi - \psi) + \rho^2\right]\ln\frac{a^4 - 2a^2 r\rho\cos(\varphi - \psi) + r^2\rho^2}{a^2\left[r^2 - 2r\rho\cos(\varphi - \psi) + \rho^2\right]}\right\}. \qquad (4.71)
\end{aligned}
$$

It is evident that (4.71) is absolutely identical to the classical formula in (4.45): the variables r and ρ in (4.71) are the moduli of the observation point z and the force application point ζ, respectively, whilst the expression

$$r^2 - 2r\rho\cos(\varphi - \psi) + \rho^2$$

represents the square of $|z - \zeta|$. For the expression

$$a^4 - 2a^2 r\rho\cos(\varphi - \psi) + r^2\rho^2$$

in the numerator of the logarithmic function, it can easily be shown that it represents the square of the modulus of $a^2 - z\overline{\zeta}$.

It is worth recalling that what is shown in (4.71) represents the only closed analytical form available in the existing texts in the field for the Green's function for the biharmonic equation. However, the technique which we suggest for its derivation, turns out to be effective in several other cases. In the following, we take advantage of the experience gained from our work in Example 4.5, displaying another closed form Green's function.

Example 4.6. We consider the homogeneous biharmonic equation on the semi-circular region $\Omega = \{(r,\varphi)|0 < r < a, 0 < \varphi < \pi\}$,

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)^2 w(r,\varphi) = 0, \quad (r,\varphi) \in \Omega, \tag{4.72}$$

subject to uniqueness conditions, which are:

(a) the boundary conditions

$$w(a,\varphi) = 0, \qquad \frac{\partial w(a,\varphi)}{\partial r} = 0, \tag{4.73}$$

$$w(r,0) = w(r,\pi) = 0, \quad \frac{\partial^2 w(r,0)}{\partial\varphi^2} = \frac{\partial^2 w(r,\pi)}{\partial\varphi^2} = 0. \tag{4.74}$$

(b) and the conditions of boundedness

$$\lim_{r\to 0} |w(r,\varphi)| < \infty, \quad \lim_{r\to 0} \left|\frac{\partial^2 w(r,\varphi)}{\partial r^2}\right| < \infty. \tag{4.75}$$

Note that the conditions (4.75) are required to make the boundary-value problem in (4.72)–(4.75) well-posed. We have discussed this issue earlier whilst considering Example 4.5. The point is that the governing equation in (4.72) is written in polar coordinates, in which case $r = 0$ represents singularity. Therefore, it is not appropriate to impose a standard set of boundary conditions at $r = 0$.

Omitting the details of the derivation procedure, the specifics of which can be clarified by following the routine developed in Example 4.5, we display only the final expression for the Green's function for the problem defined by (4.72)–(4.75):

$$G(r,\varphi;\rho,\psi)=\frac{1}{16\pi}\left\{[r^2-2r\rho\cos(\varphi+\psi)+\rho^2]\ln\frac{a^4-2a^2r\rho\cos(\varphi+\psi)+r^2\rho^2}{a^2[r^2-2r\rho\cos(\varphi+\psi)+\rho^2]}\right.$$
$$\left.-[r^2-2r\rho\cos(\varphi-\psi)+\rho^2]\ln\frac{a^4-2a^2r\rho\cos(\varphi-\psi)+r^2\rho^2}{a^2[r^2-2r\rho\cos(\varphi-\psi)+\rho^2]}\right\}. \tag{4.76}$$

Note that this closed form Green's function is not available in classical texts but has been published earlier in [45]. In order to better grasp our derivation procedure, we advise the reader to follow its details when working through the exercises at the end of this chapter.

The examples we treated so far have provided a strong confidence in the power of the proposed technique, which turns out to be productive in a number of problems where Green's functions are either not available at all or their existing representations do not meet the requirements of numerical implementations. The next example illustrates this point.

Example 4.7. Consider the problem for the biharmonic equation, the physical interpretation of which is associated with the bending of a circular simply-supported plate. An approach to the construction of the Green's function for this problem, based on complex variable theory, has been described in journal articles long ago (for example, [59]). However, an explicit expression for this Green's function has only been derived quite recently was displayed in [45].

A boundary-value problem suitable to our case is defined on the circular region $\Omega=\{(r,\varphi)|0<r<a, 0\leq\varphi<2\pi\}$ and appears as

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)^2 w(r,\varphi)=0,\quad (r,\varphi)\in\Omega, \tag{4.77}$$

$$w(a,\varphi)=0,\quad \left(\frac{\partial^2}{\partial r^2}+\frac{\sigma}{a}\left(\frac{\partial}{\partial r}+\frac{1}{a}\frac{\partial^2}{\partial\varphi^2}\right)\right)w(a,\varphi)=0, \tag{4.78}$$

$$\lim_{r\to 0}|w(r,\varphi)|<\infty,\quad \lim_{r\to 0}\left|\frac{\partial^2 w(r,\varphi)}{\partial r^2}\right|<\infty, \tag{4.79}$$

where σ represents the Poisson ratio of an elastic isotropic homogeneous material, of which the plate is made.

A specific form of the second condition in (4.78) follows from the physical interpretation of the problem, where the radial bending moment $M_r(r,\varphi)$, is expressed in terms of the deflection function $w(r,\varphi)$ as

$$M_r(r,\varphi)=\left(\frac{\partial^2}{\partial r^2}+\frac{\sigma}{r}\left(\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial^2}{\partial\varphi^2}\right)\right)w(r,\varphi).$$

Notice that in light of the first condition in (4.78) we can rewrite the second one in a more compact form: since $w(r,\varphi)$ is supposed to be equal to zero along the boundary $r=a$ of Ω, all the derivatives of $w(r,\varphi)$, with respect to the tangential variable φ must also be zero. Hence, the conditions in (4.78) reduce to

$$w(a,\varphi)=0,\qquad \frac{\partial^2 w(a,\varphi)}{\partial r^2}+\frac{\sigma}{a}\frac{\partial w(a,\varphi)}{\partial r}=0. \tag{4.80}$$

Hence, what we are searching for is the Green's function for the boundary-value problem in (4.77), (4.79) and (4.80). Following the procedure described in detail in Example 4.5, we obtain the Green's function in series form as in (4.61), the coefficients of which are, in this case, found for $r\le\rho$

$$g_0(r,\rho)=\frac{1}{8}\left\{\frac{a^2-\rho^2}{a^2}\left[(a^2+r^2)+\frac{1}{\omega}(a^2-r^2)\right]+2(r^2+\rho^2)\ln\left(\frac{\rho}{a}\right)\right\},$$

$$g_1(r,\rho)=\frac{1}{8}\left\{\frac{\omega}{1+\omega}\left[(r^2+\rho^2)-a^2\right]\frac{r\rho}{a^2}-\frac{r^3}{2\rho}\left(1-\frac{1-\omega}{1+\omega}\frac{\rho^4}{a^4}\right)-2r\rho\ln\left(\frac{\rho}{a}\right)\right\}$$

and

$$\begin{aligned}g_n(r,\rho)=-\frac{1}{8}\Bigg\{&\frac{r\rho}{n-1}\left[\left(\frac{r\rho}{a^2}\right)^{n-1}-\left(\frac{r}{\rho}\right)^{n-1}\right]+\frac{r\rho}{n+1}\left[\left(\frac{r\rho}{a^2}\right)^{n+1}-\left(\frac{r}{\rho}\right)^{n+1}\right]\\&-\frac{r^2+\rho^2}{n}\left[\left(\frac{r\rho}{a^2}\right)^{n}-\left(\frac{r}{\rho}\right)^{n}\right]+\frac{1}{n+\omega}\frac{(r^2-a^2)(a^2-\rho^2)}{a^2}\left(\frac{r\rho}{a^2}\right)^{n}\Bigg\},\end{aligned}$$

where, we introduced ω, in terms of the Poisson ratio of the material, for compactness, as $\omega=(1+\sigma)/2$.

The series in (4.61), with the coefficients shown above, cannot be summed entirely, due to the last term in $g_n(r,\rho)$, containing the parameter ω. A partial summation, though, provides a quite compact formula for the Green's function we are dealing with. That is

$$\begin{aligned}&G(r,\varphi;\rho,\psi)\\&=\frac{1}{16\pi}\Bigg\{\frac{(a^2-\rho^2)(a^2-r^2)}{a^2}\left[\frac{1+\omega}{\omega}+2\sum_{n=1}^{\infty}\frac{1}{n+\omega}\left(\frac{r\rho}{a^2}\right)^n\cos n(\varphi-\psi)\right]\\&\quad-\left[r^2-2r\rho\cos(\varphi-\psi)+\rho^2\right]\ln\frac{a^4-2a^2r\rho\cos(\varphi-\psi)+r^2\rho^2}{a^2[r^2-2r\rho\cos(\varphi-\psi)+\rho^2]}\Bigg\}.\end{aligned} \tag{4.81}$$

This formula can be convenient for computing values of the Green's function $G(r,\varphi;\rho,\psi)$ inside of the circle, if both the field point (r,φ) and the force application point (ρ,ψ) are far enough from the contour of Ω: in this case, the term $r\rho/a^2$ is significantly less than one and the series in (4.81) rapidly converges. However, notice that the rate of convergence of the series depends on the proximity of r and ρ to the contour of Ω: the convergence slows down notably if both the field and the force application point approach the contour $r = a$ of Ω (both r and ρ approach a).

Hence, to make the formula in (4.81) computer-friendly and free of its dependence on the location of the points (r,φ) and (ρ,ψ) , we must improve the convergence of its series. In the following, we will show that the practicality of (4.81) improves radically if we split off the slowly converging series component, followed by its summation. In doing so, the coefficient of the series in (4.81) is displayed in the formula

$$\frac{1}{n+\omega} = \frac{1}{n} - \frac{\omega}{n(n+\omega)}$$

yielding for the series itself

$$\begin{aligned}
2\sum_{n=1}^{\infty} &\frac{1}{n+\omega}\left(\frac{r\rho}{a^2}\right)^n \cos n(\varphi-\psi) \\
&= 2\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{\omega}{n(n+\omega)}\right]\left(\frac{r\rho}{a^2}\right)^n \cos n(\varphi-\psi) \\
&= 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r\rho}{a^2}\right)^n \cos n(\varphi-\psi) \\
&\quad - 2\omega\sum_{n=1}^{\infty}\frac{1}{n(n+\omega)}\left(\frac{r\rho}{a^2}\right)^n \cos n(\varphi-\psi).
\end{aligned}$$

The first of the above two series is summable with the help of the standard summation formula from (4.66), yielding

$$\begin{aligned}
2\sum_{n=1}^{\infty} &\frac{1}{n+\omega}\left(\frac{r\rho}{a^2}\right)^n \cos n(\varphi-\psi) \\
&= -\ln\left(1 - 2\frac{r\rho}{a^2}\cos(\varphi-\psi) + \left(\frac{r\rho}{a^2}\right)^2\right) \\
&\quad - 2\omega\sum_{n=1}^{\infty}\frac{1}{n(n+\omega)}\left(\frac{r\rho}{a^2}\right)^n \cos n(\varphi-\psi).
\end{aligned}$$

In view of the previous transformation, the expression of (4.81) for the Green's function for the problem defined by (4.77), (4.79) and (4.80) can be rewritten as

$$G(r,\varphi;\rho,\psi) \tag{4.82}$$
$$= \frac{1}{16\pi}\left\{\frac{(a^2-\rho^2)(a^2-r^2)}{a^2}\left[\frac{1+\omega}{\omega} - \ln\left(1-2\frac{r\rho}{a^2}\cos(\varphi-\psi)+\left(\frac{r\rho}{a^2}\right)^2\right)\right.\right.$$
$$\left. - 2\omega\sum_{n=1}^{\infty}\frac{1}{n(n+\omega)}\left(\frac{r\rho}{a^2}\right)^n\cos n(\varphi-\psi)\right]$$
$$\left. - \left[r^2-2r\rho\cos(\varphi-\psi)+\rho^2\right]\ln\frac{a^4-2a^2r\rho\cos(\varphi-\psi)+r^2\rho^2}{a^2\left[r^2-2r\rho\cos(\varphi-\psi)+\rho^2\right]}\right\}.$$

Upon close analysis, this version of the Green's function turns out to be superior to that in (4.81), the point being that the above series converges at a faster rate than the one in (4.81). It is evident that the rate of convergence of the series in (4.82) is of order $1/n^2$ (in other words, the convergence is uniform, and does not depend on the location of the points (r,φ) and (ρ,ψ)). In contrast, as we already mentioned, the series in (4.81) converges non-uniformly, implying that for (4.82) to be used successfully in the practical computation of $G(r,\varphi;\rho,\psi)$, it is sufficient to truncate its series appropriately.

To provide the reader with an instrumental approach to effective using of (4.82), consider the Nth partial sum

$$\sum_{n=1}^{N}\frac{1}{n(n+\omega)}\left(\frac{r\rho}{a^2}\right)^n\cos n(\varphi-\psi)$$

of its series component and estimate its Nth remainder. That is

$$|R_N(r,\varphi;\rho,\psi)| = \left|\sum_{n=N+1}^{\infty}\frac{1}{n(n+\omega)}\left(\frac{r\rho}{a^2}\right)^n\cos n(\varphi-\psi)\right| \le \sum_{n=N+1}^{\infty}\frac{1}{n(n+\omega)}$$
$$\le \sum_{n=N+1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{N}\frac{1}{n^2} = \frac{\pi^2}{6} - \sum_{n=1}^{N}\frac{1}{n^2}.$$

Hence, the compact estimate of the remainder

$$|R_N(r,\varphi;\rho,\psi)| \le \frac{\pi^2}{6} - \sum_{n=1}^{N}\frac{1}{n^2}$$

that we found, provides us with an effective tool to appropriately truncate the series in (4.82).

The expression in (4.82) can be written in a more compact form by introducing complex variable notation for the observation point $z = r(\cos\varphi + i\sin\varphi)$ and the force application point $\zeta = \rho(\cos\psi + i\sin\psi)$. After some elementary algebra, the logarithmic terms of (4.82) can be combined and rearranged, yielding

$$G(r,\varphi;\rho,\psi) = \frac{1}{8\pi}\left\{|z-\zeta|^2 \ln\frac{|z-\zeta|}{a} - \frac{|a^2 - z\bar{\zeta}|^2}{a^2}\ln\frac{|a^2 - z\bar{\zeta}|}{a^2}\right. \tag{4.83}$$
$$\left. + \frac{(a^2-\rho^2)(a^2-r^2)}{2a^2}\left[\frac{1+\omega}{\omega} - 2\omega\sum_{n=1}^{\infty}\frac{1}{n(n+\omega)}\left(\frac{r\rho}{a^2}\right)^n \cos n(\varphi-\psi)\right]\right\}.$$

Note that the first logarithmic term in the above expression contains the fundamental solution of the biharmonic equation

$$|z-\zeta|^2 \ln|z-\zeta|$$

This term represents the singular component of the Green's function. We need to clarify using the word 'singular', which is conditionally applied to the above term: the term itself is not singular, because its limit is finite for z approaching ζ and is, in fact, zero

$$\lim_{z\to\zeta} |z-\zeta|^2 \ln|z-\zeta| = 0.$$

This can be easily verified if we apply L'Hôpital's rule to the above limit. Nevertheless, we use the word 'singular' with respect to the first logarithmic term in (4.83), to highlight the fact that the stress-related components associated with this term (bending moments and shear forces) contain a logarithmic singularity and even higher order singularities. This remains in agreement with the known fact that the bending moments and shear forces, generated in a Poisson–Kirchhoff plate subject to a transverse point force, are theoretically unbounded at the force application point [72].

In the next example, we will consider another boundary-value problem for the biharmonic equation on the circle. This problem allows a meaningful interpretation within the Kirchhoff–Poisson plate theory.

Example 4.8. We formulate the problem of interest in physical terms. Let the plate's edge be clamped elastically such that the radial slope of the deflection function $w(r,\varphi)$ is zero at $r = a$, whilst the shear force $Q(r,\varphi)$, written in terms of the deflection function [72]

$$Q(r,\varphi) = D\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}\right)w(r,\varphi)$$

is directly proportional to the deflection at $r = a$. Hence, the statement of the problem is formalized as

$$\frac{\partial w(a,\varphi)}{\partial r} = 0, \quad D\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right) w(a,\varphi) = Cw(a,\varphi), \qquad (4.84)$$

where the constant parameter D is called the plate's flexural rigidity, whilst the constant parameter C represents the coefficient of the elastic edge support.

Treating the second boundary condition in (4.84) is quite cumbersome, but rewriting it as

$$\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial r^2} - \frac{2}{r^3}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial^3}{\partial r \partial \varphi^2}$$

suggests that, in compliance with the first condition in (4.84), it can be simplified, transforming the boundary conditions in (4.84) to

$$\frac{\partial w(a,\varphi)}{\partial r} = 0, \quad \left(\frac{\partial^3}{\partial r^3} + \frac{1}{a}\frac{\partial^2}{\partial r^2} - \frac{2}{a^3}\frac{\partial^2}{\partial \varphi^2}\right) w(a,\varphi) = k^* w(a,\varphi) \qquad (4.85)$$

where the parameter k^* is defined, in terms of the physical constants D and C, as $k^* = C/D$.

Hence, our aim in this example is the Green's function $G(r,\varphi;\rho,\psi)$ for the boundary-value problem in (4.77), (4.79) and (4.85). Following our procedure, as described in detail in Example 4.5, $G(r,\varphi;\rho,\psi)$ can be obtained similar to the expansion in (4.61), where the coefficients $g_0(r,\rho)$, $g_1(r,\rho)$ and $g_n(r,\rho)$ have to be found.

The coefficient $g_0(r,\rho)$, represents Green's function for the following boundary-value problem

$$\left(r^2\frac{d^4}{dr^4} + 2r\frac{d^3}{dr^3} - \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) w(r) = 0,$$

$$\lim_{r\to 0} |w(r)| < \infty, \quad \lim_{r\to 0} \left|\frac{d^2 w(r)}{dr^2}\right| < \infty$$

and

$$\frac{dw(a)}{dr} = 0, \quad \frac{d^3 w(a)}{dr^3} + \frac{1}{a}\frac{d^2 w(a)}{dr^2} - k^* w(a) = 0.$$

Due to the above problem self-adjointness, the Green's function $g_0(r,\rho)$ must be symmetric, implying $g_0(r,\rho) = g_0(\rho,r)$. Hence, it suffices to display an expression that is valid for either $r \le \rho$ or $\rho \le r$: once we have found one of them, we can find the other one by exchanging the variables r and ρ. The branch valid for $r \le \rho$ is finally found to be

$$g_0(r,\rho) = \frac{1}{8}\left[\frac{1}{a^2}(a^2 + r^2)(a^2 - \rho^2) + 2(r^2 + \rho^2)\ln\frac{\rho}{a} + \frac{8}{k^* a}\right]. \qquad (4.86)$$

The coefficient $g_1(r, \rho)$ in the expansion (4.61) represents the Green's function to another self-adjoint boundary-value problem

$$\left(r^2\frac{d^4}{dr^4} + 2r\frac{d^3}{dr^3} - 3\frac{d^2}{dr^2} + \frac{3}{r}\frac{d}{dr} + \frac{3}{r^2}\right) w(r) = 0,$$

$$\lim_{r\to 0} |w(r)| < \infty, \quad \lim_{r\to 0}\left|\frac{d^2 w(r)}{dr^2}\right| < \infty$$

and

$$\frac{dw(a)}{dr} = 0, \quad \frac{d^3 w(a)}{dr^3} + \frac{1}{a}\frac{d^2 w(a)}{dr^2} + \frac{2 - k^* a^3}{a^3} w(a) = 0.$$

An expression for $g_1(r, \rho)$, valid for $r \le \rho$ is found as

$$g_1(r, \rho) = -\frac{1}{\Delta^*}\{k^* a^3 [r(\rho^2 - a^2)(r^2(\rho^2 - a^2) - 2a^2\rho^2)] + 2ra^2[r^2(a^2 + 2\rho^2) + 2\rho^2(\rho^2 - 7a^2)]\} - \frac{r\rho}{4}\ln\frac{\rho}{a} \tag{4.87}$$

with $\Delta^* = 16a^4\rho(4 + k^* a^3)$.

In order to find the coefficient $g_n(r, \rho)$ for (4.61), we consider the self-adjoint boundary-value problem

$$\left(r^2\frac{d^4}{dr^4} + 2r\frac{d^3}{dr^3} - (1 + 2n^2)\frac{d^2}{dr^2} + \frac{1 + 2n^2}{r}\frac{d}{dr} + \frac{n^2(n^2 - 4)}{r^2}\right) w(r) = 0,$$

$$\lim_{r\to 0} |w(r)| < \infty, \quad \lim_{r\to 0}\left|\frac{d^2 w(r)}{dr^2}\right| < \infty$$

and

$$\frac{dw(a)}{dr} = 0, \quad \frac{d^3 w(a)}{dr^3} + \frac{1}{a}\frac{d^2 w(a)}{dr^2} + \frac{2n^2 - k^* a^3}{a^3} w(a) = 0$$

the Green's function of which, for $r \le \rho$, reads

$$\begin{aligned} g_n(r, \rho) = -\frac{1}{\Delta^{**}}\Bigg\{ &\frac{r^n}{n-1}\big[\underline{n(4(2 - n^2) + k^* a^3)}a^{2-2n}\rho^n \\ &- \underline{(2n^2(n+1) + k^* a^3)}\rho^{2-n} + \underline{(n-1)(2n^2 - k^* a^3)}a^{-2n}\rho^{n+2}\big] \\ &+ \big[\underline{(n+1)(2n^2 - k^* a^3)}a^{-2n}\rho^n + \underline{(2n^2(n+1) + k^* a^3)}\rho^{-n} \\ &+ nk^* a^3 a^{-2n-2}\rho^{n+2}\big]\frac{r^{n+2}}{n+1}\Bigg\} \end{aligned} \tag{4.88}$$

with $\Delta^{**} = 8n[2n^2(n+1) + k^* a^3]a^{2n-2}$.

Upon close analysis, we see that, if the parameter k^* goes to infinity, the expressions for $g_0(r,\rho)$, $g_1(r,\rho)$, and $g_n(r,\rho)$ in (4.86), (4.87) and (4.88) reduce to those found earlier in Example 4.5 (see (4.54), (4.57) and (4.60), respectively). This does not come as a surprise, because the setting in (4.77), (4.79) and (4.85) does in fact reduce, in this case, to that of (4.42)–(4.44) (if $f(r,\varphi) = 0$), the coefficients of the expansion of its Green's function in (4.61) are given by (4.54), (4.57) and (4.60)

An interesting observation arises upon analysis of the representation for the Green's function of the problem in (4.77), (4.79) and (4.85), that we found in (4.61). Analyzing the underlined components of (4.88), we assert that the series in (4.61) is computer-friendly, allowing for an accurate assessment after appropriate truncation of the series: it turns out to be uniformly convergent, with the rate of convergence of order $1/n^2$ for finite values of k^*. If, however, k^* goes to infinity, then the k^* containing terms in the underlined components of (4.88) and in Δ^{**} become dominant, reducing the rate of convergence to order $1/n$.

To complete our discussion in this section, we note that, by applying the essentials of our approach, the reader can proceed to construct Green's functions for other applied boundary-value problems for the biharmonic equation. In the next section we will implement our approach to another fourth order elliptic equation which finds important applications in the Kirchhoff–Poisson plate theory.

4.4 The equation $\nabla^2\nabla^2 w(P) + \lambda^4 w(P) = 0$

The routine developed in Sections 4.2 and 4.3 also turns out to be helpful in the construction of Green's functions for a variety of settings for the equation that is the title of the current section. This equation arises, in particular, in Poisson–Kirchhoff plate theory [72], when plates resting on a simple (single parameter) elastic foundation are considered.

Example 4.9. To set up a problem for this example, we formulate it in physical terms. We present another – classical – statement, within the scope of the Kirchhoff–Poisson plate theory. Consider bending a simply-supported rectangular plate, resting on an elastic foundation. The plate, of uniform thickness, is made from a homogeneous isotropic elastic material and is subjected to a distributed lateral load $q(x,y)$. Let the plate's middle plane occupy the region $\Omega = \{(x,y) | 0 < x < a, 0 < y < b\}$, and let the coefficient of elastic foundation be denoted by λ_0. The boundary-value problem corresponding to this physical setting is:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2}\right) + \lambda^4 w(x,y) = -f(x,y), \tag{4.89}$$

$$w = \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0,b} = 0, \quad w = \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0,a} = 0 \tag{4.90}$$

with the coefficient in (4.89) denoted, for convenience, as λ^4. and defined in terms of the elastic coefficient of the foundation λ_0, and the plate's flexural rigidity D as $\lambda^4 = \lambda_0/D$. The right-hand side function in (4.89) is defined in terms of the load function $q(x, y)$ as $f(x, y) = q(x, y)/D$.

Our approach to the construction of Green's function for the homogeneous problem corresponding to that of (4.89) and (4.90) is similar to the cases of other elliptic equations that we considered earlier in this book. We take advantage of the fact that if the solution of the problem in (4.89) and (4.90) is found in integral form

$$w(x, y) = \int_0^a \int_0^b G(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta, \tag{4.91}$$

then the kernel $G(x, y; \xi, \eta)$ in the above represents the Green's function for the corresponding homogeneous problem.

To obtain $w(x, y)$, the solution to the problem in (4.89) and (4.90), we express it as a double Fourier sine-series

$$w(x, y) = \sum_{m,n=1}^{\infty} w_{mn} \sin \mu x \sin \nu y, \quad \mu = \frac{m\pi}{a}, \ \nu = \frac{n\pi}{b}. \tag{4.92}$$

It is evident that the above formula for $w(x, y)$ satisfies all the boundary conditions in (4.90).

In addition, we express the right-hand side function of (4.89), $f(x, y)$, as the identical double sine-series

$$f(x, y) = \sum_{m,n=1}^{\infty} f_{mn} \sin \mu x \sin \nu y. \tag{4.93}$$

Now, substituting (4.92) and (4.93) into (4.89) and combining like terms in the left-hand side yields the equation

$$\sum_{m,n=1}^{\infty} \left(\mu^4 + 2\mu^2\nu^2 + \nu^4 + \lambda^4\right) w_{mn} \sin \mu x \sin \nu y = -\sum_{m,n=1}^{\infty} f_{mn} \sin \mu x \sin \nu y$$

from which, after equating the corresponding coefficients of both the series and performing some trivial algebra, we obtain

$$w_{mn} = -\frac{f_{mn}}{(\mu^2 + \nu^2)^2 + \lambda^4}.$$

Substituting this expression into (4.92) yields

$$w(x, y) = -\sum_{m,n=1}^{\infty} \frac{f_{mn}}{(\mu^2 + \nu^2)^2 + \lambda^4} \sin \mu x \sin \nu y. \tag{4.94}$$

Recalling the Euler–Fourier formula and adapting it to the Fourier double-series context, we get for the Fourier coefficients f_{mn} of the right-hand side term in (4.93)

$$f_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(\xi,\eta)\sin\mu\xi\sin\nu\eta d\xi d\eta.$$

We now substitute f_{mn} into (4.94) and exchange the order of the summation and integration in it, giving us the solution to the boundary-value problem in (4.89) and (4.90) as

$$w(x,y) = -\frac{4}{ab}\int_0^a\int_0^b\sum_{m,n=1}^{\infty}\frac{\sin\mu x\sin\mu\xi\sin\nu y\sin\nu\eta}{(\mu^2+\nu^2)^2+\lambda^4}f(\xi,\eta)d\xi d\eta. \quad (4.95)$$

Hence, in light of the relation (4.91), we conclude that the kernel in (4.95)

$$G(x,y;\xi,\eta) = -\frac{4}{ab}\sum_{m,n=1}^{\infty}\frac{\sin\mu x\sin\mu\xi\sin\nu y\sin\nu\eta}{(\mu^2+\nu^2)^2+\lambda^4} \quad (4.96)$$

represents the Green's function for the homogeneous boundary-value problem corresponding to (4.89) and (4.90).

It is clear that the above series converges at the same rate as the series in (4.11) of Section 4.2. This implies that $G(x,y;\xi,\eta)$ can be accurately computed by truncating the series in (4.96). Similar to the situation in Example 4.1 of Section 4.2, an accurate assessment of higher order derivatives of $G(x,y;\xi,\eta)$ in the vicinity of the force application point requires special attention.

Example 4.10. We now turn to the construction of the Green's function for the boundary-value problem

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 w(x,y)}{\partial x^2}+\frac{\partial^2 w(x,y)}{\partial y^2}\right)+\lambda^4 w(x,y) = f(x,y), \quad (x,y)\in\Omega, \quad (4.97)$$

$$w = \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0,b} = 0, \quad w = \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = 0 \quad (4.98)$$

on the semi-infinite strip $\Omega = \{(x,y)|0 < x < \infty, 0 < y < b\}$. Note that, to make the above setting well-posed (in other words, allowing a unique solution), the function $w(x,y)$ must, in addition to (4.98), be subjected to conditions of boundedness as x goes to infinity, which we formulate them as

$$\lim_{x\to\infty}|w(x,y)| < \infty, \quad \lim_{x\to\infty}\left|\frac{\partial w(x,y)}{\partial x}\right| < \infty. \quad (4.99)$$

It is evident that by expressing the solution $w(x, y)$ to the boundary-value problem in (4.97)–(4.99) in the Fourier sine-series

$$w(x, y) = \sum_{n=1}^{\infty} w_n(x) \sin \nu y, \quad \nu = \frac{n\pi}{b}, \tag{4.100}$$

the first two boundary conditions in (4.98) are automatically satisfied. If, in addition, the right-hand side of the equation in (4.97) is expanded as

$$f(x, y) = \sum_{n=1}^{\infty} f_n(x) \sin \nu y \tag{4.101}$$

and (4.100) and (4.101) are substituted into (4.97)–(4.99), the following set of boundary-value problems ($n = 1, 2, 3, \ldots$) arises for the coefficients $w_n(x)$ in (4.100):

$$\frac{d^4 w_n(x)}{dx^4} - 2\nu^2 \frac{d^2 w_n(x)}{dx^2} + (\nu^4 + \lambda^4) w_n(x) = -f_n(x), \quad x \in (0, \infty), \tag{4.102}$$

$$w_n(0) = \frac{d^2 w_n(0)}{dx^2} = 0, \quad \lim_{x \to \infty} |w_n(x)| < \infty, \lim_{x \to \infty} \left| \frac{dw_n(x)}{dx} \right| < \infty. \tag{4.103}$$

Following again the method of variation of parameters as applied to the problem in (4.102) and (4.103), we turn to finding a fundamental set of solutions of the homogeneous equation corresponding to (4.102). We start by writing the characteristic equation of the latter

$$k^4 - 2\nu^2 k^2 + (\nu^4 + \lambda^4) = 0$$

whose solution set is represented by the four complex numbers

$$k_j = \pm\sqrt{\nu^2 \pm i\lambda^2}, \quad j = \overline{1, 4} \tag{4.104}$$

expressed in terms of the parameters ν and λ. Note that the above square roots, as well as those in the development that follows, are understood in principal (arithmetic) values.

Upon separating real and imaginary parts of k_j, we write the expression under the square root sign in (4.104) in trigonometric form as

$$\nu^2 \pm i\lambda^2 = \sqrt{\nu^4 + \lambda^4} \left[\cos\left(\arctan \frac{\lambda^2}{\nu^2} \right) \pm i \sin\left(\arctan \frac{\lambda^2}{\nu^2} \right) \right].$$

This transforms the roots k_j of the characteristic equation to

$$k_j = \pm \sqrt[4]{\nu^4 + \lambda^4} \left[\cos\left(\frac{1}{2} \arctan \frac{\lambda^2}{\nu^2} \right) \pm i \sin\left(\frac{1}{2} \arctan \frac{\lambda^2}{\nu^2} \right) \right] \tag{4.105}$$

which can be rewritten after recalling the trigonometric half-angle identities

$$\cos\left(\frac{1}{2}\arctan\frac{\lambda^2}{\nu^2}\right) = \frac{\sqrt{2}}{2}\sqrt{1+\cos\left(\arctan\frac{\lambda^2}{\nu^2}\right)}$$
$$= \frac{\sqrt{2}}{2}\sqrt{1+\frac{\nu^2}{\sqrt{\nu^4+\lambda^4}}} = \frac{\sqrt{2}}{2}\frac{\sqrt{\sqrt{\nu^4+\lambda^4}+\nu^2}}{\sqrt[4]{\nu^4+\lambda^4}}$$

and

$$\sin\left(\frac{1}{2}\arctan\frac{\lambda^2}{\nu^2}\right) = \frac{\sqrt{2}}{2}\sqrt{1-\cos\left(\arctan\frac{\lambda^2}{\nu^2}\right)}$$
$$= \frac{\sqrt{2}}{2}\sqrt{1-\frac{\nu^2}{\sqrt{\nu^4+\lambda^4}}} = \frac{\sqrt{2}}{2}\frac{\sqrt{\sqrt{\nu^4+\lambda^4}-\nu^2}}{\sqrt[4]{\nu^4+\lambda^4}}.$$

Hence, we express the four complex numbers in (4.105) as

$$k_j = \pm\frac{\sqrt{2}}{2}\left(\sqrt{\sqrt{\nu^4+\lambda^4}+\nu^2} \pm i\sqrt{\sqrt{\nu^4+\lambda^4}-\nu^2}\right)$$

representing the solution set of the characteristic equation for the homogeneous equation corresponding to (4.102).

Hence, the following four linearly independent functions represent a fundamental set of solutions to the homogeneous equation corresponding to (4.102):

$$\begin{aligned} w_{n,1}(x) &= e^{\alpha x}\cos\beta x, \quad & w_{n,2}(x) &= e^{\alpha x}\sin\beta x, \\ w_{n,3}(x) &= e^{-\alpha x}\cos\beta x, \quad & w_{n,4}(x) &= e^{-\alpha x}\sin\beta x, \end{aligned} \tag{4.106}$$

where α and β are defined in terms of ν and λ as

$$\alpha = \frac{\sqrt{2}}{2}\sqrt{\sqrt{\nu^4+\lambda^4}+\nu^2} \quad \text{and} \quad \beta = \frac{\sqrt{2}}{2}\sqrt{\sqrt{\nu^4+\lambda^4}-\nu^2}.$$

Based on the fundamental set of solutions in (4.106), and following the method of variation of parameters, the general solution to the boundary-value problem in (4.102) and (4.103) can be written as a linear combination of the components in (4.106)

$$w_n(x) = \sum_{j=1}^{4} C_j(x) w_{n,j}(x). \tag{4.107}$$

Further following our procedure, we obtain the following system of linear algebraic equations

$$\begin{pmatrix} w_{n,1}(x) & w_{n,2}(x) & w_{n,3}(x) & w_{n,4}(x) \\ w'_{n,1}(x) & w'_{n,2}(x) & w'_{n,3}(x) & w'_{n,4}(x) \\ w''_{n,1}(x) & w''_{n,2}(x) & w''_{n,3}(x) & w''_{n,4}(x) \\ w'''_{n,1}(x) & w'''_{n,2}(x) & w'''_{n,3}(x) & w'''_{n,4}(x) \end{pmatrix} \begin{pmatrix} C'_1(x) \\ C'_2(x) \\ C'_3(x) \\ C'_4(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -f_n(x) \end{pmatrix}$$

for the derivatives of the parameters $C_1(x), C_2(x), C_3(x)$, and $C_4(x)$. The solution of this system is found in compact form as

$$C_1'(x) = \frac{\alpha \sin \beta x + \beta \cos \beta x}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{\alpha x}} f_n(x), \quad C_2'(x) = -\frac{\alpha \cos \beta x - \beta \sin \beta x}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{\alpha x}} f_n(x),$$
$$C_3'(x) = \frac{\alpha \sin \beta x - \beta \cos \beta x}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{-\alpha x}} f_n(x), \quad C_4'(x) = -\frac{\alpha \cos \beta x + \beta \sin \beta x}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{-\alpha x}} f_n(x).$$

Integrating the above, the functions $C_1(x), C_2(x), C_3(x)$,and $C_4(x)$ themselves are found to be

$$C_1(x) = \int_0^x \frac{\alpha \sin \beta \xi + \beta \cos \beta \xi}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{\alpha \xi}} f_n(\xi) d\xi + M_1,$$
$$C_2(x) = -\int_0^x \frac{\alpha \cos \beta \xi - \beta \sin \beta \xi}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{\alpha \xi}} f_n(\xi) d\xi + M_2,$$
$$C_3(x) = \int_0^x \frac{\alpha \sin \beta \xi - \beta \cos \beta \xi}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{-\alpha \xi}} f_n(\xi) d\xi + M_3,$$

and

$$C_4(x) = -\int_0^x \frac{\alpha \cos \beta \xi + \beta \sin \beta \xi}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{-\alpha \xi}} f_n(\xi) d\xi + M_4.$$

After substituting these into (4.107), we obtain $w_n(x)$ in the form

$$\begin{aligned} w_n(x) = {} & \left[\int_0^x \frac{\alpha \sin \beta \xi + \beta \cos \beta \xi}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{\alpha \xi}} f_n(\xi) d\xi + M_1\right] \mathrm{e}^{\alpha x} \cos \beta x \\ & + \left[-\int_0^x \frac{\alpha \cos \beta \xi - \beta \sin \beta \xi}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{\alpha \xi}} f_n(\xi) d\xi + M_2\right] \mathrm{e}^{\alpha x} \sin \beta x \\ & + \left[\int_0^x \frac{\alpha \sin \beta \xi - \beta \cos \beta \xi}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{-\alpha \xi}} f_n(\xi) d\xi + M_3\right] \mathrm{e}^{-\alpha x} \cos \beta x \\ & + \left[-\int_0^x \frac{\alpha \cos \beta \xi + \beta \sin \beta \xi}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{-\alpha \xi}} f_n(\xi) d\xi + M_4\right] \mathrm{e}^{-\alpha x} \sin \beta x. \end{aligned} \tag{4.108}$$

It is evident that, since the function $w_n(x)$ is assumed to be bounded as x goes to infinity, the factors

$$\int_0^x \frac{\alpha \sin \beta \xi + \beta \cos \beta \xi}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{\alpha \xi}} f_n(\xi) d\xi + M_1$$

and

$$-\int_0^x \frac{\alpha \cos \beta \xi - \beta \sin \beta \xi}{4\alpha\beta(\alpha^2+\beta^2)\mathrm{e}^{\alpha \xi}} f_n(\xi) d\xi + M_2$$

of the unbounded components $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ of the fundamental set of solutions in (4.108) must be set equal to zero when x goes to infinity. This implies

$$M_1 = -\int_0^\infty \frac{\alpha\sin\beta\xi + \beta\cos\beta\xi}{4\alpha\beta(\alpha^2+\beta^2)e^{\alpha\xi}} f_n(\xi)d\xi$$

and

$$M_2 = \int_0^\infty \frac{\alpha\cos\beta\xi - \beta\sin\beta\xi}{4\alpha\beta(\alpha^2+\beta^2)e^{\alpha\xi}} f_n(\xi)d\xi.$$

To obtain the constants M_3 and M_4, we substitute the above expressions for M_1 and M_2 into (4.108) and regroup it as

$$\begin{aligned} w_n(x) = &\int_0^x \frac{\beta\cos\beta(x-\xi)\sinh\beta(x-\xi) - \alpha\sin\beta(x-\xi)\cosh\beta(x-\xi)}{2\alpha\beta(\alpha^2+\beta^2)} f_n(\xi)d\xi \\ &+ \int_0^\infty \frac{\alpha\sin\beta(x-\xi) - \beta\cos\beta(x-\xi)}{4\alpha\beta(\alpha^2+\beta^2)e^{\alpha(\xi-x)}} f_n(\xi)d\xi \\ &+ M_3 e^{-\alpha x}\cos\beta x + M_4 e^{-\alpha x}\sin\beta x. \end{aligned} \tag{4.109}$$

From the first condition in (4.103) it follows that

$$M_3 = \int_0^\infty \frac{\alpha\sin\beta\xi + \beta\cos\beta\xi}{4\alpha\beta(\alpha^2+\beta^2)e^{\alpha\xi}} f_n(\xi)d\xi$$

whilst the second condition in (4.103) implies

$$M_4 = \int_0^\infty \frac{\alpha\cos\beta\xi - \beta\sin\beta\xi}{4\alpha\beta(\alpha^2+\beta^2)e^{\alpha\xi}} f_n(\xi)d\xi.$$

Upon substituting M_3 and M_4 into (4.109), the solution to the boundary-value problem in (4.102) and (4.103) is finally found to be

$$\begin{aligned} w_n(x) = &\int_0^x \frac{\beta\cos\beta(x-\xi)\sinh\beta(x-\xi) - \alpha\sin\beta(x-\xi)\cosh\beta(x-\xi)}{2\alpha\beta(\alpha^2+\beta^2)} f_n(\xi)d\xi \\ &+ \int_0^\infty \frac{1}{\Delta}\{e^{-\alpha(x+\xi)}[\alpha\sin\beta(x+\xi) + \beta\cos\beta(x+\xi)] \\ &\qquad - e^{-\alpha|x-\xi|}[\alpha\sin\beta|x-\xi| + \beta\cos\beta(x-\xi)]\} f_n(\xi)d\xi, \end{aligned}$$

where $\Delta = 4\alpha\beta(\alpha^2+\beta^2)$. We can rewrite this as a single integral

$$w_n(x) = \int_0^\infty g_n(x,\xi) f_n(\xi)d\xi$$

with the kernel $g_n(x,\xi)$

$$g_n(x,\xi)=\frac{1}{\Delta}\{e^{-\alpha(x+\xi)}[\alpha\sin\beta(x+\xi)+\beta\cos\beta(x+\xi)]$$
$$-e^{-\alpha|x-\xi|}[\alpha\sin\beta|x-\xi|+\beta\cos\beta(x-\xi)]\}.$$

Recall the series expansion of (4.101), where the Fourier coefficients $f_n(\xi)$ of the right-hand side term $f(x,y)$ in (4.97) can be written by means of the Euler–Fourier formula as

$$f_n(\xi)=\frac{2}{b}\int_0^b f(\xi,\eta)\sin\nu\eta d\eta$$

leading to the integral representation for $w_n(x)$

$$w_n(x)=\frac{2}{b}\int_0^b\int_0^\infty g_n(x,\xi)\sin\nu\eta f(\xi,\eta)d\xi d\eta.$$

Upon substituting this into (4.100) and changing the order of the summation and the integration, the solution of the boundary-value problem stated by (4.97)–(4.99) is now found as

$$w(x,y)=\int_0^b\int_0^\infty\left(\frac{2}{b}\sum_{n=1}^{\infty}g_n(x,\xi)\sin\nu y\sin\nu\eta\right)f(\xi,\eta)d\xi d\eta. \tag{4.110}$$

Recalling the relation displayed earlier in (4.91) and applying it to the problem in (4.97)–(4.99), we conclude that the kernel of the above integral, namely

$$G(x,y;\xi,\eta)=\frac{2}{b}\sum_{n=1}^{\infty}g_n(x,\xi)\sin\nu y\sin\nu\eta \tag{4.111}$$

represents the Green's function that we are looking for.

4.5 Elliptic Systems

In this section, we will consider specific elliptic systems of applied partial differential equations. Such systems arise in structural mechanics where they model the static equilibrium of thin elastic shells of revolution. In the following, we will generalize the technique for constructing Green's functions, that we developed so far, and adjust it to the construction of Green's matrices for the systems mentioned above.

There is an interesting historical observation to be made, with regard to the application of the Green's function method to problems arising in plate and shell theory. The static equilibrium of thin elastic shells provides us with a class of boundary-value problems in structural mechanics. Efficient computational algorithms, based on this method, have been developed significantly earlier for this class of problems than for other areas of the applied sciences. Indeed, pioneering works on the application of the Green's function method to shell problems were published more than four decades ago (see, for example, [26]), whereas intensive work on this method in other related areas of science was only started at least a decade later.

4.5.1 Construction of Green's Matrices

We consider the geometrically linear elastic equilibrium [72] of a thin shell of revolution, with its middle surface is closed in the longitudinal direction, and the meridian being a smooth (differentiable) curve. Let $x \in (0, l)$ and $\varphi \in (0, 2\pi)$ represent the meridional (latitudinal) and the circumferential (longitudinal), respectively, geographical coordinates of a point of this surface. We formulate a system of partial differential equations modeling the equilibrium state of the shell, to be solved for the displacements

$$\mathbf{\Lambda}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial \varphi}, x\right)\mathbf{U}(x,\varphi) = -\mathbf{F}(x,\varphi) \quad \text{in } \Omega(0 < x < l, 0 < \varphi < 2\pi) \tag{4.112}$$

with

$$\mathbf{U}(x,\varphi) = \begin{pmatrix} u(x,\varphi) \\ v(x,\varphi) \\ w(x,\varphi) \end{pmatrix} \quad \text{and} \quad \mathbf{F}(x,\varphi) = \begin{pmatrix} X(x,\varphi) \\ Y(x,\varphi) \\ Z(x,\varphi) \end{pmatrix}$$

being the vector-function for the displacement of the point (x, φ) on the middle surface and the vector-function for the external load, respectively. $u(x, \varphi)$ and $v(x, \varphi)$ in $\mathbf{U}(x, \varphi)$ represent the components of the displacement vector in the latitudinal and longitudinal direction, respectively, whilst $w(x, \varphi)$ is a component normal to the middle surface. It will be referred to later as the deflection of the shell. $X(x, \varphi)$, $Y(x, \varphi)$ and $Z(x, \varphi)$ represent components of the load vector in the corresponding directions. The coefficients of the elements Λ_{ij} of the matrix-operator $\mathbf{\Lambda}$

$$\mathbf{\Lambda}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial \varphi}, x\right) = \left(\Lambda_{ij}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial \varphi}, x\right)\right)_{3\times 3}$$

in (4.112) are functions of the latitudinal coordinate x.

The total order of the system in (4.112) is eight. In accordance with that, we impose boundary conditions on the boundary segments $x = 0$ and $x = l$ of Ω in the form

$$B_0\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial \varphi}\right)\mathbf{U}(0,\varphi) = 0, \quad B_l\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial \varphi}\right)\mathbf{U}(l,\varphi) = 0 \tag{4.113}$$

with each of B_0 and B_l a 4×3 matrix-operator. So, we impose four boundary conditions on each segment of the boundary of Ω. We assume these conditions to be linearly independent and the boundary-value problem in (4.112) and (4.113) to be well-posed allowing the existence of a unique solution $\mathbf{U}(x, \varphi)$.

In the following, we will focus on the development of an efficient procedure for construction of the Green's matrix $G(x, \varphi; s, \psi)$ for the homogeneous boundary-value problem corresponding to (4.112) and (4.113). In pursuing this goal, we will take advantage of the fact that the solution $\mathbf{U}(x, \varphi)$ of the problem in (4.112) and (4.113) itself can be expressed in terms of $G(x, \varphi; s, \psi)$and the right-hand side vector-function $\mathbf{F}(x, \varphi)$ of (4.112) as

$$\mathbf{U}(x, \varphi) = \iint_{\Omega} G(x, \varphi; s, \psi)\mathbf{F}(s, \psi)d\Omega(s, \psi). \tag{4.114}$$

Note that our problem is 2π-periodic with respect to φ. In developing a procedure for the construction of the Green's function, we take advantage of this feature, and expand the vector-functions $\mathbf{U}(x, \varphi)$ and $\mathbf{F}(x, \varphi)$ into the following trigonometric series

$$\mathbf{U}(x, \varphi) = \sum_{n=0}^{\infty} Q_n(\varphi)\mathbf{U}_n(x), \quad \mathbf{F}(x, \varphi) = \sum_{n=0}^{\infty} Q_n(\varphi)\mathbf{F}_n(x) \tag{4.115}$$

with respect to the φ variable. For the transformation matrix $Q_n(\varphi)$ we choose

$$Q_n(\varphi) = \begin{pmatrix} \cos n\varphi & 0 & 0 \\ 0 & \sin n\varphi & 0 \\ 0 & 0 & \cos n\varphi \end{pmatrix}.$$

Clearly, the expansions (4.115) conform to the 2π-periodicity of our problem, with respect to the φ variable.

The vector-functions $\mathbf{U}_n(x)$ and $\mathbf{F}_n(x)$ in (4.115) are written as

$$\mathbf{U}_n(x) = \begin{pmatrix} u_n(x) \\ v_n(x) \\ w_n(x) \end{pmatrix}, \quad \mathbf{F}_n(x) = \begin{pmatrix} X_n(x) \\ Y_n(x) \\ Z_n(x) \end{pmatrix}$$

with $u_n(x)$, $v_n(x)$ and $w_n(x)$ representing the corresponding Fourier coefficients of $u(x, \varphi)$, $v(x, \varphi)$ and $w(x, \varphi)$, respectively, and $X_n(x)$, $Y_n(x)$ and $Z_n(x)$ the Fourier coefficients of $X(x, \varphi)$, $Y(x, \varphi)$ and $Z(x, \varphi)$, respectively.

Upon substitution of (4.115) into (4.112), we obtain

$$\mathbf{\Lambda}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial \varphi}, x\right)\left[\sum_{n=0}^{\infty} Q_n(\varphi)\mathbf{U}_n(x)\right] = -\sum_{n=0}^{\infty} Q_n(\varphi)\mathbf{F}_n(x)$$

or

$$\sum_{n=0}^{\infty} Q_n(\varphi)\left[\mathbf{\Lambda}_n\left(\frac{d}{dx}, x\right)\mathbf{U}_n(x)\right] = -\sum_{n=0}^{\infty} Q_n(\varphi)\mathbf{F}_n(x).$$

From this, it follows that the vector-functions $\mathbf{U}_n(x)$ must satisfy the following system of ordinary differential equations

$$\mathbf{\Lambda}_n\left(\frac{d}{dx}, x\right)\mathbf{U}_n(x) = -\mathbf{F}_n(x), \quad n = 0, 1, 2, \ldots. \tag{4.116}$$

Since the displacement vector-function $\mathbf{U}(x, \varphi)$ is expressed in terms of the series (4.115), the boundary conditions of (4.113) transform to

$$\sum_{n=0}^{\infty} Q_n(\varphi)\left[B_{0n}\left(\frac{d}{dx}\right)\mathbf{U}_n(0)\right] = 0,$$
$$\sum_{n=0}^{\infty} Q_n(\varphi)\left[B_{ln}\left(\frac{d}{dx}\right)\mathbf{U}_n(l)\right] = 0$$

or

$$B_{0n}\left(\frac{d}{dx}\right)\mathbf{U}_n(0) = 0, \quad B_{ln}\left(\frac{d}{dx}\right)\mathbf{U}_n(l) = 0. \tag{4.117}$$

Hence, the original boundary-value problem in (4.112) and (4.113) has been reduced to a set of the eighth order systems of linear ordinary differential equations ($n = 0, 1, 2, \ldots$).

Proceeding with the construction of the Green's function $G(x, \varphi; s, \psi)$, the solution to the boundary-value problem in (4.116) and (4.117), $\mathbf{U}_n(x)$, must be expressed in integral form.

One feature of the system (4.116) introduces a notable complication: it has (generally speaking) variable coefficients. This does not allow us to find an analytical solution $\mathbf{U}_n(x)$. Hence, we can only obtain a fundamental set of solutions to the homogeneous system corresponding to that in (4.116), as is required in our development, using numerical methods. In order to provide a backdrop to this, we convert the system in (4.116) to the normal form

$$\frac{dy_i(x)}{dx} = \sum_{j=1}^{8} \alpha_{ij}(x) y_j(x) + f_i(x), \quad i = \overline{1, 8}, \tag{4.118}$$

where the unknown functions $y_i(x)$ are defined in terms of the components of the vector-function $\mathbf{U}_n(x)$ as

$$y_1(x) = u_n(x), \quad y_2(x) = \frac{du_n(x)}{dx},$$
$$y_3(x) = v_n(x), \quad y_4(x) = \frac{dv_n(x)}{dx},$$
$$y_5(x) = w_n(x), \quad y_6(x) = \frac{dw_n(x)}{dx},$$

and

$$y_7(x) = \frac{d^2 w_n(x)}{dx^2}, \quad y_8(x) = \frac{d^3 w_n(x)}{dx^3}. \tag{4.119}$$

The coefficients $\alpha_{ij}(x)$ of the system (4.118), and its right-hand side functions $f_i(x)$, are defined by the operator $\mathbf{\Lambda}_n$, and the right-hand side vector-function $\mathbf{F}_n(x)$, respectively.

We will provide the reader with a hint as to how we can express the boundary conditions (4.117) in terms of the newly introduced functions $y_i(x)$: consider a particular case of the problem, and express, for example, the matrix-operators $B_{0n}(d/dx)$ and $B_{ln}(d/dx)$ as

$$B_{0n}\left(\frac{d}{dx}\right) \equiv \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & d/dx \end{pmatrix}, \quad B_{ln}\left(\frac{d}{dx}\right) \equiv \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & d^2/dx^2 \end{pmatrix}.$$

In physical terms, this statement models the case where the shell's edge $x = 0$ is clamped whilst the edge $x = l$ is assumed simply-supported.

Now, in light of the relations from (4.119), we can reformulate the boundary conditions imposed by (4.117) in terms of $y_i(x)$ as

$$\begin{aligned} y_i(0) &= 0, \quad \text{for } i = 1, 3, 5, 6, \\ y_i(l) &= 0, \quad \text{for } i = 1, 3, 5, 7. \end{aligned} \tag{4.120}$$

To obtain the Green's matrix for the homogeneous system

$$\frac{dy_i(x)}{dx} = \sum_{j=1}^{8} \alpha_{ij}(x) y_j(x), \quad i = \overline{1, 8}, \tag{4.121}$$

subject to the boundary conditions of (4.120), a fundamental set of solutions for (4.121) is required. That is, we need a set ($j = 1, 2, \ldots, 8$) of its eight linearly independent vector-solutions $\{y_{ij}(x)\}$. Of course, it can be computed with the aid of standard subroutines for obtaining numerical solutions of Cauchy problems, for systems of the type in (4.121), which are widely available in contemporary computer software. We should, however, put high requirements on numerical methods on which the subroutines are based: the diagonal elements α_{ii} in the coefficient matrix of the system in (4.121) are much smaller than its peripheral elements, which makes the system rigid and numerically unstable. It is worth noting that this phenomenon is well-known to numerical analysts working in shell theory; a number of highly effective numerical methods are already available in the field (see, for example, [2]).

Note that, for cylindrical and conical shells, fundamental sets of solutions can be obtained analytically, since the corresponding systems of differential equations have constant coefficients and hence allow an analytical solution. In Section 4.5.2 the reader will encounter several important comments with regard to possible forms of fundamental sets of solutions that can be used for a cylindrical shell.

In compliance with the Lagrange method of variation of parameters, we can write the general solution of the system in (4.118), in terms of a fundamental set of solutions $\{y_{ij}(x)\}$, as

$$y_i(x) = \sum_{j=1}^{8} y_{ij}(x) C_j(x), \quad i = \overline{1,8}, \tag{4.122}$$

with $C_j(x)$ unknown functions whose derivatives $C_j'(x)$ must satisfy the following system of linear algebraic equations

$$\sum_{j=1}^{8} y_{ij}(x) C_j'(x) = f_i(x), \quad i = \overline{1,8}. \tag{4.123}$$

Recall that the functions $f_i(x)$ are the right-hand side terms of the system in (4.118).

We now write the solution to the system (4.123) in the form

$$C_i'(x) = \sum_{j=1}^{8} y_{ij}^{-1}(x) f_j(x), \quad i = \overline{1,8},$$

with $y_{ij}^{-1}(x)$ representing elements of the inverse of the coefficient matrix of the system in (4.123).

Upon integration of the above expressions for $C_i'(x)$, we obtain the following integral representations for $C_i(x)$

$$C_i(x) = \int_0^x \sum_{j=1}^{8} y_{ij}^{-1}(s) f_j(s) ds + D_i, \quad i = \overline{1,8},$$

with D_i being arbitrary constants of integration, allowing the general solution of the system in (4.118)

$$y_i(x) = \int_0^x \sum_{j=1}^{8} T_{ij}(x,s) f_j(s) ds + \sum_{j=1}^{8} y_{ij}(x) D_j \tag{4.124}$$

with

$$T_{ij}(x,s) = \sum_{k=1}^{8} y_{ik}(x) y_{kj}^{-1}(s).$$

By virtue of the boundary conditions in (4.120), we obtain the following system of linear algebraic equations

$$\sum_{j=1}^{8} r_{ij} D_j = S_i, \quad i = \overline{1,8}, \tag{4.125}$$

for the constants D_i.

The elements r_{ij} of the first four rows in the 8×8 coefficient matrix $R(r_{ij})$ of the system in (4.125) are defined as

$$r_{ij} = y_{mj}(0), \quad i = \overline{1,4}, \ m = 1, 3, 5, 6,$$

whilst the elements of the last four rows of that matrix are defined as

$$r_{ij} = y_{mj}(l), \quad i = \overline{5,8}, \ m = 1, 3, 5, 7.$$

The components S_i of the right-hand side vector of the system in (4.125) are expressed as

$$S_i = \int_0^l \sum_{j=1}^{8} T_{ij}^*(l,s) f_j(s) ds, \quad i = \overline{1,8}, \tag{4.126}$$

with $T_{ij}^*(l,s)$ defined as

$$T_{ij}^*(x,s) = \begin{cases} 0, & i = \overline{1,4}, \\ -T_{ij}(l,s), & i = \overline{5,8}, \end{cases}$$

where $T_{ij}(l,s)$ represent the boundary-values of the functions $T_{ij}(x,s)$ defined earlier through $y_{ik}(x)$ and $y_{kj}^{-1}(s)$ in the equation that immediately follows that of (4.124).

Hence, the solution of the linear system in (4.125) can be found in terms of the inverse of its coefficient matrix $R(r_{ij})$ as

$$D_i = \sum_{j=1}^{8} r_{ij}^{-1} S_j, \quad i = \overline{1,8},$$

where r_{ij}^{-1} represent the elements of the inverse $R^{-1}(r_{ij})$ of $R(r_{ij})$.

Substituting S_j from (4.126) into the above relation, we obtain

$$D_i = \int_0^l \sum_{j=1}^{8} P_{ij}(l,s) f_j(s) ds,$$

with

$$P_{ij}(l,s) = \sum_{k=1}^{8} r_{ik}^{-1} T_{kj}^{*}(l,s).$$

Substitution these D_i into (4.124) yields, for the general solution of the system from (4.118)

$$y_i(x) = \int_0^x \sum_{j=1}^{8} T_{ij}(x,s) f_j(s) ds + \int_0^l \sum_{j=1}^{8} H_{ij}(x,s) f_j(s) ds, \tag{4.127}$$

with

$$H_{ij}(x,s) = \sum_{j=1}^{8} y_{ik}(x) P_{kj}(l,s).$$

The expression for $y_i(x)$ in (4.127) can be written in compact integral form

$$y_i(x) = \int_0^l \sum_{j=1}^{8} g_{ij}(x,s) f_j(s) ds, \quad i = \overline{1,8}, \tag{4.128}$$

in which

$$g_{ij}(x,s) = \begin{cases} H_{ij}(x,s), & x \leq s, \\ H_{ij}(x,s) + T_{ij}(x,s), & x \geq s. \end{cases} \tag{4.129}$$

Hence, the solution of the boundary-value problem stated by (4.118) and (4.120) is found in the form of the definite integral in (4.128). Subsequently, in light of the relations from (4.119), the vector-function

$$\mathbf{U}_n(x) = \begin{pmatrix} u_n(x) \\ v_n(x) \\ w_n(x) \end{pmatrix}$$

representing the solution to the problem in (4.116) and (4.117) can be written as

$$\mathbf{U}_n(x) = \int_0^l G_n(x,s) \mathbf{F}_n(s) ds$$

or, in an extended form

$$\begin{pmatrix} u_n(x) \\ v_n(x) \\ w_n(x) \end{pmatrix} = \int_0^l \begin{pmatrix} g_{12}(x,s) & g_{14}(x,s) & g_{18}(x,s) \\ g_{32}(x,s) & g_{34}(x,s) & g_{38}(x,s) \\ g_{52}(x,s) & g_{54}(x,s) & g_{58}(x,s) \end{pmatrix} \begin{pmatrix} X_n(s) \\ Y_n(s) \\ Z_n(s) \end{pmatrix} ds. \tag{4.130}$$

The elements $g_{ij}(x,s)$ (with $i = 1,3,5$ and $j = 2,4,8$) of the kernel-matrix $G_n(x,s)$ in the integral representation of (4.130) can be found in the 8×8 kernel-matrix from (4.128) and are shown in (4.129).

Upon substitution of the components of $\mathbf{U}_n(x)$ from (4.130) into the first of the expansions in (4.115), and successively applying the Euler–Fourier formula for the coefficients of the second of the expansions from (4.115),

$$\mathbf{F}_n(s) = \frac{\varepsilon_n}{2\pi}\int_0^{2\pi} Q_n(\psi)\mathbf{F}(s,\psi)d\psi, \quad \varepsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n > 0, \end{cases}$$

we obtain the following integral representation for the solution $\mathbf{U}(x,\varphi)$ to the original boundary-value problem defined by (4.112) and (4.113).

$$\mathbf{U}(x,\varphi) = \int_0^l \int_0^{2\pi} G(x,\varphi;s,\psi)\mathbf{F}(s,\psi)dsd\psi. \tag{4.131}$$

Observing the form of the solution $\mathbf{U}(x,\varphi)$, we conclude that, in light of the relation in (4.114), the kernel-matrix

$$G(x,\varphi;s,\psi) = \sum_{n=0}^{\infty}\frac{\varepsilon_n}{2\pi}Q_n(\varphi)G_n(x,s)Q_n(\psi)$$

of the integral in (4.131) is what we set out to find: it represents the Green's matrix of the homogeneous problem corresponding to (4.112) and (4.113).

In the following subsection, we will implement the procedure that we described above to several specific problems defined on a circular cylindrical shell.

4.5.2 Cylindrical Shells

The previous subsection touches upon shells of revolution with an arbitrarily smooth meridian. The appearance of the system (4.112) modeling the elastic equilibrium of a particular shell of revolution depends upon the shape of its meridian $\varphi = \text{const}$. To find an example of a particular form of the system and to give the reader a clear sense of the form in which the Green's matrix might appear, we consider a circular cylindrical shell of radius a, subject to a transverse distributed load $Z = Z(x,\varphi)$.

As mentioned before, in the case of a cylindrical shell loaded with a transverse load, the system in (4.112) has constant coefficients, and appears [72] as

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{1-\sigma}{2a^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1+\sigma}{2a}\frac{\partial^2 v}{\partial x \partial \varphi} - \frac{\sigma}{a}\frac{\partial w}{\partial x} &= 0, \\
\frac{1+\sigma}{2}\frac{\partial^2 u}{\partial x \partial \varphi} + a\frac{1-\sigma}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{a}\frac{\partial^2 v}{\partial \varphi^2} - \frac{1}{a}\frac{\partial w}{\partial \varphi} &= 0, \\
\sigma\frac{\partial u}{\partial x} + \frac{1}{a}\frac{\partial v}{\partial \varphi} - \frac{h^2}{12}\left(a\frac{\partial^4 w}{\partial x^4} + \frac{2}{a}\frac{\partial^4 w}{\partial x^2 \partial \varphi^2} + \frac{1}{a^3}\frac{\partial^4 w}{\partial \varphi^4}\right) - \frac{w}{a} &= -F
\end{aligned} \tag{4.132}$$

with the constants E, σ, and h representing the modulus of elasticity, the Poisson ratio of the material of which the shell is made, and the shell's thickness, respectively. The right-hand side $F = F(x, \varphi)$ is expressed in terms of the load function Z as $F = a(1-\sigma^2)Z/(Eh)$.

Example 4.11. We begin the construction of the Green's matrix to a boundary-value problem for the system in (4.132), modeling the static equilibrium of a section of a cylindrical shell with its middle surface occupying the region $\Omega = \{0 < x < l, 0 < \varphi < b\}$, in which $b < \pi$, an all edges $x = 0$, $x = l$, $\varphi = 0$ and $\varphi = b$ simply-supported.

To set up the corresponding boundary-value problem, we need to know how the boundary conditions of a simple support can be written in terms of the components $u = u(x, \varphi)$, $v = v(x, \varphi)$and $w = w(x, \varphi)$ of the displacement vector-function $\mathbf{U}(x, \varphi)$. In doing so, recall the expressions

$$N_x(x, \varphi) = \frac{Eh}{1-\sigma^2}\left[\frac{\partial u}{\partial x} + \frac{\sigma}{a}\left(\frac{\partial v}{\partial \varphi} - w\right)\right],$$
$$N_\varphi(x, \varphi) = \frac{Eh}{1-\sigma^2}\left[\sigma\frac{\partial u}{\partial x} + \frac{1}{a}\left(\frac{\partial v}{\partial \varphi} - w\right)\right],$$
$$M_x(x, \varphi) = -\frac{Eh^3}{12(1-\sigma^2)}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\sigma}{a^2}\frac{\partial^2 w}{\partial \varphi^2}\right),$$
$$M_\varphi(x, \varphi) = -\frac{Eh^3}{12(1-\sigma^2)}\left(\sigma\frac{\partial^2 w}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2 w}{\partial \varphi^2}\right)$$

for the normal forces $N_x(x, \varphi)$ and $N_\varphi(x, \varphi)$, and the bending moments $M_x(x, \varphi)$ and $M_\varphi(x, \varphi)$ generated in the middle surface of the shell [72].

For the edges $x = 0$ and $x = l$, their simple support implies that the deflection function $w(x, \varphi)$, the normal force $N_\varphi(x, \varphi)$ and the bending moment $M_x(x, \varphi)$ must vanish along these edges:

$$w(x, \varphi)|_{x=0,l} = 0, \quad N_\varphi(x, \varphi)|_{x=0,l} = 0, \quad M_x(x, \varphi)|_{x=0,l} = 0. \tag{4.133}$$

Similarly, we impose the following boundary conditions of simple-support of the edges $\varphi = 0$ and $\varphi = b$

$$w(x, \varphi)|_{\varphi=0,b} = 0, \quad N_x(x, \varphi)|_{\varphi=0,b} = 0, \quad M_\varphi(x, \varphi)|_{\varphi=0,b} = 0. \tag{4.134}$$

In accordance with (4.114), we state that the solution of the inhomogeneous boundary-value problem in (4.132)–(4.134) can be written in the form

$$\begin{pmatrix} u(x, \varphi) \\ v(x, \varphi) \\ w(x, \varphi) \end{pmatrix} = \int_0^b \int_0^l G(x, \varphi; s, \psi) \begin{pmatrix} 0 \\ 0 \\ F(s, \psi) \end{pmatrix} ds d\psi \tag{4.135}$$

with the kernel-matrix

$$G(x,\varphi;s,\psi) = \left(g_{ij}(x,\varphi;s,\psi)\right)_{i,j=\overline{1,3}}$$

representing the Green's matrix to the homogeneous boundary-value problem corresponding to (4.132)–(4.134).

Since the first and the second components of the right-hand side vector in (4.135) are zero, the components of the solution are defined completely in terms of the components of the third column of $G(x,\varphi;s,\psi)$. With this in mind, the matrix form in (4.135) transforms to the three scalar relations

$$\begin{aligned} u(x,\varphi) &= \int_0^b \int_0^l g_{13}(x,\varphi;s,\psi)F(s,\psi)ds d\psi, \\ v(x,\varphi) &= \int_0^b \int_0^l g_{23}(x,\varphi;s,\psi)F(s,\psi)ds d\psi, \\ w(x,\varphi) &= \int_0^b \int_0^l g_{33}(x,\varphi;s,\psi)F(s,\psi)ds d\psi \end{aligned} \tag{4.136}$$

Now, in order to obtain the elements $g_{13}(x,\varphi;s,\psi)$, $g_{23}(x,\varphi;s,\psi)$ and $g_{33}(x,\varphi;s,\psi)$ of the Green's matrix $G(x,\varphi;s,\psi)$, required for the components $u = u(x,\varphi)$, $v = v(x,\varphi)$and $w = w(x,\varphi)$ of the displacement vector $\mathbf{U}(x,\varphi)$, we will require the integral form of (4.136).

Our approach to the solution of the boundary-value problem in (4.132)–(4.134) is based on the specificity of the boundary conditions in (4.133) and (4.134). The latter allow us to express the components of the displacement vector $\mathbf{U}(x,\varphi)$, and the right-hand side function $F(x,\varphi)$ in series-form

$$\begin{aligned} u(x,\varphi) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}\cos\mu x\sin\nu\varphi, \\ v(x,\varphi) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{mn}\sin\mu x\cos\nu\varphi, \\ w(x,\varphi) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}\sin\mu x\sin\nu\varphi \end{aligned} \tag{4.137}$$

and

$$F(x,\varphi) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn}\sin\mu x\sin\nu\varphi \tag{4.138}$$

with μ and ν defined as

$$\mu = \frac{m\pi}{l} \quad \text{and} \quad \nu = \frac{n\pi}{b}.$$

Upon substitution of (4.137) and (4.138) into the system (4.132), we obtain the system of linear algebraic equations

$$\begin{aligned}
&\left(\mu^2+\frac{1-\sigma}{2a^2}\nu^2\right)u_{mn}+\frac{1+\sigma}{2a}\mu\nu v_{mn}+\frac{\sigma}{a}\mu w_{mn}=0,\\
&\frac{1+\sigma}{2}\mu\nu u_{mn}+\left(a\frac{1-\sigma}{2}\mu^2+\frac{1}{a}\nu^2\right)v_{mn}+\frac{1}{a}\nu w_{mn}=0, \qquad (4.139)\\
&\sigma\mu u_{mn}+\frac{\nu}{a}v_{mn}-\left[\frac{1}{a}+\frac{h^2}{12}\left(a\mu^4+\frac{2\mu^2\nu^2}{a}+\frac{\nu^4}{a^3}\right)\right]w_{mn}=-F_{mn}
\end{aligned}$$

for the coefficients of the expansions in (4.137) u_{mn}, v_{mn},and w_{mn}.

The system in (4.139) reduces to a more compact form after multiplying its first equation by a^2, whilst the second and the third equations are multiplied by a. This reduces the above system to

$$\begin{aligned}
&\left(\widetilde{\mu}^2+\frac{1-\sigma}{2}\nu^2\right)u_{mn}+\frac{1+\sigma}{2}\widetilde{\mu}\nu v_{mn}+\sigma\widetilde{\mu}w_{mn}=0,\\
&\frac{1+\sigma}{2}\widetilde{\mu}\nu u_{mn}+\left(\frac{1-\sigma}{2}\widetilde{\mu}^2+\nu^2\right)v_{mn}+\nu w_{mn}=0, \qquad (4.140)\\
&\sigma\widetilde{\mu}u_{mn}+\nu v_{mn}+\left[1+\frac{h^2}{12a^2}\left(\widetilde{\mu}^2+\nu^2\right)^2\right]w_{mn}=-aF_{mn},
\end{aligned}$$

where $\widetilde{\mu}$ is introduced as $\widetilde{\mu}=\mu a$. Now, the determinant Δ of the coefficient matrix

$$\begin{pmatrix}
\widetilde{\mu}^2+\frac{1-\sigma}{2}\nu^2 & \frac{1+\sigma}{2}\widetilde{\mu}\nu & \sigma\widetilde{\mu}\\
\frac{1+\sigma}{2}\widetilde{\mu}\nu & \frac{1-\sigma}{2}\widetilde{\mu}^2+\nu^2 & \nu\\
\sigma\widetilde{\mu} & \nu & 1+\frac{h^2}{12a^2}\left(\widetilde{\mu}^2+\nu^2\right)^2
\end{pmatrix}$$

of the system in (4.140) is expressed as

$$\Delta=12\widetilde{\mu}^4a^2(1-\sigma^2)+h^2(\widetilde{\mu}^2+\nu^2)^4.$$

Based on this we might assume that, since the above determinant can be zero, the system in (4.140), mathematically speaking, is not necessarily well-posed. However, as we learn from the mechanics of materials [19, 56, 71], the Poisson ratio σ of an elastic material ranges from 0 to 0.5. This guarantees the determinant Δ to be nonzero: the expression for Δ represents the sum of two non-negative terms $12\widetilde{\mu}^4a^2(1-\sigma^2)$ and $h^2(\widetilde{\mu}^2+\nu^2)^4$, implying that the system in (4.140) always has a unique solution which is ultimately found as

$$\begin{aligned}
u_{mn}&=\frac{12\widetilde{\mu}a^3}{\Delta}\left[\widetilde{\mu}^2(1+\sigma)-(\widetilde{\mu}^2+\nu^2)\right]F_{mn},\\
v_{mn}&=\frac{12\nu a^3}{\Delta}\left[\widetilde{\mu}^2(1+\sigma)+(\widetilde{\mu}^2+\nu^2)\right]F_{mn}
\end{aligned}$$

and

$$w_{mn} = -\frac{12a^3}{\Delta}(\widetilde{\mu}^2 + \nu^2)^2 F_{mn}.$$

Substituting u_{mn}, v_{mn}, and w_{mn} into (4.137), we can rewrite the components of the displacement vector as

$$u(x,\varphi) = 12a^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\widetilde{\mu}}{\Delta}\left[\widetilde{\mu}^2(1+\sigma) - (\widetilde{\mu}^2+\nu^2)\right] F_{mn}\cos\mu x \sin\nu\varphi,$$

$$v(x,\varphi) = 12a^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\nu}{\Delta}\left[\widetilde{\mu}^2(1+\sigma) + (\widetilde{\mu}^2+\nu^2)\right] F_{mn}\sin\mu x \sin\nu\varphi \quad (4.141)$$

and

$$w(x,\varphi) = -12a^3 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(\widetilde{\mu}^2+\nu^2)^2}{\Delta} F_{mn}\sin\mu x \sin\nu\varphi.$$

The coefficients F_{mn} of the trigonometric series in (4.138) can be expressed in terms of the right-hand side function $F(x,\varphi)$ as

$$F_{mn} = \frac{4}{lb}\int_0^b\int_0^l F(s,\psi)\sin\mu s \sin\nu\psi\, ds\, d\psi.$$

With this for F_{mn} available, the components of the displacement vector become

$$u(x,\varphi) = \int_0^b\int_0^l \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{48\widetilde{\mu}a^3}{lb\Delta}\left[\widetilde{\mu}^2(1+\sigma) - (\widetilde{\mu}^2+\nu^2)\right]$$
$$\times \cos\mu x \sin\nu\varphi \sin\mu s \sin\nu\psi F(s,\psi)\, ds\, d\psi,$$

$$v(x,\varphi) = \int_0^b\int_0^l \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{48\nu a^3}{lb\Delta}\left[\widetilde{\mu}^2(1+\sigma) + (\widetilde{\mu}^2+\nu^2)\right]$$
$$\times \sin\mu x \cos\nu\varphi \sin\mu s \sin\nu\psi F(s,\psi)\, ds\, d\psi \quad (4.142)$$

and

$$w(x,\varphi) = -\int_0^b\int_0^l \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{48a^3}{lb\Delta}(\widetilde{\mu}^2+\nu^2)^2$$
$$\times \sin\mu x \sin\nu\varphi \sin\mu s \sin\nu\psi F(s,\psi)\, ds\, d\psi.$$

Hence, it follows from (4.135) that the components of the third column in the Green's matrix $G(x,\varphi;s,\psi)$ of the homogeneous boundary-value problem corresponding to that in (4.132)–(4.134) are

$$g_{13}(x,\varphi;s,\psi) = \frac{48a^3}{lb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\widetilde{\mu}\left[\widetilde{\mu}^2(1+\sigma)-(\widetilde{\mu}^2+\nu^2)\right] \times \frac{\cos\mu x\sin\nu\varphi\sin\mu s\sin\nu\psi}{12\widetilde{\mu}^4a^2(1-\sigma^2)+h^2(\widetilde{\mu}^2+\nu^2)^4},$$

$$g_{23}(x,\varphi;s,\psi) = \frac{48a^3}{lb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\nu\left[\widetilde{\mu}^2(1+\sigma)+(\widetilde{\mu}^2+\nu^2)\right] \times \frac{\sin\mu x\cos\nu\varphi\sin\mu s\sin\nu\psi}{12\widetilde{\mu}^4a^2(1-\sigma^2)+h^2(\widetilde{\mu}^2+\nu^2)^4} \quad (4.143)$$

and

$$g_{33}(x,\varphi;s,\psi) = -\frac{48a^3}{lb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\widetilde{\mu}^2+\nu^2)^2\frac{\sin\mu x\sin\nu\varphi\sin\mu s\sin\nu\psi}{12\widetilde{\mu}^4a^2(1-\sigma^2)+h^2(\widetilde{\mu}^2+\nu^2)^4}.$$

In terms of a physical interpretation, these represent components of the displacement vector of a point (x,φ) on the middle surface of the shell due to the transverse unit point force acting at a point (s,ψ).

In the following two brief examples, we will demonstrate how we can employ the Green's matrix, the elements of which we just obtained, to determine the components of the displacement vector for the simply supported section Ω of a cylindrical shell loaded with different distributed transverse loads.

Example 4.12. Let the simply supported cylindrical shell section $\{0 < x < l, 0 < \varphi < b\}$ with radius a be loaded with a uniform transverse load Z_0.

After substituting the load function $Z(s,\psi) = Z_0$ into (4.132) and performing the double integration with respect to s and ψ in (4.135), we obtain the components of the displacement vector as

$$u(x,\varphi) = \frac{48a^4(1-\sigma^2)Z_0}{Ehlb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\widetilde{\mu}^2(1+\sigma)-(\widetilde{\mu}^2+\nu^2)\right] \times \frac{(\cos\mu l-1)(\cos\nu b-1)\cos\mu x\sin\nu\varphi}{\nu\left[12\widetilde{\mu}^4a^2(1-\sigma^2)+h^2(\widetilde{\mu}^2+\nu^2)^4\right]},$$

$$v(x,\varphi) = \frac{48a^4(1-\sigma^2)Z_0}{Ehlb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\widetilde{\mu}^2(1+\sigma)+(\widetilde{\mu}^2+\nu^2)\right] \times \frac{(\cos\mu l-1)(\cos\nu b-1)\sin\mu x\cos\nu\varphi}{\widetilde{\mu}\left[12\widetilde{\mu}^4a^2(1-\sigma^2)+h^2(\widetilde{\mu}^2+\nu^2)^4\right]}$$

and

$$w(x,\varphi) = -\frac{48a^4(1-\sigma^2)Z_0}{Ehlb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\widetilde{\mu}^2+\nu^2)^2$$
$$\times \frac{(\cos\mu l - 1)(\cos\nu b - 1)\sin\mu x\sin\nu\varphi}{\widetilde{\mu}\nu\left[12\widetilde{\mu}^4 a^2(1-\sigma^2)+h^2(\widetilde{\mu}^2+\nu^2)^4\right]}.$$

Using the above expressions for the components of the displacement vector, we can obtain the stress-related components of the stress-strain state, caused by the uniform transverse load Z_0. For example, we have, for the the bending moment M_x

$$M_x(x,\varphi) = -\frac{Eh^3}{12(1-\sigma^2)}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\sigma}{a^2}\frac{\partial^2 w}{\partial\varphi^2}\right)$$

after the corresponding differentiation and some elementary algebra, we finally arrive at

$$M_x(x,\varphi) = \frac{4a^2h^2Z_0}{lb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(\widetilde{\mu}^2+\nu^2)^2(\widetilde{\mu}^2+\sigma\nu^2)$$
$$\times \frac{(\cos\mu l - 1)(\cos\nu b - 1)\sin\mu x\sin\nu\varphi}{\widetilde{\mu}\nu\left[12\widetilde{\mu}^4 a^2(1-\sigma^2)+h^2(\widetilde{\mu}^2+\nu^2)^4\right]}.$$

Example 4.13. Consider again the section $\{0 < x < l, 0 < \varphi < b\}$ of a cylindrical shell with radius a and simply-supported edges. Let it be loaded with load $Z(x,\varphi) = Z_0\varphi + Z_1$, representing a linear function of φ.

The integration with respect to s and ψ in (4.135), with $Z(s,\psi) = Z_0\psi + Z_1$, yields the following expressions

$$u(x,\varphi) = -\frac{48a^4(1-\sigma^2)}{Ehlb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\widetilde{\mu}^2(1+\sigma)-(\widetilde{\mu}^2+\nu^2)\right]$$
$$\times\frac{1}{\nu^2}\left[Z_0(\nu b\cos\nu b-\sin\nu b)+\nu Z_1(\cos\nu b-1)\right]$$
$$\times \frac{(\cos\mu l - 1)\cos\mu x\sin\nu\varphi}{\left[12\widetilde{\mu}^4 a^2(1-\sigma^2)+h^2(\widetilde{\mu}^2+\nu^2)^4\right]},$$

$$v(x,\varphi) = -\frac{48a^4(1-\sigma^2)}{Ehlb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\widetilde{\mu}^2(1+\sigma)+(\widetilde{\mu}^2+\nu^2)\right]$$
$$\times\frac{1}{\widetilde{\mu}\nu}\left[Z_0(\nu b\cos\nu b-\sin\nu b)+\nu Z_1(\cos\nu b-1)\right]$$
$$\times \frac{(\cos\mu l - 1)\sin\mu x\cos\nu\varphi}{\left[12\widetilde{\mu}^4 a^2(1-\sigma^2)+h^2(\widetilde{\mu}^2+\nu^2)^4\right]}$$

and

$$w(x,\varphi) = \frac{48a^4(1-\sigma^2)Z_0}{Ehlb} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\widetilde{\mu}^2 + \nu^2)^2$$
$$\times \frac{1}{\widetilde{\mu}\nu^2} [Z_0(\nu b \cos \nu b - \sin \nu b) + \nu Z_1(\cos \nu b - 1)]$$
$$\times \frac{(\cos \mu l - 1) \sin \mu x \sin \nu\varphi}{\left[12\widetilde{\mu}^4 a^2(1-\sigma^2) + h^2(\widetilde{\mu}^2 + \nu^2)^4\right]}$$

for the components of the displacement vector.

We will complete this section by emphasizing an important particular case for a cylindrical shell, namely the so-called axially symmetric problem. We will explain particularities of process of finding a fundamental set of solutions, required for the construction of Green's matrix for the governing system of differential equations. In doing so, we consider the homogeneous system

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} + \frac{1-\sigma}{2a^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1+\sigma}{2a}\frac{\partial^2 v}{\partial x \partial \varphi} - \frac{\sigma}{a}\frac{\partial w}{\partial x} &= 0, \\
\frac{1+\sigma}{2}\frac{\partial^2 u}{\partial x \partial \varphi} + a\frac{1-\sigma}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1}{a}\frac{\partial^2 v}{\partial \varphi^2} - \frac{1}{a}\frac{\partial w}{\partial \varphi} &= 0, \\
\frac{\sigma}{a}\frac{\partial u}{\partial x} + \frac{1}{a^2}\frac{\partial v}{\partial \varphi} - \frac{h^2}{12}\left(\frac{\partial^4 w}{\partial x^4} + \frac{2}{a^2}\frac{\partial^4 w}{\partial x^2 \partial \varphi^2} + \frac{1}{a^4}\frac{\partial^4 w}{\partial \varphi^4}\right) - \frac{w}{a^2} &= 0
\end{aligned} \tag{4.144}$$

corresponding to that in (4.132) and show how the axial symmetry affects its form.

First, the axial symmetry implies that the components of the displacement vector do not depend on the circumferential variable φ. Therefore, the system in (4.144) becomes one-dimensional, with the meridional coordinate x representing a single independent variable. Another simplification stems from the fact that, in an axially symmetric problem, a point on a meridian $\varphi = \text{const}$ remains on this meridian, also after deformation, implying that: (i) the component v of the displacement vector $\mathbf{U}$ is zero, and (ii) all derivatives in (4.144) with respect to φ vanish. With all this taken into account, the system in (4.144) transforms to

$$\begin{aligned}
\frac{d^2 u(x)}{dx^2} - \frac{\sigma}{a}\frac{dw(x)}{dx} &= 0, \\
\sigma\frac{du(x)}{dx} - \frac{w(x)}{a} - \frac{ah^2}{12}\frac{d^4 w(x)}{dx^4} &= 0
\end{aligned} \tag{4.145}$$

which is a system of two ordinary differential equations for $u(x)$ and $w(x)$.

It is worth noting, that the total order of the above system is six. This implies that its fundamental set of solutions must consist of six linearly independent vector-functions. The manual derivation of such a set is highly cumbersome, but we can recommend computer algebra software (for example, **Maple** or **Mathematica**), in order to facilitate the time consuming work. Omitting the details, we display only the final result, i.e. the general solution of the system in (4.145), as:

$$u(x) = C_1 + C_2x + C_3i\sigma e^{-i\beta x} - C_4\sigma e^{-\beta x} - C_5i\sigma e^{i\beta x} + C_6\sigma e^{\beta x}$$

and

$$w(x) = C_2\sigma a + C_3a\beta e^{-i\beta x} + C_4a\beta e^{-\beta x} + C_5a\beta e^{i\beta x} + C_6a\beta e^{\beta x}$$

with β is defined as

$$\beta = \frac{\sqrt[4]{12a^2h^2(\sigma^2-1)}}{ah}. \tag{4.146}$$

This implies that the set of vector-functions

$$\begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}x\\\sigma a\end{pmatrix}, \quad \begin{pmatrix}i\sigma e^{-i\beta x}\\a\beta e^{-i\beta x}\end{pmatrix},$$
$$\begin{pmatrix}-\sigma e^{-\beta x}\\a\beta e^{-\beta x}\end{pmatrix}, \quad \begin{pmatrix}-i\sigma e^{i\beta x}\\a\beta e^{i\beta x}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\sigma e^{\beta x}\\a\beta e^{\beta x}\end{pmatrix} \tag{4.147}$$

can be taken as a fundamental set of solutions to the system in (4.145).

Since the components of the vectors in (4.147) are complex-valued functions, it is inconvenient to use them for practical construction of Green's matrices for boundary-value problems for the governing system in (4.145). This deficiency can be eliminated by separating the real and imaginary parts in the components of the vector-functions in (4.147). Upon doing this, the latter transform to

$$\begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}x\\\sigma a\end{pmatrix}, \quad \begin{pmatrix}\sigma\sin\beta x\\a\beta\cos\beta x\end{pmatrix},$$
$$\begin{pmatrix}-\sigma\ \cos\beta x\\a\beta\sin\beta x\end{pmatrix}, \quad \begin{pmatrix}-\sigma e^{-\beta x}\\a\beta e^{-\beta x}\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\sigma e^{\beta x}\\a\beta e^{\beta x}\end{pmatrix}. \tag{4.148}$$

The above set looks more attractive as compared to (4.147) because it contains no explicit imaginary components. However, upon close analysis we find a deficiency in (4.148) similar to that of (4.147): as mentioned before, the Poisson ratio σ of an elastic material ranges from 0 to 0.5, which makes the factor $(\sigma^2 - 1)$ in (4.146) negative. This implies that the number under the square root is negative, making β complex-valued. Hence, the fundamental set of solutions in (4.148) is also impractical.

To circumvent the deficiency taking place in (4.147) and (4.148), we search for another fundamental set of solutions to the system in (4.145). In doing so, we use the De Moivre's formula expressing $\sqrt[4]{-1}$ as

$$\sqrt[4]{-1} = \cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}, \quad k = 0, 1, 2, 3,$$

and isolate its principal ($k = 0$) value

$$\sqrt[4]{-1} = \frac{\sqrt{2}}{2}(1+i)$$

transforming β in (4.146) to

$$\beta = (1+i)\frac{\sqrt[4]{3a^2h^2(1-\sigma^2)}}{ah}$$

allowing us to obtain a fundamental set of solutions to the system in (4.145) of the form

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x \\ \sigma a \end{pmatrix}, \quad \begin{pmatrix} \sigma \mathrm{e}^{\omega x}\overline{S}(x) \\ 2a\omega \mathrm{e}^{\omega x}\cos\omega x \end{pmatrix},$$

$$\begin{pmatrix} -\sigma \mathrm{e}^{-\omega x}\underline{S}(x) \\ 2a\omega \mathrm{e}^{-\omega x}\cos\omega x \end{pmatrix}, \quad \begin{pmatrix} -\sigma \mathrm{e}^{\omega x}\underline{S}(x) \\ 2a\omega \mathrm{e}^{\omega x}\sin\omega x \end{pmatrix}, \quad \begin{pmatrix} -\sigma \mathrm{e}^{-\omega x}\overline{S}(x) \\ 2a\omega \mathrm{e}^{-\omega x}\cos\omega x \end{pmatrix}, \tag{4.149}$$

where ω represents the real-valued parameter

$$\omega = \frac{\sqrt[4]{3a^2h^2(1-\sigma^2)}}{ah}$$

and $\overline{S}(x)$ and $\underline{S}(x)$ are defined as

$$\overline{S}(x) = \cos\omega x + \sin\omega x \quad \text{and} \quad \underline{S}(x) = \cos\omega x - \sin\omega x.$$

The set of vector-functions in (4.149) turns out to be helpful in the construction of Green's matrices for a variety of boundary-value problems modeling axially symmetric deformation of cylindrical shells: an efficient procedure has been described in detail in Section 4.5.1, where we obtained the solution of the boundary-value problem in (4.116) and (4.117) in integral form in (4.130). The kernel of the latter represents the sought-after Green's matrix. We encourage the reader to implement the set of vector-functions in (4.149) as a fundamental set of solutions when solving Chapter Exercises 11 and 12 below.

4.6 Chapter Exercises

1. Construct the Green's function for the boundary-value problem for the biharmonic equation, which models the bending of a simply supported infinite strip-shaped Poisson–Kirchhoff plate, the middle plane of which occupies the region $\Omega = \{(x, y) | -\infty < x < \infty, 0 < y < b\}$.
2. Construct the Green's functions for the rectangular Poisson–Kirchhoff plate with its middle plane occupying the region $\Omega = \{(x, y)|0 < x < a, 0 < y < b\}$, with the following boundary conditions imposed:

(a) $w(0,y) = \partial^2 w(0,y)/\partial x^2 = 0, \partial^2 w(a,y)/\partial x^2 = \partial^3 w(a,y)/\partial x^3 = 0$
$w(x,0) = \partial^2 w(x,0)/\partial y^2 = 0, w(x,b) = \partial^2 w(x,b)/\partial y^2 = 0$
(the edges $x = 0, y = 0$,and $y = b$ are simply supported, while the edge $x = a$ is free of tension);

(b) $w(0,y) = \partial w(0,y)/\partial x = 0, w(a,y) = \partial w(a,y)/\partial x = 0$
$w(x,0) = \partial^2 w(x,0)/\partial y^2 = 0, w(x,b) = \partial^2 w(x,b)/\partial y^2 = 0$
(two opposite edges are clamped, while the other two are simply supported);

(c) $w(0,y) = \partial w(0,y)/\partial x = 0, \partial^2 w(a,y)/\partial x^2 = \partial^3 w(a,y)/\partial x^3 = 0$
$w(x,0) = \partial^2 w(x,0)/\partial y^2 = 0, w(x,b) = \partial^2 w(x,b)/\partial y^2 = 0$
(one edge is clamped, two other opposite edges are simply supported, whilst the fourth is not subject to any tension).

3. Use the Green's function constructed in Section 4.2 (see (4.11)) to determine the deflection $w(x,y)$, and the bending moments $M_x(x,y)$ and $M_y(x,y)$ in a simply-supported rectangular $\{(x,y)|0 < x < a, 0 < y < b\}$ Poisson–Kirchhoff plate, subject to a uniform transverse load of magnitude Q_0 applied to a rectangular region $\{(x,y)|a_1 < x < a_2, b_1 < y < b_2\}$, with $a_2 < a$ and $b_2 < b$.

4. Use the Green's function in (4.11) to determine the deflection $w(x,y)$ of the simply supported rectangular Poisson–Kirchhoff plate, where $\{(x,y)|0 < x < a, 0 < y < b\}$, loaded with:

   (a) $f(x,y) = Q_0 =$ const applied to the region $\{(x,y)|0 < x < a, 0 < y < b/2\}$;

   (b) $f(x,y) = Q_0 xy$, with $Q_0 =$ const applied to the region $\{(x,y)|0 < x < a/2, 0 < y < b\}$;

   (c) $f(x,y) = Q_0 x(a-x)y(b-y)$, with $Q_0 =$ const applied to the region $\{(x,y)|0 < x < a, 0 < y < b/2\}$;

   (d) $f(x,y) = Q_0 \exp(xy)$, , with $Q_0 =$ const applied to the region $\{(x,y)| 0 < x < a/2, 0 < y < b/2\}$.

5. Use the Green's function defined by the series in (4.20), with coefficients shown in (4.34), to determine the deflection $w(x,y)$ of the semi-infinite strip-shaped Poisson–Kirchhoff plate, where $\{(x,y)|0 < x < \infty, 0 < y < b\}$, loaded with:

   (a) $f(x,y) = Q_0 =$ const applied to the region $\{(x,y)|0 < x < a, 0 < y < b\}$;

   (b) $f(x,y) = Q_0 xy$, with $Q_0 =$ const applied to the region $\{(x,y)|0 < x < a, 0 < y < b\}$;

(c) $f(x, y) = Q_0 x(a - x)y(b - y)$, with $Q_0 =$ const applied to the region $\{(x, y)|0 < x < a, 0 < y < b/2\}$;

(d) $f(x, y) = Q_0 \exp(xy)$, , with $Q_0 =$ const applied to the region $\{(x, y)| 0 < x < a/2, 0 < y < b/2\}$.

6. Use the Green's function in (4.96) to determine the deflection $w(x, y)$ of the simply supported rectangular Poisson–Kirchhoff plate, with $\{(x, y)|0 < x < a, 0 < y < b\}$, resting on an elastic foundation (λ) and loaded with:

   (a) $f(x, y) = Q_0$, with $Q_0 =$ const applied to the region $\{(x, y)|0 < x < a/2, 0 < y < b\}$;

   (b) $f(x, y) = Q_0$, with $Q_0 =$ const applied to the region $\{(x, y)|0 < x < a/2, 0 < y < b/2\}$;

   (c) $f(x, y) = Q_0 xy$, with $Q_0 =$ const applied to the region $\{(x, y)|0 < x < a, 0 < y < b\}$.

7. Use the Green's function in (4.71) to determine the deflection $w(x, y)$ of the clamped circular plate with radius a, subject to a uniform transverse load Q_0.

8. Use the Green's function in (4.76) to determine the deflection $w(x, y)$ of the semi-circular plate with radius a, subject to a uniform transverse load Q_0.

9. Use the Green's function in (4.83) to determine the deflection $w(x, y)$ of the simply supported circular plate with radius a, subject to a uniform transverse load Q_0 applied to the region $\{(r, \varphi)|0 < r < a_1, 0 < \varphi < 2\pi\}$, where $a_1 < a$.

10. Use the Green's matrix whose elements are shown in (4.143) to find the components of the displacement vector for the section $\Omega = \{(x, \varphi)|0 < x < l, 0 < \varphi < b\}$ of a cylindrical shell with radius a, loaded with a uniform transverse load Z_0, applied to the half-section $\{0 < x < l/2, 0 < \varphi < b\}$.

11. Construct the Green's matrix for a boundary-value problem for the system in (4.145), modeling axially symmetric deformation of a semi-infinite cylindrical shell with a simply-supported edge $x = 0$.

12. Construct the Green's matrix for a boundary-value problem for the system in (4.145), modeling axially symmetric deformation of a semi-infinite cylindrical shell with a clamped edge $x = 0$.

Chapter 5
Multi-Point-Posed Problems

The opening chapters of this book were conceived as a review of the theory of the Green's function in its traditional sense. We also wanted to familiarize the reader with several applications of that theory. The review underwrites the fundamental importance of this subject. It represents a powerful instrument in the qualitative as well as quantitative analysis of boundary-value problems for linear ordinary and partial differential equations. One of the necessary conditions for the existence of the Green's function, requires the coefficients of the governing differential equation to be smooth functions. This implies that the coefficients should not only be continuous functions of the independent variables, but should also be differentiable, up to a certain order.

However, it is worth noting that the coefficients of differential equations modeling many physical phenomena are not necessarily smooth functions. They might, for example, be discontinuous, in which case the Green's function formalism is not directly applicable. This creates a situation where it is desirable to adjust the formalism to such irregular differential equations accordingly. Earlier in [44, 45, 47, 50], we have reported on our work and the progress made in the area. In this chapter, we will discuss this indicated adjustment: we will introduce a novel notion of the *matrix of Green's type* for specific systems of ordinary differential equations, and will present a number of applications. Later, in Chapter 6, the notion of the matrix of Green's type will be extended further, in order to make it applicable to partial differential equations.

In the first section below, we will extend the sphere of applicability of the Green's function to the so-called *multi-point-posed* boundary-value problems specified for specific sets of linear ordinary differential equations. We will see how the notion of the matrix of Green's type already arises from this extension.

Section 5.2 will deal with applications of matrices of Green's type to multi-point-posed boundary-value problems, modeling the static stress-strain state of multi-span Poisson–Kirchhoff beams. In Section 5.3 we will also provide a generalization of the material in the first Section, when we will treat sets of linear ordinary differential equations with individual domains. In contrast to the problems in Section 5.1, every governing equation will be defined on a single edge of a finite weighted graph. The continuity conditions and boundary conditions imposed at the vertexes and at the endpoints of the graph respectively, order our problem as a specific system of differential equations. matrices of Green's type for the latter will be constructed using one version of the method proposed in Section 5.1.

5.1 Matrix of Green's Type

In this section, we set out to present several non-traditional implementations of the Green's function method. We will deal with problem statements that reduce to multi-point-posed boundary-value problems for specific sets of linear ordinary differential equations. Those sets do not, however, represent systems of equations in the usual sense, where several unknown functions are assumed to have a common domain, and at least one of the equations in the system involves more than one of the unknown functions.

In this case, each of the involved differential equations governs a single unknown function, each of which is defined on an individual domain that represents a subinterval of a certain basic interval. The equations are transformed to a system by imposing contact conditions at the endpoints of the subintervals.

5.1.1 Definition

We introduce the *matrix of Green's type* notion for a piecewise homogeneous medium of a *sandwich type*. Later, in Section 5.3, we will extend this notion to problems defined on more complex assemblies of one-dimensional elements.

We now present a typical setting of a multi-point-posed boundary-value problem of the kind to be considered. In doing so, let a closed interval $[a_0, a_k]$ be partitioned by a set of $k-1$ distinct internal points a_i, $i = \overline{1, k-1}$, into k arbitrary subintervals (a_{i-1}, a_i). Consider a set of k inhomogeneous nth order linear differential equations

$$L_i[y_i(x)] \equiv \sum_{j=0}^{n} p_{ij}(x) \frac{dy_i^{(n-j)}(x)}{dx^{n-j}} = f_i(x), \quad x \in (a_{i-1}, a_i),\ i = \overline{1, k}, \tag{5.1}$$

each of which is defined on an individual subinterval. The coefficients $p_{ij}(x)$ of the operators L_i represent continuous functions on $[a_{i-1}, a_i]$, with the leading coefficients $p_{i0}(x)$ being nonzero on $[a_{i-1}, a_i]$. In addition, we assume each of the right-hand side functions $f_i(x)$ in (5.1) to be continuous on the corresponding subinterval (a_{i-1}, a_i).

Clearly, the equations in (5.1) cannot be viewed as a system of differential equations in the usual sense. They should rather be considered as a set of k separate equations. However, since every two next-door subintervals (a_{i-1}, a_i) and (a_i, a_{i+1}) in $[a_0, a_k]$ share the end-point a_i, we can treat the set in (5.1) as a system if we impose the following set of boundary and contact conditions

$$M_q[y_1(a_0), y_1(a_1), y_2(a_1), \ldots, y_k(a_k)] = 0, \quad q = \overline{1, n \times k}, \tag{5.2}$$

with M_q linearly independent forms, representing the conditions imposed at the points $a_i, (i = \overline{0, k})$. Of the total number of $n \times k$ relations in (5.2), only n represent boundary conditions imposed at the end-point a_0 and a_k of $[a_0, a_k]$, whereas the remaining relations represent contact conditions imposed at the internal points a_i, $i = \overline{1, k-1}$.

In the following, a setting of the kind in (5.1) and (5.2) will be referred to as a multi-point-posed boundary-value problem. Assuming that the problem in (5.1) and (5.2) is well-posed, this implies that the corresponding homogeneous (all $f_i(x) \equiv 0$) problem only has the trivial solution.

To provide a backdrop for extending the Green's function formalism to the setting in (5.1) and (5.2), we introduce a vector-function $\mathbf{Y}(x)$ the components of which $Y_i(x)$ are defined in terms of the functions $y_i(x)$ that are to be found, as follows

$$Y_i(x) = \begin{cases} y_i(x), & \text{for } x \in (a_{i-1}, a_i), \\ 0, & \text{for } x \in (a_0, a_k) \setminus (a_{i-1}, a_i). \end{cases} \tag{5.3}$$

We introduce another vector-function $\mathbf{F}(x)$ for the right-hand side functions $f_i(x)$ in (5.1), whose components $F_i(x)$ are defined as

$$F_i(x) = \begin{cases} f_i(x), & \text{for } x \in (a_{i-1}, a_i), \\ 0, & \text{for } x \in (a_0, a_k) \setminus (a_{i-1}, a_i). \end{cases} \tag{5.4}$$

We are now in a position to formally extend the definition of Green's function in order to make it valid for the multi-point-posed boundary-value problem displayed in (5.1) and (5.2).

Definition. If, for any allowable vector-function $\mathbf{F}(x)$, the vector-function $\mathbf{Y}(x)$ is expressed in integral form

$$\mathbf{Y}(x) = -\int_{a_0}^{a_k} G(x, \xi)\mathbf{F}(\xi)d\xi \tag{5.5}$$

then we can refer to the kernel matrix

$$G(x, \xi) = \big(g_{ij}(x, \xi)\big)_{i,j=\overline{1,k}} \tag{5.6}$$

in (5.5) as the *matrix of Green's type* for the homogeneous multi-point-posed boundary-value problem, corresponding to (5.1) and (5.2).

Note that the first subscript i in the element $g_{ij}(x, \xi)$ of $G(x, \xi)$ matches the domain of the variable x, $x \in [a_{i-1}, a_i]$, whilst the second subscript j matches the domain of the variable ξ ,$\xi \in [a_{j-1}, a_j]$.

For any fixed value of ξ, the components $g_{ij}(x, \xi)$ are assumed to meet the following properties:

1. For $i \neq j$ (meaning that the domains of x and ξ never overlap), the elements $g_{ij}(x, \xi)$ are continuous along with their derivatives with respect to x up to the nth order.

2. For $i = j$ (x and ξ share the domain), when $x \neq \xi$, the elements $g_{ii}(x, \xi)$ are continuous, along with their derivatives with respect to x up to the nth order. However, for $x = \xi$, the elements$g_{ii}(x, \xi)$ are continuous along with their derivatives with respect to x up to the $(n - 2)$nd order, whereas their $(n-1)$st derivatives make a discontinuous jump, the magnitude of which equals $-p_{i0}^{-1}(\xi)$.
3. For $x \neq \xi$, the elements $g_{ij}(x, \xi)$,as functions of x, satisfy the homogeneous equations

$$L_i[g_{ij}(x, \xi)] = 0, \quad x \in (a_{i-1}, a_i),\ i = \overline{1, k},$$

on the domain of x.
4. The elements $g_{ij}(x, \xi)$ satisfy both the boundary and the contact conditions in (5.2) i.e.:

$$M_q[g_{ij}(a_0, \xi), g_{ij}(a_1, \xi), \ldots, g_{ij}(a_k, \xi)] = 0, \quad q = \overline{1, n \times k},$$

which are applicable to them.

5.1.2 Construction

Our objective in this section is to provide the reader with a straightforward guide for constructing matrices of Green's type. We will try to be as clear and as transparent as possible and will not describe the construction procedure in general terms. Instead, we consider several specific problems of the kind in (5.1) and (5.2), and show how we can practically construct their matrices of Green's type. We believe that this methodology will work best.

Example 5.1. Consider the simplest three-point posed boundary-value problem

$$\frac{d^2y_1(x)}{dx^2} = f_1(x), \quad x \in (-1, 0), \tag{5.7}$$

$$\frac{d^2y_2(x)}{dx^2} = f_2(x), \quad x \in (0, 1), \tag{5.8}$$

$$y_1(-1) = 0, \quad y_2(1) = 0, \tag{5.9}$$

$$y_1(0) = y_2(0), \quad \frac{dy_1(0)}{dx} = \lambda \frac{dy_2(0)}{dx}. \tag{5.10}$$

This problem may be interpreted as a model for the steady-state heat conduction in a compound bar consisting of two segments of unit length, each made out of a physically homogeneous material. The parameter λ represents the ratio λ_2/λ_1 of the heat conductivities of the materials out of which the bars are made.

It can be easily shown, that the homogeneous problem corresponding to (5.7)–(5.10) has only the trivial solution. In Chapter Exercise 1(a), we invite the reader to verify this.

Following the method of variation of parameters procedure, we now represent the general solution of (5.7) as

$$y_1(x) = C_1(x) + xC_2(x) \tag{5.11}$$

yielding the following system of linear algebraic equations

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ f_1(x) \end{pmatrix}$$

for the derivatives of $C_1(x)$ and $C_2(x)$, with a well-posed coefficient matrix. From this, it follows that

$$C_1'(x) = -xf_1(x), \quad C_2'(x) = f_1(x).$$

Expressions for $C_1(x)$ and $C_2(x)$ are obtained from the above, by straightforward integration, as

$$C_1(x) = -\int_{-1}^{x} \xi f_1(\xi)d\xi + M_1, \quad C_2(x) = \int_{-1}^{x} f_1(\xi)d\xi + M_2.$$

Substituting $C_1(x)$ and $C_2(x)$ into (5.11), and combining the two integral terms in one, we obtain

$$y_1(x) = \int_{-1}^{x} (x-\xi)f_1(\xi)d\xi + M_1 + M_2x, \quad x \in [-1, 0]. \tag{5.12}$$

It is evident we can obtain an expression for $y_2(x)$ by following an analogous procedure. The only difference with the case for $y_1(x)$ is the domain for $y_1(x)$. Hence, we have respectively

$$y_2(x) = \int_{0}^{x} (x-\xi)f_2(\xi)d\xi + N_1 + N_2x, \quad x \in [0, 1]. \tag{5.13}$$

We use the boundary and contact conditions in (5.9) and (5.10), applied to $y_1(x)$ and $y_2(x)$ to calculate the constants of integration M_1, M_2, N_1, and N_2. The boundary conditions in (5.9) yield

$$M_1 - M_2 = 0, \tag{5.14}$$

$$N_1 + N_2 = \int_{0}^{1} (\xi - 1)f_2(\xi)d\xi, \tag{5.15}$$

whilst the contact conditions in (5.10) result in

$$M_1 - N_1 = \int_{-1}^{0} \xi f_1(\xi) d\xi, \tag{5.16}$$

$$\lambda N_2 - M_2 = \int_{-1}^{0} f_1(\xi) d\xi. \tag{5.17}$$

The relations in (5.14) through (5.17) form a well-posed system of linear algebraic equations for M_1, M_2, N_1, and N_2, with the solution

$$M_1 = M_2 = \frac{1}{1+\lambda}\int_{-1}^{0}(\lambda\xi - 1) f_1(\xi) d\xi + \frac{\lambda}{1+\lambda}\int_{0}^{1}(\xi - 1) f_2(\xi) d\xi,$$

$$N_1 = -\frac{1}{1+\lambda}\int_{-1}^{0}(\xi + 1) f_1(\xi) d\xi + \frac{\lambda}{1+\lambda}\int_{0}^{1}(\xi - 1) f_2(\xi) d\xi,$$

and

$$N_2 = \frac{1}{1+\lambda}\int_{-1}^{0}(\xi + 1) f_1(\xi) d\xi + \frac{1}{1+\lambda}\int_{0}^{1}(\xi - 1) f_2(\xi) d\xi.$$

Upon substituting M_1, M_2, N_1, and N_2 into (5.12) and (5.13), and after performing a few elementary transformations, we obtain

$$y_1(x) = \int_{-1}^{0} \frac{(x+1)(\lambda\xi - 1)}{1+\lambda} f_1(\xi) d\xi + \int_{-1}^{x} (x - \xi) f_1(\xi) d\xi + \int_{0}^{1} \frac{\lambda(x+1)(\xi - 1)}{1+\lambda} f_2(\xi) d\xi \tag{5.18}$$

and

$$y_2(x) = \int_{-1}^{0} \frac{(x-1)(\xi + 1)}{1+\lambda} f_1(\xi) d\xi + \int_{0}^{x} (x - \xi) f_2(\xi) d\xi + \int_{0}^{1} \frac{(x+\lambda)(\xi - 1)}{1+\lambda} f_2(\xi) d\xi. \tag{5.19}$$

If we write the first and the second integrals in (5.18) as a single integral, we get

$$y_1(x) = -\int_{-1}^{0} g_{11}(x, \xi) f_1(\xi) d\xi - \int_{0}^{1} g_{12}(x, \xi) f_2(\xi) d\xi \tag{5.20}$$

whilst after combining the second and the third integrals in (5.19), we obtain

$$y_2(x) = -\int_{-1}^{0} g_{21}(x, \xi) f_1(\xi) d\xi - \int_{0}^{1} g_{22}(x, \xi) f_2(\xi) d\xi \tag{5.21}$$

with the kernel functions $g_{ij}(x,\xi)$ in (5.20) and (5.21) are found as

$$g_{11}(x,\xi) = \begin{cases} \beta(x+1)(1-\lambda\xi), & \text{for } -1 \le x \le s < 0, \\ \beta(\xi+1)(1-\lambda x), & \text{for } -1 < s \le x \le 0, \end{cases}$$

$$g_{12}(x,\xi) = \lambda\beta(x+1)(1-\xi), \quad \text{for } -1 \le x \le 0 < s < 1,$$

$$g_{21}(x,\xi) = \beta(1-x)(\xi+1), \quad \text{for } -1 < s < 0 \le x \le 1,$$

and

$$g_{22}(x,\xi) = \begin{cases} \beta(x+\lambda)(1-\xi), & \text{for } 0 \le x \le s < 1, \\ \beta(\xi+\lambda)(1-x), & \text{for } 0 < s \le x \le 1, \end{cases}$$

with $\beta = (1+\lambda)^{-1}$.

In accordance with the approach suggested earlier in this section (see the vector-functions introduced in (5.3) and (5.4)), we introduce the vector-function $\mathbf{Y}(x)$ with the components

$$Y_1(x) = \begin{cases} y_1(x), & \text{for } x \in (-1,0), \\ 0, & \text{for } x \in (0,1), \end{cases}$$

and

$$Y_2(x) = \begin{cases} 0, & \text{for } x \in (-1,0), \\ y_2(x), & \text{for } x \in (0,1). \end{cases}$$

Similarly, components of the vector-function $\mathbf{F}(x)$ are defined as

$$F_1(x) = \begin{cases} f_1(x), & \text{for } x \in (-1,0), \\ 0, & \text{for } x \in (0,1), \end{cases}$$

and

$$F_2(x) = \begin{cases} 0, & \text{for } x \in (-1,0), \\ f_2(x), & \text{for } x \in (0,1). \end{cases}$$

We can now rewrite the integrals in (5.20) and (5.21), in terms of the vector-functions $\mathbf{Y}(x)$ and $\mathbf{F}(x)$, as the single integral

$$\mathbf{Y}(x) = -\int_{-1}^{1} G(x,\xi)\mathbf{F}(\xi)d\xi.$$

Hence, it follows from the definition we introduced earlier in this section, that the functions $g_{ij}(x,\xi)$ can be referred to as the elements of the matrix of Green's type $G(x,\xi)$ of the three-point posed homogeneous problem corresponding to (5.7)–(5.10).

Example 5.2. We consider the system of Cauchy–Euler equations

$$\frac{d}{dx}\left(x\frac{dy_1(x)}{dx}\right) - \frac{1}{x}y_1(x) = f_1(x), \quad x \in (0,a), \tag{5.22}$$

$$\frac{d}{dx}\left(x\frac{dy_2(x)}{dx}\right) - \frac{1}{x}y_2(x) = f_2(x), \quad x \in (a,\infty), \tag{5.23}$$

with boundary and contact conditions imposed as

$$\lim_{x\to 0}|y_1(x)| < \infty, \quad \lim_{x\to\infty}|y_2(x)| < \infty, \tag{5.24}$$

$$y_1(a) = y_2(a), \quad \frac{dy_1(a)}{dx} = \lambda\frac{dy_2(a)}{dx}. \tag{5.25}$$

We have chosen this problem on purpose: whilst solving it, we will show how the method of variation of parameters works for multi-point-posed boundary-value problems, with singular points for the governing equations, which in turn may be defined on unbounded domains. Clearly, $x = 0$ represents a singular point for the equation in (5.22), whereas the domain for the second governing equation in (5.23) is unbounded.

In Chapter Exercise 1 (b), we ask the reader to verify that the problem in (5.22)–(5.25) is well-posed, thus justifying the existence and uniqueness of its matrix of Green's type.

Evidently, the functions $Y_1(x) \equiv x$ and $Y_2(x) \equiv x^{-1}$ may constitute a fundamental set of solutions for the homogeneous equation corresponding to that in (5.22). Thus, following our procedure, we seek the general solution to that equation in the form

$$y_1(x) = C_1(x)x + C_2(x)x^{-1}. \tag{5.26}$$

This yields the following well-posed system of linear algebraic equations

$$\begin{pmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{pmatrix} \times \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ f_1(x)/x \end{pmatrix}$$

in $C_1'(x)$ and $C_2'(x)$. The solution of this system is

$$C_1'(x) = \frac{f_1(x)}{2x}, \quad C_2'(x) = -\frac{xf_1(x)}{2}.$$

After integrating these expressions and substituting $C_1(x)$ and $C_2(x)$ into (5.26), we have

$$y_1(x) = \int_0^x \frac{x^2 - \xi^2}{2x\xi} f_1(\xi)d\xi + D_{11}x + D_{12}x^{-1}. \tag{5.27}$$

Analogously, for $y_2(x)$ one obtains

$$y_2(x) = \int_a^x \frac{x^2 - \xi^2}{2x\xi} f_2(\xi)d\xi + D_{21}x + D_{22}x^{-1}. \tag{5.28}$$

Notice again that the lower limits of the two integrals in (5.27) and (5.28) represent the left-end points of the intervals $(0, a)$ and (a, ∞), respectively.

The constants of integration in (5.27) and (5.28) can be obtained by imposing the boundary and contact conditions in (5.24) and (5.25). Clearly, the first condition in (5.24) requires $D_{12} = 0$, since its factor x^{-1} is unbounded as x goes to zero. To satisfy the second condition in (5.24), we regroup the terms in (5.28) as follows

$$y_2(x) = \left(\int_a^x \frac{1}{2\xi} f_2(\xi)d\xi + D_{21}\right)x + \left(-\int_a^x \frac{\xi}{2} f_2(\xi)d\xi + D_{22}\right)x^{-1}. \tag{5.29}$$

It can be readily seen that for the second condition in (5.24) to hold, the integral-containing factor of x in (5.29) must be equal to zero, resulting in

$$D_{21} = -\int_a^\infty \frac{1}{2\xi} f_2(\xi)d\xi.$$

Taking into account the values of D_{12} and D_{21} we just found, the first condition in (5.25) provides us with

$$D_{11}a - D_{22}a^{-1} = -\int_0^a \frac{a^2 - \xi^2}{2a\xi} f_1(\xi)d\xi - \int_a^\infty \frac{a}{2\xi} f_2(\xi)d\xi. \tag{5.30}$$

To treat the second condition in (5.25) properly, we first differentiate $y_1(x)$ and $y_2(x)$ in (5.27) and (5.28), which yields

$$y_1'(x) = \int_0^x \frac{x^2 + \xi^2}{2\xi x^2} f_1(\xi)d\xi + D_{11}$$

and

$$y_2'(x) = \int_a^x \frac{x^2 + \xi^2}{2\xi x^2} f_2(\xi)d\xi + D_{21} - D_{22}x^{-2}.$$

It now follows from the second condition in (5.25) that

$$D_{11} + \lambda D_{22}a^{-2} = -\int_0^a \frac{a^2 + \xi^2}{2\xi a^2} f_1(\xi)d\xi - \int_a^\infty \frac{\lambda}{2\xi} f_2(\xi)d\xi. \tag{5.31}$$

The relations (5.30) and (5.31) form a well-posed system of linear algebraic equations in D_{11} and D_{22}, the solution of which is found as

$$D_{11} = -\int_a^\infty \frac{\lambda}{(1+\lambda)\xi} f_2(\xi)d\xi - \int_0^a \frac{(a^2 + \xi^2) + \lambda(a^2 - \xi^2)}{2(1+\lambda)a^2\xi} f_1(\xi)d\xi$$

and

$$D_{22} = -\int_0^a \frac{(a^2+\xi^2) - \lambda(a^2-\xi^2)}{4\lambda\xi} f_1(\xi)d\xi.$$

Substituting all four values of D_{ij}, $i, j = 1, 2$, into (5.27) and (5.28), we obtain the solution to the problem in (5.22)–(5.25) as

$$y_1(x) = -\int_0^a \frac{x[(a^2+\xi^2)+\lambda(a^2-\xi^2)]}{2(1+\lambda)a^2\xi} f_1(\xi)d\xi$$
$$+\int_0^x \frac{x^2-\xi^2}{2x\xi} f_1(\xi)d\xi - \int_a^\infty \frac{\lambda x}{(1+\lambda)\xi} f_2(\xi)d\xi$$

and

$$y_2(x) = -\int_0^a \frac{(a^2+\xi^2)-\lambda(a^2-\xi^2)}{4\lambda x\xi} f_1(\xi)d\xi$$
$$+\int_a^x \frac{x^2-\xi^2}{2x\xi} f_2(\xi)d\xi - \int_a^\infty \frac{x}{2\xi} f_2(\xi)d\xi.$$

From these integral representations of $y_1(x)$ and $y_2(x)$, it follows that, in accordance with the definition we introduced earlier, the elements $g_{ij}(x,\xi)$ of the matrix of Green's type to the homogeneous three-point posed boundary-value problem corresponding to (5.22)–(5.25) are finally found as:

$$g_{11}(x,\xi) = \begin{cases} x[(a^2+\xi^2)+\lambda(a^2-\xi^2)][2(1+\lambda)a^2\xi]^{-1}, & \text{for } 0 \le x \le s < a, \\ \xi[(a^2+x^2)+\lambda(a^2-x^2)][2(1+\lambda)a^2x]^{-1}, & \text{for } 0 < s \le x \le a, \end{cases}$$

$$g_{12}(x,\xi) = \lambda x[(1+\lambda)\xi]^{-1}, \quad \text{for } 0 \le x \le a < s < \infty,$$

$$g_{21}(x,\xi) = [(a^2+\xi^2)-\lambda(a^2-\xi^2)](4\lambda x\xi)^{-1}, \quad \text{for } 0 < s < a \le x < \infty,$$

$$g_{22}(x,\xi) = \begin{cases} x(2\xi)^{-1}, & \text{for } a \le x \le s < \infty, \\ \xi(2x)^{-1}, & \text{for } a < s \le x < \infty. \end{cases}$$

Before shifting our attention to the next problem, we note that in the settings presented so far in this section, we considered multi-point-posed boundary-value problems in which domains of independent variables form a *sandwich type* assembly. The four-point-posed problem in the following example is different: three segments form an assembly with their left-end points contacting in the way shown in Figure 5.1.

Example 5.3. Consider the following problem:

$$\frac{d^2y_i(x)}{dx^2} = -f_i(x), \quad x \in (0,1), \quad i = 1,2,3, \tag{5.32}$$

$$y_1(0) = y_2(0) = y_3(0), \quad h_1y_1'(0) + h_2y_2'(0) + h_3y_3'(0) = 0, \tag{5.33}$$

$$y_1(1) = 0, \quad y_2(1) = 0, \quad y_3(1) = 0. \tag{5.34}$$

Note that, for each element of the assembly, the problems are formulated in several local coordinate systems: it can be viewed, for example, as modeling a steady-state heat conduction in the assembly of three rods, each with unit length as shown in Figure 5.1. The rods are made out of homogeneous conducting materials whose heat conductivities are defined by the constants h_1, h_2, and h_3. Within this interpretation, we can refer to the relations in (5.33) as conditions of ideal thermal contact.

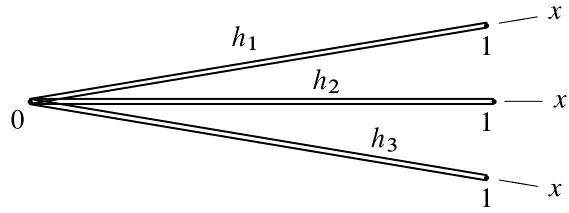

Figure 5.1. Heat conduction in an assembly of rods.

In the following, we will show how we can apply the technique we described earlier in this section to the construction of matrices of Green's type for the class of problems in (5.32)–(5.34).

In Chapter Exercise 1 (c), we invite the reader to verify that the homogeneous problem corresponding to that in (5.32)–(5.34) only has the trivial solution, justifying, consequently, the existence and uniqueness of its matrix of Green's type.

We now write the general solution to each of the equations in (5.32) as

$$y_i(x) = C_i(x) + D_i(x)x, \quad i = 1, 2, 3,$$

reducing, through method of variation of parameters, to the following integral formula

$$y_i(x) = \int_0^x (\xi - x) f_i(\xi) d\xi + M_i + N_i x, \quad i = 1, 2, 3. \tag{5.35}$$

Satisfying the first group $y_1(0) = y_2(0) = y_3(0)$ of conditions in (5.33), we derive the following two equations

$$M_1 = M_2 = M_3 \tag{5.36}$$

in M_1, M_2 and M_3. The last condition in (5.33) yields

$$h_1 N_1 + h_2 N_2 + h_3 N_3 = 0. \tag{5.37}$$

The boundary conditions in (5.34) provide us with three additional relations for M_i and N_i, namely

$$M_i + N_i = \int_0^1 (1 - \xi) f_i(\xi) d\xi, \quad i = 1, 2, 3. \tag{5.38}$$

The relations (5.36)–(5.38) form a well-posed system of six linear algebraic equations in six unknowns, from which the latter are found as

$$M_1 = M_2 = M_3 = H\int_0^1 (1-\xi)[h_1 f_1(\xi) + h_2 f_2(\xi) + h_3 f_3(\xi)]d\xi,$$

$$N_1 = H\int_0^1 (1-\xi)[(h_2+h_3) f_1(\xi) - h_2 f_2(\xi) - h_3 f_3(\xi)]d\xi,$$

$$N_2 = H\int_0^1 (1-\xi)[(h_1+h_3) f_2(\xi) - h_1 f_1(\xi) - h_3 f_3(\xi)]d\xi,$$

and

$$N_3 = H\int_0^1 (1-\xi)[(h_1+h_2) f_3(\xi) - h_1 f_1(\xi) - h_2 f_2(\xi)]d\xi$$

where $H = (h_1 + h_2 + h_3)^{-1}$.

Substituting these into (5.35), we obtain the solution to the problem in (5.32)–(5.34) in matrix form

$$\begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = \int_0^1 \begin{pmatrix} g_{11}(x,\xi) & g_{12}(x,\xi) & g_{13}(x,\xi) \\ g_{21}(x,\xi) & g_{22}(x,\xi) & g_{23}(x,\xi) \\ g_{31}(x,\xi) & g_{32}(x,\xi) & g_{33}(x,\xi) \end{pmatrix} \begin{pmatrix} f_1(\xi) \\ f_2(\xi) \\ f_3(\xi) \end{pmatrix} d\xi \qquad (5.39)$$

The elements $g_{ij}(x,\xi)$ of the kernel-matrix in the above integral are as follows

$$g_{11}(x,\xi) = \begin{cases} H(1-\xi)[h_1 + x(h_2+h_3)], & \text{for } x \le s, \\ H(1-x)[h_1 + \xi(h_2+h_3)], & \text{for } s \le x, \end{cases}$$

$$g_{12}(x,\xi) = Hh_2(1-\xi)(1-x), \quad g_{13}(x,\xi) = Hh_3(1-\xi)(1-x),$$

$$g_{22}(x,\xi) = \begin{cases} H(1-\xi)[h_2 + x(h_1+h_3)], & \text{for } x \le s, \\ H(1-x)[h_2 + \xi(h_1+h_3)], & \text{for } s \le x, \end{cases}$$

$$g_{21}(x,\xi) = Hh_1(1-\xi)(1-x), \quad g_{23}(x,\xi) = Hh_3(1-\xi)(1-x),$$

$$g_{31}(x,\xi) = Hh_1(1-\xi)(1-x), \quad g_{32}(x,\xi) = Hh_2(1-\xi)(1-x),$$

and

$$g_{33}(x,\xi) = \begin{cases} H(1-\xi)[h_3 + x(h_1+h_2)], & \text{for } x \le s, \\ H(1-x)[h_3 + \xi(h_1+h_2)], & \text{for } s \le x. \end{cases}$$

It follows from the definition we introduced in the opening part of this section, that the kernel-matrix in (5.39) represents the matrix of Green's type $G(x,\xi)$ for the homogeneous problem corresponding to (5.32)–(5.34). With respect to the physical interpretation of the setting, we can call $G(x,\xi)$ the influence function of a point source for the entire assembly of rods shown in Figure 5.1.

Similar to the Green's function formalism applied to a single equation, matrices of Green's type can naturally be utilized to solve multi-point-posed boundary-value problems for inhomogeneous systems of equations, subject to homogeneous boundary conditions. In a number of the Chapter Exercises, we will instruct the reader to use this approach when solving particular problems.

5.2 Influence Function of a Multi-Span Beam

With the introduction of the notion of the matrix of Green's type and after developing a procedure for the construction of such matrices, we are now ready to bring this notion to a specific area in structural mechanics. We will show how the notion of the matrix of Green's type helps to develop a solid basis for an efficient approach to a wide range of problems [45, 47], by using it to tackle the problem of static equilibrium of multi-span Poisson–Kirchhoff beams.

Our discussion will be based on a relationship between the matrix of Green's type for a multi-point-posed boundary-value problem, and the influence function of a transverse point force for a multi-span beam, which our problem models mathematically. Upon considering a number of particular problems, we will explain, specific features of the method proposed for the construction of influence functions for different multi-span beams.

Example 5.4. Consider a compound cantilever beam overhanging an intermediate simple support. The beam is comprised of two spans having uniform flexural rigidities EI_1 and EI_2, as depicted in Figure 5.2.

To obtain the influence function (matrix) of the transverse unit force for the beam in the statement, we set up the following three-point posed boundary-value problem

$$\frac{d^4 w_1(x)}{dx^4} = -\frac{q_1(x)}{EI_1} = -f_1(x), \quad x \in (0, b), \tag{5.40}$$

$$\frac{d^4 w_2(x)}{dx^4} = -\frac{q_2(x)}{EI_2} = -f_2(x), \quad x \in (b, a), \tag{5.41}$$

$$w_1(0) = \frac{dw_1(0)}{dx} = 0, \quad \frac{d^2 w_2(a)}{dx^2} = \frac{d^3 w_2(a)}{dx^3} = 0, \tag{5.42}$$

$$w_1(b) = w_2(b) = 0, \quad \frac{dw_1(b)}{dx} = \frac{dw_2(b)}{dx}, \tag{5.43}$$

$$EI_1 \frac{d^2 w_1(b)}{dx^2} = EI_2 \frac{d^2 w_2(b)}{dx^2}, \tag{5.44}$$

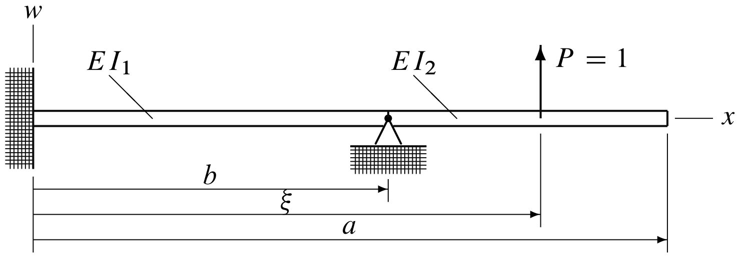

Figure 5.2. A compound beam overhanging a simple support.

with $w_1(x)$ and $w_2(x)$ representing the deflection functions of the beam for the corresponding span, $q_1(x)$ and $q_2(x)$ in (5.40), and (5.41) are continuously distributed transverse loads, applied to the left-hand and the right-hand span respectively.

With regard to the boundary and contact conditions imposed in (5.42)–(5.44), we note that the first two relations in (5.42) model the clamped edge conditions at $x = 0$, while the second pair of the relations in (5.42) represent the free edge conditions at $x = a$. The four relations in (5.43) and (5.44) model the continuity conditions imposed at the point of intermediate support, where the deflection function must be set to zero for $x = b$, whilst the slope of the deflection function and the bending moment in that point are assumed to be continuous.

In mechanics, the matrix of Green's type

$$G(x, \xi) = \left(g_{ij}(x, \xi)\right)_{i,j=\overline{1,2}}$$

for the homogeneous ($f_1(x) = f_2(x) = 0$) three-point posed boundary-value problem corresponding to (5.40)–(5.44) is called the influence function of a transverse unit force, for the beam in Figure 5.2. With this in mind, we interpret the first row elements $g_{11}(x, \xi)$ and $g_{12}(x, \xi)$ in $G(x, \xi)$ as the deflection in the left-hand span ($0 \le x \le b$) of the beam, caused by a transverse unit force applied within the left-hand span ($\xi \in (0, b)$) and the right-hand span ($\xi \in (b, a)$), respectively. The second row elements $g_{21}(x, \xi)$ and $g_{22}(x, \xi)$ represent the deflection in the right-hand span ($b \le x \le a$), caused by a transverse unit force applied within the left-hand and the right-hand span, respectively.

We will now use the method of variation of parameters, as developed earlier for second order equations, to obtain the matrix $G(x, \xi)$. The reader might recall from our experience with the application of this method, that the load functions $q_1(x)$ and $q_2(x)$ should not be specified. We introduce the functions $f_1(x)$ and $f_2(x)$ to aid notational convenience in the development that follows.

Following the method of variation of parameters, we represent the general solutions of the equations in (5.40) and (5.41) as

$$w_i(x) = A_i(x) + B_i(x)x + C_i(x)x^2 + D_i(x)x^3, \quad i = 1, 2. \tag{5.45}$$

The resulting system of linear equations for $A_i'(x)$, $B_i'(x)$, $C_i'(x)$, and $D_i'(x)$, $i = 1, 2$, now appears in the form

$$\begin{pmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 2 & 6x \\ 0 & 0 & 0 & 6 \end{pmatrix} \times \begin{pmatrix} A_i'(x) \\ B_i'(x) \\ C_i'(x) \\ D_i'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -f_i(x) \end{pmatrix}.$$

The specific shape of the above system makes its solution very simple, since the coefficient matrix of the system is in upper triangular form. Because of this, we find the system's solution by backward substitution, yielding

$$\begin{aligned} A_i'(x) &= \frac{x^3}{6} f_i(x), \quad & B_i'(x) &= -\frac{x^2}{2} f_i(x), \\ C_i'(x) &= \frac{x}{2} f_i(x), \quad & D_i'(x) &= -\frac{1}{6} f_i(x). \end{aligned} \tag{5.46}$$

Based on that, the coefficients $A_1(x), \ldots, D_1(x)$ for $w_1(x)$ in (5.45) can be obtained by integrating the above derivatives over the interval $[0, x]$, which yields

$$A_1(x) = \int_0^x \frac{\xi^3}{6} f_1(\xi) d\xi + H_1, \quad B_1(x) = -\int_0^x \frac{\xi^2}{2} f_1(\xi) d\xi + K_1,$$

$$C_1(x) = \int_0^x \frac{\xi}{2} f_1(\xi) d\xi + L_1, \quad D_1(x) = -\int_0^x \frac{1}{6} f_1(\xi) d\xi + M_1.$$

Analogously, the coefficients $A_2(x), \ldots, D_2(x)$ for $w_2(x)$ can be found as integrals of (5.46) over the interval $[b, x]$. That is,

$$A_2(x) = \int_b^x \frac{\xi^3}{6} f_2(\xi) d\xi + H_2, \quad B_2(x) = -\int_b^x \frac{\xi^2}{2} f_2(\xi) d\xi + K_2,$$

$$C_2(x) = \int_b^x \frac{\xi}{2} f_2(\xi) d\xi + L_2, \quad D_2(x) = -\int_b^x \frac{1}{6} f_2(\xi) d\xi + M_2.$$

Upon substituting these functions in (5.45) and grouping all the integral-containing terms together, we obtain the following expressions

$$w_1(x) = \int_0^x \frac{(\xi - x)^3}{6} f_1(\xi) d\xi + H_1 + K_1 x + L_1 x^2 + M_1 x^3, \quad x \in [0, b], \tag{5.47}$$

$$w_2(x) = \int_b^x \frac{(\xi - x)^3}{6} f_2(\xi) d\xi + H_2 + K_2 x + L_2 x^2 + M_2 x^3, \quad x \in [b, a], \tag{5.48}$$

for the general solutions of (5.40) and (5.41).

The parameters $H_1, \ldots, M_2$ in (5.47) and (5.48) can be found by imposing the boundary and contact conditions from (5.42)–(5.44): the first condition $w_1(0) = 0$ of a clamped edge in (5.42) yields $H_1 = 0$, whilst the second condition $w'(0) = 0$ results in $K_1 = 0$.

Through satisfying the first condition for a free edge, as imposed in (5.42), which represents the physical fact that the bending moment vanishes at $x = a$, we obtain

$$2L_2 + 6M_2 a = -\int_b^a (\xi - a) f_2(\xi) d\xi \tag{5.49}$$

whilst the second of the free edge conditions, which simulates the shear force vanishing at $x = a$, yields

$$M_2 = \int_b^a \frac{1}{6} f_2(\xi) d\xi.$$

Substituting M_2 into (5.49), we obtain

$$L_2 = -\int_b^a \frac{\xi}{2} f_2(\xi) d\xi.$$

The first contact condition $w_1(b) = 0$ from (5.43) yields

$$L_1 b^2 + M_1 b^3 = -\int_0^b \frac{(\xi - b)^3}{6} f_1(\xi) d\xi \tag{5.50}$$

while the contact condition in (5.44) provides

$$EI_1 \left[2L_1 + 6M_1 b + \int_0^b (\xi - b) f_1(\xi) d\xi \right] = EI_2 \int_b^a (b - \xi) f_2(\xi) d\xi. \tag{5.51}$$

Clearly, the relations (5.50) and (5.51) form a well-posed system of linear algebraic equations for L_1 and M_1, which gives us

$$L_1 = \int_0^b \frac{1}{4b^2} \xi (b - \xi)(\xi - 2b) f_1(\xi) d\xi - \int_b^a \frac{\lambda}{4} (b - \xi) f_2(\xi) d\xi$$

and

$$M_1 = \int_0^b \frac{1}{12b^3} (\xi - b)(\xi^2 - 2b\xi - 2b^2) d\xi + \int_b^a \frac{\lambda}{4b} (b - \xi) f_2(\xi) d\xi$$

with λ representing the ratio of the flexural rigidities EI_2 and EI_1, that is $\lambda = EI_2/EI_1$.

Since we already have H_1, K_1, L_1, and M_1 available, we rewrite (5.47) to give us a final expression for $w_1(x)$,

$$w_1(x) = \int_0^b \frac{x^2}{12b^3}(\xi - b)[\xi(3b - x)(2b - \xi) - 2b^2 x] f_1(\xi)d\xi$$
$$+ \int_b^a \frac{\lambda x^2}{4b}(b - \xi)(x - b) f_2(\xi)d\xi + \int_0^x \frac{(\xi - x)^3}{6} f_1(\xi)d\xi, \quad (5.52)$$

which represents the deflection function for the left-hand span of the beam, caused by the two continuously distributed transverse loads (with $q_1(x)$ applied to the left-hand span and $q_2(x)$ applied to the right-hand span).

We now turn our attention to the deflection function $w_2(x)$ in (5.48), and recall the second contact condition $w_2(b) = 0$ in (5.43), giving

$$H_2 + K_2 b + L_2 b^2 + M_2 b^3 = 0.$$

Using the expressions for L_2 and M_2 that we found earlier, the last equation transforms to

$$H_2 + K_2 b = -\int_b^a \frac{b^2(b - 3\xi)}{6} f_2(\xi)d\xi. \quad (5.53)$$

Through satisfying the third contact condition $w_1'(b) = w_2'(b)$ in (5.43), we obtain

$$2L_1 b + 3M_1 b^2 - \int_0^b \frac{(\xi - b)^2}{2} f_1(\xi)d\xi = K_2 + \int_b^a \frac{b(b - 2\xi)}{2} f_2(\xi)d\xi. \quad (5.54)$$

Note that, since we already know L_1 and M_1, equation (5.54) yields

$$K_2 = \int_0^b \frac{1}{4b}\xi^2(b - \xi) f_1(\xi)d\xi + \int_b^a \frac{b}{4}(b - \xi)(\lambda - 2) f_2(\xi)d\xi.$$

With this expression, we return to equation (5.53), and obtain

$$H_2 = -\int_0^b \frac{1}{4}\xi^2(b - \xi) f_1(\xi)d\xi + \int_b^a \frac{b^2}{12}[3\lambda(\xi - b) - 2(3\xi - 2b)] f_2(\xi)d\xi.$$

After substituting H_2, K_2, L_2, and M_2 into (5.48), and doing some trivial algebra, we obtain an explicit expression for $w_2(x)$ as

$$w_2(x) = \int_0^b \frac{1}{4b}\xi^2(b - \xi)(b - x) f_1(\xi)d\xi + \int_b^x \frac{(\xi - x)^3}{6} f_2(\xi)d\xi \quad (5.55)$$
$$+ \int_b^a \frac{1}{12}(x - b)[2(x - b)^2 - 3(\xi - b)(b(\lambda - 2) + 2x)] f_2(\xi)d\xi$$

representing, in physical terms, the deflection function for the right-hand span of the beam, caused by the transverse loads $q_1(x)$ applied to the left-hand span and $q_2(x)$ to the right-hand span.

Recalling $w_1(x)$ in (5.52), we express the functions $f_1(x)$ and $f_2(x)$ from that equation in terms of the loading functions $q_1(x)$ and $q_2(x)$ (see (5.40) and (5.41)), and rewrite $w_1(x)$ in the form

$$w_1(x) = \int_0^b g_{11}(x,\xi)q_1(\xi)d\xi + \int_b^a g_{12}(x,\xi)q_2(\xi)d\xi, \tag{5.56}$$

where

$$g_{11}(x,\xi) = \frac{1}{12b^3EI_1}\begin{cases} x^2(\xi-b)[\xi(3b-x)(2b-\xi)-2b^2x], & \text{for } x \le s, \\ \xi^2(x-b)[x(3b-\xi)(2b-x)-2b^2\xi], & \text{for } s \le x, \end{cases} \tag{5.57}$$

with both the variables x and ξ ranging between 0 and b. For $g_{12}(x,\xi)$, with $0 \le x \le b$ and $b \le \xi \le a$, we find

$$g_{12}(x,\xi) = \frac{\lambda x^2}{4bEI_2}(b-\xi)(x-b). \tag{5.58}$$

Similarly, the function $w_2(x)$ transforms to

$$w_2(x) = \int_0^b g_{21}(x,\xi)q_1(\xi)d\xi + \int_b^a g_{22}(x,\xi)q_2(\xi)d\xi, \tag{5.59}$$

where

$$g_{21}(x,\xi) = \frac{\xi^2}{4bEI_1}(b-\xi)(x-b) \tag{5.60}$$

with $b \le x \le a$ and $0 \le \xi \le b$. For $g_{22}(x,\xi)$, with both the variables x and ξ ranging between b and a, we obtain

$$g_{22}(x,\xi) = \frac{1}{12EI_2}\begin{cases} (x-b)[2(x-b)^2+3(b-\xi)(2x-b(2-\lambda))], & x \le \xi, \\ (\xi-b)[2(\xi-b)^2+3(b-x)(2\xi-b(2-\lambda))], & \xi \le x. \end{cases} \tag{5.61}$$

Observing the solutions $w_1(x)$ and $w_2(x)$ in (5.56) and (5.59), to the three-point posed boundary-value problem in (5.40)–(5.44), we can guess that $g_{ij}(x,\xi)$ in (5.57), (5.58) and (5.60), (5.61) represent elements of the matrix of Green's type $G(x,\xi)$ for the homogeneous boundary-value problem corresponding to (5.40)–(5.44). In physical terms, we may call $G(x,\xi)$ the influence matrix for a transverse point concentrated unit force for the compound beam shown in Figure 5.2.

It is worth noting that there is a match between the variables x and ξ in $g_{ij}(x,\xi)$ and the subscripts i and j. To clarify this, we recall that the first subscript i specifies the span number of the observation point, whereas the second subscript j represents the span number of the force application point. This implies that the match goes with the fact that the x variable in $g_{ij}(x,\xi)$ is located in the ith span, while the ξ variable belongs to the jth span.

Example 5.5. Construct the influence function (matrix) of a transverse unit force for the compound cantilever beam overhanging an intermediate elastic support, with elastic spring constant k^*, as shown in Figure 5.3.

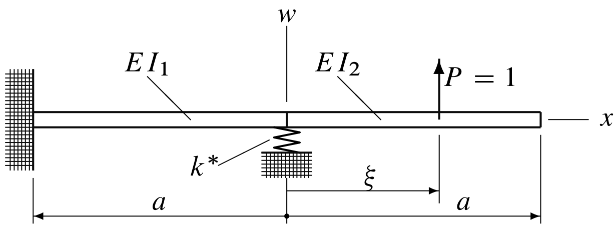

Figure 5.3. A beam overhanging an elastic support.

The influence matrix that we set out to find represents the matrix of Green's type for the homogeneous problem corresponding to the following three-point-posed boundary-value problem

$$\frac{d^4 w_1(x)}{dx^4} = -\frac{q_1(x)}{EI_1} = -f_1(x), \quad x \in (-a, 0), \tag{5.62}$$

$$\frac{d^4 w_2(x)}{dx^4} = -\frac{q_2(x)}{EI_2} = -f_2(x), \quad x \in (0, a), \tag{5.63}$$

$$w_1(-a) = \frac{dw_1(-a)}{dx} = 0, \quad \frac{d^2 w_2(a)}{dx^2} = \frac{d^3 w_2(a)}{dx^3} = 0, \tag{5.64}$$

$$w_1(0) = w_2(0), \quad \frac{dw_1(0)}{dx} = \frac{dw_2(0)}{dx}, \tag{5.65}$$

$$EI_1 \frac{d^2 w_1(0)}{dx^2} = EI_2 \frac{d^2 w_2(0)}{dx^2}, \tag{5.66}$$

$$EI_1 \frac{d^3 w_1(0)}{dx^3} - k w_1(0) = EI_2 \frac{d^3 w_2(0)}{dx^3} + k w_2(0), \quad k = 2k^*, \tag{5.67}$$

for the deflection functions $w_1(x)$ and $w_2(x)$, to be determined on the intervals $[-a, 0]$ and $[0, a]$, respectively.

Following the procedure described in detail in Example 5.4, we find the solution to the problem in (5.62)–(5.67) of the form

$$w_1(x) = \int_{-a}^{x} \frac{(\xi - x)^3}{6} f_1(\xi)d\xi + H_1 + K_1 x + L_1 x^2 + M_1 x^3$$

and

$$w_2(x) = \int_{0}^{x} \frac{(\xi - x)^3}{6} f_2(\xi)d\xi + H_2 + K_2 x + L_2 x^2 + M_2 x^3.$$

The constants of integration H_i, K_i, L_i and M_i, $i = 1, 2$, can be determined by taking advantage of the set of boundary and contact conditions specified in (5.64)–(5.67), yielding a well-posed system of eight linear algebraic equations for H_i, K_i, L_i and M_i. After obtaining the latter, and substituting them into the above expressions for $w_1(x)$ and $w_2(x)$, we get

$$\begin{aligned} w_1(x) = &\int_{-a}^{x} \frac{(\xi - x)^3}{6} f_1(\xi)d\xi + \int_{-a}^{0} \frac{(a+x)^2}{6p}\{k\xi[x(\xi^2 - a^2) - 2a(\xi^2 + ax)] \\ &+ 3EI_1[(x+a) - 3(\xi + a)]\} f_1(\xi)d\xi \\ &+ \lambda \int_{0}^{a} \frac{(a+x)^2}{2p}\{EI_1[(a+x) - 3(a+\xi)] - ka^2 x\xi\} f_2(\xi)d\xi, \quad x \in [-a, 0], \end{aligned}$$

and

$$\begin{aligned} w_2(x) = &\int_{-a}^{0} \frac{(a+\xi)^2}{2p}\{EI_1[(a+\xi) - 3(x+a)] - ka^2\xi x\} f_1(\xi)d\xi \\ &+ \int_{0}^{x} \frac{(\xi - x)^3}{6} f_2(\xi)d\xi + \int_{0}^{a} \frac{1}{6p}\{px^2(x - 3\xi) - 3\lambda k a^4 x\xi \\ &- 3EI_2 a[a(2a + 3\xi) + 3x(a + 2\xi)]\} f_2(\xi)d\xi, \quad x \in [0, a], \end{aligned}$$

where p and λ are introduced as $p = (2a^3 k + 3EI_1)$ and $\lambda = EI_2/EI_1$. The above can be displayed in compact form

$$w_1(x) = \int_{-a}^{0} g_{11}(x,\xi) f_1(\xi)d\xi + \int_{0}^{a} g_{12}(x,\xi) f_2(\xi)d\xi, \quad x \in [-a, 0], \qquad (5.68)$$

and

$$w_2(x) = \int_{-a}^{0} g_{21}(x,\xi) f_1(\xi)d\xi + \int_{0}^{a} g_{22}(x,\xi) f_2(\xi)d\xi, \quad x \in [0, a], \qquad (5.69)$$

with the kernels $g_{ij}(x,\xi)$ of the above integrals representing elements of the influence matrix of a transverse unit force of the beam under consideration. The element $g_{11}(x,\xi)$ is expressed in two segments, with

$$g_{11}^{+}(x,\xi) = \frac{(a+x)^2}{6pEI_1}\{k\xi[x(\xi^2 - a^2) - 2a(\xi^2 + ax)] + 3EI_1[(x+a) - 3(\xi + a)]\}$$

representing the first branch of $g_{11}(x,\xi)$, valid for $-a \le x \le \xi \le 0$. For the second branch of $g_{11}(x,\xi)$, valid for $-a \le \xi \le x \le 0$, we find

$$g_{11}^{-}(x,\xi) = \frac{(a+\xi)^2}{6pEI_1}\{kx[\xi(x^2-a^2)-2a(x^2+a\xi)]+3EI_1[(\xi+a)-3(x+a)]\}.$$

The element $g_{12}(x,\xi)$, defined for $x \in [-a,0]$ and $\xi \in [0,a]$, is found as

$$g_{12}(x,\xi) = \frac{(a+x)^2}{2pEI_1}\{EI_1[(a+x)-3(a+\xi)]-ka^2x\xi\}$$

and the element $g_{21}(x,\xi)$, with $\xi \in [-a,0]$ and $x \in [0,a]$, reads

$$g_{21}(x,\xi) = \frac{(a+\xi)^2}{2pEI_1}\{EI_1[(a+\xi)-3(x+a)]-ka^2\xi x\}.$$

Finally, for the branch of $g_{22}^{+}(x,\xi)$, with both variables x and ξ in the interval $[0,a]$ and $x \le \xi$, we obtain

$$g_{22}^{+}(x,\xi) = \frac{1}{6pEI_2}\{px^2(x-3\xi)-3\lambda ka^4x\xi-3EI_2a[a(2a+3\xi)+3x(a+2\xi)]\}$$

while for the branch $g_{22}^{-}(x,\xi)$, with $x \ge \xi$, we have

$$g_{22}^{-}(x,\xi) = \frac{1}{6pEI_2}\{p\xi^2(\xi-3x)-3\lambda ka^4x\xi-3EI_2a[a(2a+3x)+3\xi(a+2x)]\}.$$

Note that, after obtaining the influence matrix of a transverse point concentrated unit force, the elements of which are displayed above, we can determine the response of the beam depicted in Figure 5.3 to any applied transverse load. All components of the stress-strain state of the beam can be computed by using the integral representations for the deflection function in (5.68) and(5.69). However, a word of caution is appropriate with regard to a feasible implementation of this approach: the user has to make sure that the problem in (5.62)–(5.67) represents an adequate mathematical model for the actual mechanical setting. That is, the latter is assumed physically and geometrically linear [7, 19, 56, 71].

Example 5.6. Determine the deflection function caused by a transverse concentrated unit force applied to the left-hand span at an arbitrary point ξ for the compound triple-span beam depicted in Figure 5.4.

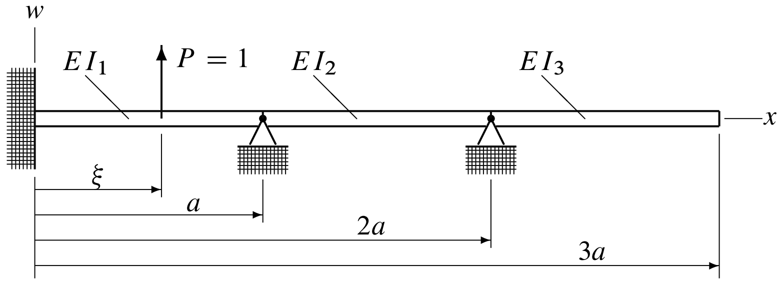

Figure 5.4. A compound beam overhanging two supports.

From the mechanics of materials, it follows that the four-point-posed boundary-value problem

$$\frac{d^4w_1(x)}{dx^4} = -\frac{q_1(x)}{EI_1}, \quad x \in (0,a), \tag{5.70}$$

$$\frac{d^4w_2(x)}{dx^4} = -\frac{q_2(x)}{EI_2}, \quad x \in (a,2a), \tag{5.71}$$

$$\frac{d^4w_3(x)}{dx^4} = -\frac{q_3(x)}{EI_3}, \quad x \in (2a,3a), \tag{5.72}$$

$$w_1(0) = \frac{dw_1(0)}{dx} = 0, \quad \frac{d^2w_3(3a)}{dx^2} = \frac{d^3w_3(3a)}{dx^3} = 0, \tag{5.73}$$

$$w_1(a) = w_2(a) = 0, \quad \frac{dw_1(a)}{dx} = \frac{dw_2(a)}{dx}, \tag{5.74}$$

$$EI_1\frac{d^2w_1(a)}{dx^2} = EI_2\frac{d^2w_2(a)}{dx^2}, \quad EI_2\frac{d^2w_2(2a)}{dx^2} = EI_3\frac{d^2w_3(2a)}{dx^2}, \tag{5.75}$$

$$w_2(2a) = w_3(2a) = 0, \quad \frac{dw_2(2a)}{dx} = \frac{dw_3(2a)}{dx} \tag{5.76}$$

models the bending of the beam depicted in Figure 5.4, with $w_1(x)$, $w_2(x)$ and $w_3(x)$ representing the deflection functions of the corresponding span of the beam.

In one of the Chapter Exercises, we challenge the reader to determine whether the above boundary-value problem is well-posed or ill-posed.

It is evident that once the matrix of Green's type of the homogeneous problem corresponding to (5.70)–(5.76)

$$G(x,\xi) = \left(g_{ij}(x,\xi)\right)_{i,j=\overline{1,3}} \tag{5.77}$$

is obtained, the elements of its first column $g_{11}(x,\xi)$, $g_{21}(x,\xi)$, and $g_{31}(x,\xi)$ represent the deflection of the beam, caused by a unit force applied at an arbitrary point $\xi \in [0,a]$.

In mechanical terms, the matrix of Green's type $G(x, \xi)$ in (5.77) can be referred to, as the influence matrix for a transverse unit point force for the beam in our problem. We can find it by following again the method of variation of parameters, in order to solve the problem in (5.70)–(5.76). We described the specifics of this method earlier in Example 5.4. We leave it as an exercise to the reader, to follow the procedure thoroughly, in order to grasp the details of the method.

Following now the variation of parameters procedure, we obtain the deflection function $w_1(x)$ for the left-hand span $(0, a)$ of the beam in the form

$$w_1(x) = \int_0^a g_{11}(x,\xi)q_1(\xi)d\xi + \int_a^{2a} g_{12}(x,\xi)q_2(\xi)d\xi \\ + \int_{2a}^{3a} g_{13}(x,\xi)q_3(\xi)d\xi, \quad x \in [0,a], \tag{5.78}$$

with the kernel of the first integral term in (5.78) representing the element $g_{11}(x, \xi)$ of the first column of $G(x, \xi)$. The second element $g_{21}(x, \xi)$ of the first column of $G(x, \xi)$ can be found once we find the deflection function $w_2(x)$ as

$$w_2(x) = \int_0^a g_{21}(x,\xi)q_1(\xi)d\xi + \int_a^{2a} g_{22}(x,\xi)q_2(\xi)d\xi \\ + \int_{2a}^{3a} g_{23}(x,\xi)q_3(\xi)d\xi, \quad x \in [a,2a]. \tag{5.79}$$

Hence, $g_{21}(x, \xi)$ is the kernel of the first integral term in (5.79). Finally, the element $g_{31}(x, \xi)$ can be read from the first integral term in

$$w_3(x) = \int_0^a g_{31}(x,\xi)q_1(\xi)d\xi + \int_a^{2a} g_{32}(x,\xi)q_2(\xi)d\xi \\ + \int_{2a}^{3a} g_{33}(x,\xi)q_3(\xi)d\xi, \quad x \in [2a,3a]. \tag{5.80}$$

Omitting the details of the algorithm, we display the final result. Note that $g_{11}(x, \xi)$ is defined in two segments, one of them valid for $0 \le x \le \xi \le a$, and obtained as

$$g_{11}^{+}(x,\xi) = \frac{x^2(a-\xi)}{6pa^3}\{2[\xi(2a-\xi)(x-3a) + 2a^2x] \\ + 3\lambda_1(a-\xi)[x(a+\xi) + \xi(x-3a)]\}, \quad 0 \le x \le \xi \le a.$$

The other branch $g_{11}^{-}(x, \xi)$ of $g_{11}(x, \xi)$, valid for $0 \le \xi \le x \le a$, is found to be

$$g_{11}^{-}(x,\xi) = \frac{\xi^2(a-x)}{6pa^3}\{2[x(2a-x)(\xi-3a) + 2a^2\xi] \\ + 3\lambda_1(a-x)[\xi(a+x) + x(\xi-3a)]\}, \quad 0 \le \xi \le x \le a,$$

where $p = 4EI_1 + 3EI_2$ and $\lambda_1 = EI_2/EI_1$.

Since the arguments $x \in [a, 2a]$ and $\xi \in [0, a]$ of $g_{21}(x, \xi)$ are specified on different domains, this element of $G(x, \xi)$ is defined in a single piece, found as

$$g_{21}(x, \xi) = \frac{1}{2pa^3}\xi^2(\xi - a)(3a - x)(a - x)(2a - x),$$

whilst for the element $g_{31}(x, \xi)$, with $x \in [2a, 3a]$ and $\xi \in [0, a]$, we similarly find

$$g_{31}(x, \xi) = \frac{1}{2pa}\xi^2(a - \xi)(2a - x).$$

Hence, the problem in Exercise 5.6 has been solved formally: the response to the force $P = 1$ applied at an arbitrary point ξ on the left-hand span of the beam depicted in Figure 5.4 is already found.

If the beam is loaded with a continuously distributed load $q_1(x)$ applied within the left-hand span only, then the deflection functions $w_1(x)$, $w_2(x)$ and $w_3(x)$ for all the three spans are defined exclusively by the first integral terms in (5.78), (5,79) and (5.80), respectively. If, on the other hand, the beam is loaded with three continuously distributed loads $q_1(x)$, $q_2(x)$ and $q_3(x)$, we need to compute all nine integral terms in (5.78), (5,79) and (5.80). To achieve that, we need the remaining elements of $G(x, \xi)$. Hence, we display them all in what follows.

We obtain $g_{12}(x, \xi)$, with $x \in [0, a]$ and $\xi \in [a, 2a]$, in the form

$$g_{12}(x, \xi) = \frac{1}{2pa^3}x^2(a - \xi)(3a - \xi)(x - a)(2a - \xi),$$

whilst the branch $g_{22}^{+}(x, \xi)$ of $g_{22}(x, \xi)$, with both variables x and ξ in $[a, 2a]$, and $x \leq \xi$, is expressed as

$$g_{22}^{+}(x, \xi) = \frac{1}{6\lambda_1 pa^3}(2a - \xi)(a - x)\{2(x - a)[\xi(\xi - 4a)(x - 4a) + a^2(x - 10a)] - 3\lambda_1 a^2[(\xi - 3a)(\xi - a) + (x - a)^2]\}, \quad x \leq \xi.$$

We find the other branch $g_{22}^{-}(x, \xi)$, for $\xi \leq x$, as

$$g_{22}^{-}(x, \xi) = \frac{1}{6\lambda_1 pa^3}(2a - x)(a - \xi)\{2(\xi - a)[x(x - 4a)(\xi - 4a) + a^2(\xi - 10a)] - 3\lambda_1 a^2[(x - 3a)(x - a) + (\xi - a)^2]\},$$

whilst $g_{32}(x, \xi)$, with $x \in [2a, 3a]$ and $\xi \in [a, 2a]$, reads

$$g_{32}(x, \xi) = \frac{1}{2\lambda_1 pa}(\xi - a)(x - 2a)(\xi - 2a)[2(a - \xi) - \lambda_1 \xi].$$

For $g_{13}(x,\xi)$, with $x \in [0,a]$ and $\xi \in [2a,3a]$, we obtain

$$g_{13}(x,\xi) = \frac{1}{2pa}x^2(a-x)(2a-\xi).$$

The element $g_{23}(x,\xi)$, with $x \in [a,2a]$ and $\xi \in [2a,3a]$, is expressed as

$$g_{23}(x,\xi) = \frac{1}{2\lambda_1 pa}(x-2a)(x-a)(\xi-2a)[2(a-x)-\lambda_1 x].$$

For the branch $g_{33}^{+}(x,\xi)$ of $g_{33}(x,\xi)$, for $2a \le x \le \xi \le 3a$, we obtain

$$g_{33}^{+}(x,\xi) = (x-2a)\left\{\frac{1+\lambda_1}{\lambda_1 p}a(2a-\xi) + \frac{(x-2a)}{6EI_3}[(x+a)+3(a-\xi)]\right\},$$

whilst the other branch $g_{33}^{-}(x,\xi)$ of $g_{33}(x,\xi)$, for $2a \le \xi \le x \le 3a$, is found as

$$g_{33}^{-}(x,\xi) = (\xi-2a)\left\{\frac{1+\lambda_1}{\lambda_1 p}a(2a-x) + \frac{(\xi-2a)}{6EI_3}[(\xi+a)+3(a-x)]\right\}.$$

The influence matrix, the elements of which we just displayed, enables us to analytically obtain components of the stress-strain state, caused in the beam by any combination of transverse and bending loads. Note that, if the load functions $q_1(x), q_2(x)$, and $q_3(x)$ have a simple form (polynomial, exponential, trigonometric, or their elementary combination), the integration in (5.78)–(5.80) can be performed analytically. If this is not the case, we can approximate it by using appropriate quadrature formulas.

The examples we analyzed in this section are helpful to comprehending the material, but they do not clarify all the possible peculiarities of the influence function method, when applied to multi-spanned compound beams. We encourage the reader to work through the Chapter Exercises in order to gain further experience, required for work in this field.

5.3 Further Extension of the Formalism

With the exception of Example 5.3, our involvement in the previous sections of the present Chapter has been limited to applications of the matrix of Green's type formalism to the so-called *sandwich type inhomogeneity* of the material out of which the assembly is composed. To further extend the range of possible applications of this formalism, we intend to target, in the current section, multi-point-posed boundary-value problems of a more general type. We have chosen the framework of graph theory to provide a backdrop for such an extension.

We will consider specific sets of linear ordinary differential equations, set up on a finite weighted graph such that each of the equations governs a single unknown function and is defined on a single edge of the graph. The individual equations are treated as a system by imposing contact and boundary conditions at the vertexes and endpoints of the graph. Based on this system, we introduce a novel definition of the matrix of Green's type. We will address the existence and uniqueness of such matrices and we will propose two analytical methods for their practical construction. We will consider a number of specific problems, describing phenomena and processes in continuum mechanics.

Numerous authors have recommended, and actually used computational implementations of the approach based on Green's function to solving problems in applied mathematical physics (see, for example, [6, 13, 14, 17, 26, 30, 32, 40, 41, 42, 45, 47, 48, 64, 69, 70]). However, as we mentioned earlier in this book, practical use of Green's functions for actual computations in engineering and science is significantly restricted. We can list many reasons to explain this situation. One of the most significant of them is the lack of available computer-friendly formulas for Green's functions in the literature.

The Green's function formalism is only applicable to settings with linear ordinary or partial differential equations with continuous coefficients. However, the requirement of continuity of coefficients can be significantly relaxed. In [44], successful attempts were undertaken to extend this formalism to boundary-value problems in continuum mechanics, formulated throughout piecewise homogeneous regions, yielding a discontinuity of the coefficients in the governing differential equations. An effort has been put forth to implement the Green's function formalism to treat the so-called multi-point-posed boundary-value problems that model various settings in continuum mechanics for piecewise homogeneous media. This is where the notion of the matrix of Green's type was first introduced.

In Sections 5.1 and 5.2, we have followed the concept proposed in [44]. It is worth noting, however, that the range of implementation of that notion is limited to the sandwich type of material inhomogeneity. It is our intention in the present section to introduce notion of the matrix of Green's type in a different way. The objective is to provide an extension of the Green's function formalism to multi-point-posed boundary-value problems occurring in complex assemblies, consisting of different homogeneous fragments.

For notational convenience, we specify boundary-value problems for systems of linear ordinary differential equations on finite weighted graphs. This allows us a systematic analysis of a variety of problems for assemblies of one-dimensional fragments.

Consider a finite weighted graph R (see Figure 5.5). For terminological purposes, vertexes of degree one will be referred to as the *endpoints*. Let the graph have n edges denoted with $e_i, i = \overline{1,n}$, m endpoints, $E_h, h = \overline{1,m}$, and r vertexes, $V_k, k = \overline{1,r}$. In addition, let d_k represent the degree of the vertex V_k and let the positive real numbers $l_i, i = \overline{1,n}$, each representing the length of the edge e_i, be regarded as its weight.

Suppose that all edges e_i of R (each fragment in the assembly) is occupied with

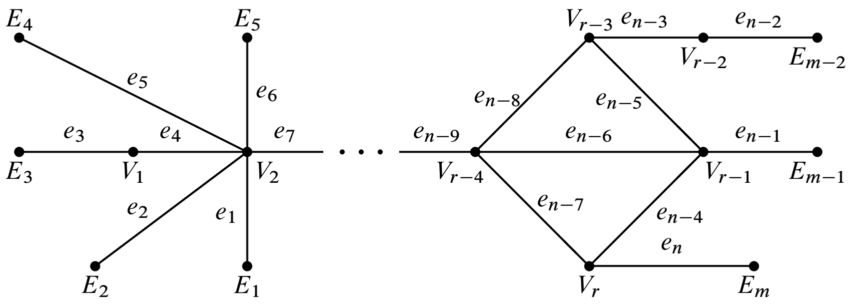

Figure 5.5. Graph R hosting a system of equations.

a conducting material (of either thermal or electrical or any other relevant nature), the conductivity $p_i(x)$ of which is a continuously differentiable function of a local coordinate x, which is longitudinal for all edges.

Let $u_i(x)$ represent the unknown function (temperature, electric potential, etc.) to be determined on the edge e_i of R. We will determine a set of these functions using the following set of linear second order differential equations

$$\frac{d}{dx}\left(p_i(x)\frac{du_i(x)}{dx}\right) + q_i(x)u_i(x) = -f_i(x), \quad x \in (0, l_i),\ i = \overline{1, n}, \tag{5.81}$$

each of which is defined on an individual edge e_i of the graph R.

These individual equations are arranged into a system by imposing a set of contact conditions

$$u_1(V_k) = \cdots = u_{d_k}(V_k), \quad \sum_{j=1}^{d_k} p_j(V_k)\frac{du_j(V_k)}{dx} = 0, \quad k = \overline{1, r}, \tag{5.82}$$

at each of the vertexes V_k, with d_k the degree of V_k. In formulating the above conditions, we use, for notational convenience, a 'local' numbering of the edges incident to the vertex V_k. It can be easily seen that the number of contact conditions imposed at each vertex is equal to the degree of the vertex. Clearly, the contact conditions in (5.82) simulate the conservation of energy law at each vertex V_k of R. In addition, the boundary conditions

$$\alpha_h\frac{du_i(E_h)}{dx} + \beta_h u_i(E_h) = 0, \quad h = \overline{1, m}, \tag{5.83}$$

are imposed in each of the endpoints E_h of R, implying that the functions $u_i(x)$ in (5.83) are defined on the edges e_i incident to E_h.

Observe that the number of contact conditions imposed at a vertex V_k is equal to the degree of the vertex, whilst a single boundary condition is imposed at each endpoint E_h. This implies that the total number N of uniqueness conditions imposed in (5.82) and (5.83) is

$$N = m + \sum_{k=1}^{r} d_k$$

which is, according to graph theory [62], twice the number of edges n in R, i.e. $N = 2n$. This makes equations (5.81)–(5.83) a well-posed problem.

In this section, we will focus on finding the influence matrix representing the response of the entire assembly to a unit source acting, at an arbitrary point ξ within an arbitrary edge of R. Notice that the emphasis will be on multi-point-posed boundary-value problems similar to (5.81)–(5.83). However, we can readily extend the results of this section to problems formulated for higher order linear differential equations.

We are now in a position to extend the conventional definition of the Green's function so as to make it valid for a multi-point-posed boundary-value problem similar to (5.81)–(5.83).

Definition. An $n \times n$ matrix $G(x, \xi)$, whose elements $g_{ij}(x, \xi)$ are defined for $x \in e_i$ and $\xi \in e_j$ on R, is referred to as the *matrix of Green's type* of the homogeneous multi-point-posed boundary-value problem corresponding to (5.81)–(5.83), if for any fixed value of ξ, the elements $g_{ij}(x, \xi)$ have the following properties:

1. For $x \neq \xi$, the elements $g_{ii}(x, \xi)$ of the principal diagonal ($i = j$) are continuous functions of x on e_i, they have continuous partial derivatives with respect to x up to the second order, and satisfy the homogeneous equations corresponding to those in (5.81).
2. As $x = \xi$, the elements $g_{ii}(x, \xi)$ of the principal diagonal are continuous functions of x, whereas their first order partial derivatives with respect to x are discontinuous functions, providing

$$\lim_{x \to \xi^+} \frac{\partial g_{ii}(x, \xi)}{\partial x} - \lim_{x \to \xi^-} \frac{\partial g_{ii}(x, \xi)}{\partial x} = -\frac{1}{p_i(\xi)}$$

and

$$\lim_{\xi \to x^+} \frac{\partial g_{ii}(x, \xi)}{\partial x} - \lim_{\xi \to x^-} \frac{\partial g_{ii}(x, \xi)}{\partial x} = \frac{1}{p_i(\xi)}.$$

3. The peripheral ($i \neq j$) elements $g_{ij}(x, \xi)$ of $G(x, \xi)$ are continuous functions of x for any value of $\xi \in e_j$, they have continuous partial derivatives with respect to x up to the second order, and satisfy the homogeneous equations corresponding to those in (5.81).
4. All elements $g_{ij}(x, \xi)$ of $G(x, \xi)$ satisfy the contact and end conditions (which pertain to them) in (5.82) and (5.83), in the sense that each of these conditions is satisfied for ξ on any of the edges e_j, $j = \overline{1, n}$.

In the following discussion, the arguments x and ξ of the matrix of Green's type (analogously to those in the Green's function) will be referred to as the *observation* (*field*) *point* and the *source point*, respectively.

Before formulating a theorem that stipulates the existence and uniqueness of the matrix of Green's type, we note that if the problem in (5.81)–(5.83) is well-posed, having a unique solution, then the trivial solution

$$u_i(x) \equiv 0, \quad x \in (0, l_i),\ i = \overline{1,n},$$

represents the only solution of the corresponding homogeneous problem.

Theorem 5.1. *If the multi-point-posed boundary-value problem in* (5.81)–(5.83) *has a unique solution, there exists a unique matrix of Green's type $G(x,\xi)$ for the corresponding homogeneous problem.*

Proof. Let $u_{i1}(x)$ and $u_{i2}(x)$, $i = \overline{1,n}$, be pairs of linearly independent particular solutions on e_i (fundamental sets of solutions), of the homogeneous equations corresponding to those in (5.81). Now, by virtue of defining property 1, the diagonal elements $g_{ii}(x,\xi)$ of $G(x,\xi)$ can be found in the formula

$$g_{ii}(x,\xi) = \begin{cases} a_{i1}(\xi)u_{i1}(x) + a_{i2}(\xi)u_{i2}(x), & \text{for } x \le s, \\ b_{i1}(\xi)u_{i1}(x) + b_{i2}(\xi)u_{i2}(x), & \text{for } x \ge s, \end{cases} \tag{5.84}$$

whereas, in accordance with defining property 3, the peripheral ($i \neq j$) elements $g_{ij}(x,\xi)$ of $G(x,\xi)$ can be written as

$$g_{ij}(x,\xi) = c_{ij}(\xi)u_{i1}(x) + d_{ij}(\xi)u_{i2}(x). \tag{5.85}$$

The coefficients $a_{i1}(\xi)$, $a_{i2}(\xi)$, $b_{i1}(\xi)$, $b_{i2}(\xi)$, $c_{ij}(\xi)$, and $d_{ij}(\xi)$ are to be determined through application of the remaining defining properties of the matrix of Green's type. With regard to whether this problem is well-posed or not, notice that the total number of coefficients equals $2n(n+1)$ whilst the total number of the relations provided by properties 2 and 4 is also equal to $2n(n+1)$.

By virtue of property 2, we obtain the following well-posed systems of linear algebraic equations in two unknowns

$$\begin{pmatrix} u_{i1}(\xi) & u_{i2}(\xi) \\ u'_{i1}(\xi) & u'_{i2}(\xi) \end{pmatrix} \times \begin{pmatrix} C_{i1}(\xi) \\ C_{i2}(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ p_i^{-1}(\xi) \end{pmatrix}, \quad i = \overline{1,n}, \tag{5.86}$$

providing n of the total $2n$ equations in $2n$ unknowns $C_{i1}(\xi)$ and $C_{i2}(\xi)$, $i = \overline{1,n}$. We express these unknowns in terms of the coefficient functions $g_{ii}(x,\xi)$ in (5.84) as

$$C_{ik}(\xi) = b_{ik}(\xi) - a_{ik}(\xi), \quad k = 1,2. \tag{5.87}$$

It follows from the fact that the determinant of its coefficient matrix represents Wronskian for the linearly independent functions $u_{i1}(x)$ and $u_{i2}(x)$, that the problem is indeed well-posed. Hence, a unique expression for $C_{i1}(\xi)$ and $C_{i2}(\xi)$ can be readily obtained. Subsequently, in accordance with (5.87), the functions $a_{i1}(\xi)$ and $a_{i2}(\xi)$ can be uniquely written in terms of $b_{i1}(\xi)$ and $b_{i2}(\xi)$ and vice versa.

Hence, the number of undetermined coefficients in (5.84) and (5.85) reduces to $2n^2$ and they can ultimately be found after applying defining property 4. After satisfying the entire set of boundary and contact conditions stated in (5.82) and (5.83) n times (once for each location of the source point $\xi \in e_j$, $j = \overline{1,n}$), we finally obtain an inhomogeneous system of $2n^2$ linear algebraic equations in $2n^2$ unknowns. The coefficient matrix of this system reduces to the following partitioned diagonal form

$$M = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{nn} \end{pmatrix}$$

in which A_{ii}, $i = \overline{1,n}$, represent the $2n \times 2n$ matrices, whose regularity follows from the fact that the original multi-point-posed boundary-value problem in (5.81)–(5.83) is well-posed. The peripheral sub-matrices of M represent $2n \times 2n$ null matrices. Thus, M represents a nonsingular matrix, meaning that all coefficient functions in (5.84) and (5.85) can be uniquely determined.

This completes the proof of Theorem 5.1 since, once the values of $a_{i1}(\xi)$, $a_{i2}(\xi)$, $b_{i1}(\xi)$, $b_{i2}(\xi)$, $c_{ij}(\xi)$, and $d_{ij}(\xi)$ have been found, we can immediately obtain explicit expressions for the elements of $G(x,\xi)$ by substituting them into (5.84) and (5.85). □

Notice that the proof, that we have just completed, can be called constructive: it offers a straightforward procedure for the practical construction of matrices of Green's type for multi-point-posed boundary-value problems defined on graphs.

Later in this section, we introduce an alternative procedure for obtaining matrices of Green's type for homogeneous boundary-value problems similar to the one defined on graphs in (5.81)–(5.83). This procedure is based on the method of variation of parameters. To aid its description, we introduce a vector-function $\mathbf{U}(x)$ the components of which $U_i(x)$, $i = \overline{1,n}$, are defined in terms of the solutions $u_i(x)$ of the governing equation in (5.81) as

$$U_i(x) = \begin{cases} u_i(x), & \text{for } x \in e_i, \\ 0, & \text{for } x \in R \setminus e_i. \end{cases} \tag{5.88}$$

We additionally introduce a vector-function $\mathbf{F}(x)$ the components of which $F_i(x)$ are defined in terms of the right-hand side functions $f_i(x)$ of (5.81) in the form

$$F_i(x) = \begin{cases} f_i(x), & \text{for } x \in e_i, \\ 0, & \text{for } x \in R \setminus e_i. \end{cases} \tag{5.89}$$

The following theorem will be proved to determine the solution of the problem specified in (5.81)–(5.83), in terms of the matrix of Green's type of the corresponding homogeneous problem.

Theorem 5.2. *If $G(x, \xi)$ represents the matrix of Green's type of the homogeneous multi-point-posed boundary-value problem corresponding to* (5.81)–(5.83)*, then the solution of the problem in* (5.81)–(5.83) *on R can be written as*

$$\mathbf{U}(x) = \int_R G(x, \xi)\mathbf{F}(\xi)dR(\xi), \quad x \in R, \tag{5.90}$$

where the integration is carried out over the entire graph R. The converse is also true: if the solution of the problem in (5.81)–(5.83) *on R is obtained in the integral formula in* (5.90)*, then the kernel $G(x, \xi)$ of the integral represents the matrix of Green's type for the homogeneous problem corresponding to* (5.81)–(5.83).

Proof. For the components of the vector-functions in (5.88) and (5.89), the integral in (5.90) gives us the scalar formula

$$u_i(x) = \sum_{j=1}^{n} \int_{e_j} g_{ij}(x, \xi) f_j(\xi) de_j(\xi), \quad i = \overline{1, n},$$

which can be rewritten in terms of the local coordinates as

$$u_i(x) = \sum_{j=1}^{n} \int_0^{l_j} g_{ij}(x, \xi) f_j(\xi) d\xi, \quad x \in [0, l_i], \ i = \overline{1, n}. \tag{5.91}$$

Since the diagonal elements $g_{ii}(x, \xi)$ and the peripheral elements $g_{ij}(x, \xi)$ of the matrix of Green's type are defined differently (see (5.84) and (5.85)), we isolate the ith term of the finite sum in (5.91)

$$u_i(x) = \sum_{j=1}^{i-1} \int_0^{l_j} g_{ij}(x, \xi) f_j(\xi) d\xi + \int_0^{l_i} g_{ii}(x, \xi) f_i(\xi) d\xi$$
$$+ \sum_{j=i+1}^{n} \int_0^{l_j} g_{ij}(x, \xi) f_j(\xi) d\xi, \quad x \in [0, l_i], \ i = \overline{1, n}.$$

As soon as we have defined the diagonal elements of $G(x, \xi)$ in segments, we can split the integral containing $g_{ii}(x, \xi)$ into two additive terms as follows

$$u_i(x) = \sum_{j=1}^{i-1} \int_0^{l_j} g_{ij}(x, \xi) f_j(\xi) d\xi$$
$$+ \int_0^{x} g_{ii}^{-}(x, \xi) f_i(\xi) d\xi + \int_x^{l_i} g_{ii}^{+}(x, \xi) f_i(\xi) d\xi$$
$$+ \sum_{j=i+1}^{n} \int_0^{l_j} g_{ij}(x, \xi) f_j(\xi) d\xi, \quad x \in [0, l_i], \ i = \overline{1, n},$$

with $g_{ii}^{-}(x,\xi)$ and $g_{ii}^{+}(x,\xi)$ representing the lower and the upper branches of the diagonal elements of $G(x,\xi)$, valid for $x \geq \xi$ and $x \leq \xi$, respectively (see (5.84)).

To properly differentiate the functions $u_i(x)$ in the above equation, we recall the defining properties of the elements of $G(x,\xi)$ and note that the above expression contains integrals involving parameters and has variable limits. With this in mind, we obtain

$$\begin{aligned}\frac{du_i(x)}{dx} &= \sum_{j=1}^{i-1}\int_0^{l_j}\frac{\partial g_{ij}(x,\xi)}{\partial x}f_j(\xi)d\xi + \int_0^x \frac{\partial g_{ii}^{-}(x,\xi)}{\partial x}f_i(\xi)d\xi \\ &\quad + g_{ii}(x,x^-)f_i(x) + \int_x^{l_i}\frac{\partial g_{ii}^{+}(x,\xi)}{\partial x}f_i(\xi)d\xi - g_{ii}(x,x^+)f_i(x) \\ &\quad + \sum_{j=i+1}^{n}\int_0^{l_j}\frac{\partial g_{ij}(x,\xi)}{\partial x}f_j(\xi)d\xi, \quad x\in[0,l_i],\ i=\overline{1,n}.\end{aligned}$$

The above formula transforms to the more compact form

$$\begin{aligned}\frac{du_i(x)}{dx} &= \sum_{j=1}^{i-1}\int_0^{l_j}\frac{\partial g_{ij}(x,\xi)}{\partial x}f_j(\xi)d\xi \\ &\quad + \int_0^x \frac{\partial g_{ii}^{-}(x,\xi)}{\partial x}f_i(\xi)d\xi + \int_x^{l_i}\frac{\partial g_{ii}^{+}(x,\xi)}{\partial x}f_i(\xi)d\xi \\ &\quad + \sum_{j=i+1}^{n}\int_0^{l_j}\frac{\partial g_{ij}(x,\xi)}{\partial x}f_j(\xi)d\xi, \quad x\in[0,l_i],\ i=\overline{1,n},\end{aligned}$$

since the sum

$$g_{ii}(x,x^-)f_i(x) - g_{ii}(x,x^+)f_i(x)$$

of the non-integral terms equals zero, in accordance with property 1 of the definition of $G(x,\xi)$. Hence, the derivative of $u_i(x)$ reads as

$$\frac{du_i(x)}{dx} = \sum_{j=1}^{n}\int_0^{l_j}\frac{\partial g_{ij}(x,\xi)}{\partial x}f_j(\xi)d\xi, \quad x\in[0,l_i],\ i=\overline{1,n}, \tag{5.92}$$

implying that the first order derivatives of the integral representations of $u_i(x)$ in (5.91) can be obtained by straightforward differentiation of the integrands. Consequently, these formulas for $u_i(x)$ and $du_i(x)/dx$, satisfy the boundary conditions in (5.82) and (5.83).

To find out whether the integral formulas for $u_i(x)$, as shown in (5.91), satisfy the governing differential equations, we obtain the second order derivatives of $u_i(x)$

$$\begin{aligned}\frac{d^2u_i(x)}{dx^2} &= \sum_{j=1}^{i-1}\int_0^{l_j}\frac{\partial^2 g_{ij}(x,\xi)}{\partial x^2}f_j(\xi)d\xi \\ &+ \int_0^x \frac{\partial^2 g_{ii}^-(x,\xi)}{\partial x^2}f_i(\xi)d\xi + \frac{\partial g_{ii}(x,x^-)}{\partial x}f_i(x) \\ &+ \int_x^{l_i}\frac{\partial^2 g_{ii}^+(x,\xi)}{\partial x^2}f_i(\xi)d\xi - \frac{\partial g_{ii}(x,x^+)}{\partial x}f_i(x) \\ &+ \sum_{j=i+1}^{n}\int_0^{l_j}\frac{\partial^2 g_{ij}(x,\xi)}{\partial x^2}f_j(\xi)d\xi, \quad x\in[0,l_i], \quad i=\overline{1,n}.\end{aligned}$$

In accordance with property 2 of the definition of $G(x,\xi)$, we have

$$\frac{\partial g_{jj}(x,x^-)}{\partial x}f_i(x) - \frac{\partial g_{jj}(x,x^+)}{\partial x}f_i(x) = -\frac{f_i(x)}{p_i(\xi)}.$$

And for the second order derivative of $u_i(x)$, we finally obtain a compact formula

$$\frac{d^2u_i(x)}{dx^2} = \sum_{j=1}^{n}\int_0^{l_j}\frac{\partial^2 g_{ij}(x,\xi)}{\partial x^2}f_j(\xi)d\xi - \frac{f_i(x)}{p_i(x)}, \quad x\in[0,l_i],\ i=\overline{1,n}. \tag{5.93}$$

After substituting $u_i(x)$ and their derivatives from (5.91)–(5.93) into (5.81), we finally obtain

$$\sum_{j=1}^{n}\int_0^{l_j} L[g_{ij}(x,\xi)]f_j(\xi)d\xi - f_i(x) = -f_i(x), \quad x\in(0,l_i),$$

with L representing the differential operator of (5.81).

Thus, the integral representations of $u_i(x)$ in (5.91) satisfy the governing differential equations, because the elements of the matrix of Green's type satisfy the homogeneous equations corresponding to (5.81). That is, $L[g_{ij}(x,\xi)] = 0$, which makes the integral terms in the above equation vanish. With this, we have proven Theorem 5.2. □

Theorem 5.2 clearly suggests that, once the solution to the original problem in (5.81)–(5.83) is expressed in terms of the integral in (5.90), the kernel $G(x,\xi)$ of the integral represents the matrix of Green's type for the corresponding homogeneous problem.

We propose a procedure based on a version of the method of variation of parameters in order to obtain an integral representation of (5.90) for the solution of the inhomogeneous multi-point-posed boundary-value problem specified in (5.81)–(5.83).

To sketch the procedure briefly, we recall the fundamental sets of solutions $u_{i1}(x)$ and $u_{i2}(x)$ of the homogeneous equations corresponding to those in (5.81). We search for the general solution of (5.81) $u_i(x)$ following to the method of variation of parameters, as

$$u_i(x) = D_{i1}(x)u_{i1}(x) + D_{i2}(x)u_{i2}(x), \quad i = \overline{1,n}. \tag{5.94}$$

Based on this and following the standard routine of the method of variation of parameters, we obtain the following well-posed systems of linear algebraic equations

$$\begin{pmatrix} u_{i1}(x) & u_{i2}(x) \\ u'_{i1}(x) & u'_{i2}(x) \end{pmatrix} \times \begin{pmatrix} D'_{i1}(x) \\ D'_{i2}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -f_i(x)/p_i(x) \end{pmatrix}, \quad i = \overline{1,n},$$

for the derivatives of the coefficients $D_{i1}(x)$ and $D_{i2}(x)$ of $u_i(x)$. From this it follows that

$$D'_{i1}(x) = \frac{u_{i2}(x)f_i(x)}{p_i(x)W_i(x)}, \quad D'_{i2}(x) = -\frac{u_{i1}(x)f_i(x)}{p_i(x)W_i(x)}, \quad i = \overline{1,n},$$

where $W_i(x) = u_{i1}(x)u'_{i2}(x) - u_{i2}(x)u'_{i1}(x)$ represent the Wronskians of the fundamental solution sets $u_{i1}(x)$ and $u_{i2}(x)$.

Integration of the derivatives $D'_{i1}(x)$ and $D'_{i2}(x)$ yields

$$D_{i1}(x) = \int_0^x \frac{u_{i2}(\xi)f_i(\xi)}{p_i(\xi)W_i(\xi)}d\xi + E_{i1}, \quad i = \overline{1,n},$$

and

$$D_{i2}(x) = -\int_0^x \frac{u_{i1}(\xi)f_i(\xi)}{p_i(\xi)W_i(\xi)}d\xi + E_{i2}, \quad i = \overline{1,n},$$

where E_{i1} and E_{i2} represent constants of integration. Upon substituting $D_{i1}(x)$ and $D_{i2}(x)$ into (5.94), we can rewrite the latter as

$$\begin{aligned} u_i(x) = u_{i1}(x)\int_0^x \frac{u_{i2}(\xi)f_i(\xi)}{p_i(\xi)W_i(\xi)}d\xi - u_{i2}(x)\int_0^x \frac{u_{i1}(\xi)f_i(\xi)}{p_i(\xi)W_i(\xi)}d\xi \\ + E_{i1}u_{i1}(x) + E_{i2}u_{i2}(x), \quad i = \overline{1,n}. \end{aligned}$$

After combining the integral terms in the above equation, we finally obtain the general solution of (5.81) in the form

$$\begin{aligned} u_i(x) = \int_0^x \frac{u_{i1}(x)u_{i2}(\xi) - u_{i2}(x)u_{i1}(\xi)}{p_i(\xi)W_i(\xi)} f_i(\xi)d\xi \\ + E_{i1}u_{i1}(x) + E_{i2}u_{i2}(x), \quad x \in (0, l_i),\ i = \overline{1,n}. \end{aligned} \tag{5.95}$$

The $2n$ constants of integration E_{i1} and E_{i2} can be obtained by imposing the contact and boundary conditions in (5.82) and (5.83). The total number of linear equations resulting from the substitution also equals $2n$. This yields a well-posed system of linear algebraic equations. After solving the latter, we reduce the formula in (5.95) to the integral representation of (5.90), and the sought-after matrix of Green's type consequently appears as the kernel of the integral in (5.90).

In the following example we will apply the described formalism to a problem for a medium whose property is a discontinuous function of a spatial variable.

Example 5.7. Construct the matrix of Green's type for the steady-state heat conduction problem stated in an assembly of rods (see Figure 5.6), each of which is composed of a homogeneous material with thermal conductivity p_i.

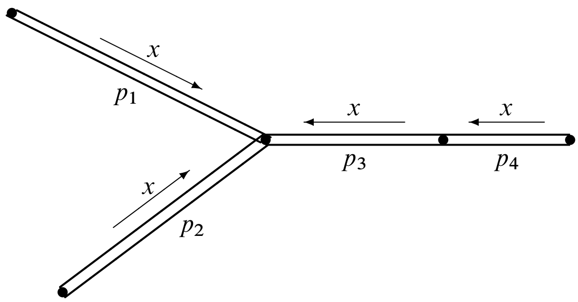

Figure 5.6. An assembly of heat conducting rods.

On the weighted graph associated with the assembly depicted above, we formulate the following five-point-posed boundary-value problem

$$p_i \frac{d^2u_i(x)}{dx^2} = -f_i(x), \quad x \in (0, l_i), \quad i = \overline{1,4}, \tag{5.96}$$

$$u_1(l_1) = u_2(l_2) = u_3(l_3), \tag{5.97}$$

$$p_1 \frac{du_1(l_1)}{dx} + p_2 \frac{du_2(l_2)}{dx} + p_3 \frac{du_3(l_3)}{dx} = 0, \tag{5.98}$$

$$u_3(0) = u_4(l_4), \quad p_3 \frac{du_3(0)}{dx} - p_4 \frac{du_4(l_4)}{dx} = 0, \tag{5.99}$$

$$u_1(0) = u_2(0) = u_4(0) = 0 \tag{5.100}$$

describing the steady-state heat conduction phenomenon in this assembly. Here l_i, $i = \overline{1,4}$, represent the lengths of the rods.

In one of the Chapter Exercises we ask the reader to determine whether the above boundary-value problem is well-posed or not.

Following again the method of variation of parameters, we search for a general solution of (5.96) of the form

$$u_{i,g}(x) = D_{i1}(x) + D_{i2}(x)x, \quad i = \overline{1,4},$$

in this case (analogous to the formula in (5.95)) ultimately reducing to

$$u_i(x) = \int_0^x \frac{\xi - x}{p_i} f_i(\xi)d\xi + E_{i1} + E_{i2}x, \quad x \in (0, l_i), \quad i = \overline{1,4}. \tag{5.101}$$

The constants of integration E_{i1} and E_{i2}, $i = \overline{1,4}$, in (5.101) are to be determined by imposing the contact and boundary conditions (5.97)–(5.100). In particular, the conditions (5.101) yield

$$E_{11} = E_{21} = E_{41} = 0.$$

For the remaining constants of integration, we obtain a well-posed system of linear algebraic equations appearing as

$$\begin{pmatrix} l_1 & -l_2 & 0 & 0 & 0 \\ l_1 & 0 & -1 & -l_3 & 0 \\ p_1 & p_2 & 0 & p_3 & 0 \\ 0 & 0 & 1 & 0 & -l_4 \\ 0 & 0 & 0 & p_3 & -p_4 \end{pmatrix} \times \begin{pmatrix} E_{12} \\ E_{22} \\ E_{31} \\ E_{32} \\ E_{42} \end{pmatrix} = \begin{pmatrix} A_2 - A_1 \\ A_3 - A_1 \\ B_1 + B_2 + B_3 \\ A_4 \\ -B_4 \end{pmatrix} \tag{5.102}$$

with the constants on the right-hand side found as

$$A_i = \int_0^{l_i} \frac{\xi - l_i}{p_i} f_i(\xi)d\xi, \quad B_i = \int_0^{l_i} f_i(\xi)d\xi, \quad i = \overline{1,4}.$$

In the following, we assume, for the sake of simplicity, that the rods in the assembly (the edges of the graph) are of equal length, i.e. $l_1 = l_2 = l_3 = l_4 = l$, converting the determinant Δ of the coefficient matrix of the system in (5.102) into

$$\Delta = l^2[(p_1 + p_2)(p_3 + p_4) + p_3 p_4].$$

Upon solving the system in (5.102) and subsequent substitution of the constants E_{i1} and E_{i2} into (5.101), we finally obtain

$$u_1(x) = \int_0^l \frac{x}{\Delta^* p_1}\{\Delta^* - \xi[p_2(p_3+p_4)+p_3p_4]\}f_1(\xi)d\xi \tag{5.103}$$
$$+\int_0^x \frac{\xi - x}{p_1} f_1(\xi)d\xi + \int_0^l \frac{x\xi}{\Delta^*}(p_3+p_4)f_2(\xi)d\xi$$
$$+\int_0^l \frac{x}{\Delta^*}(lp_3+\xi p_4)f_3(\xi)d\xi + \int_0^l \frac{x\xi}{\Delta^*}p_3 f_4(\xi)d\xi,$$

$$u_2(x) = \int_0^l \frac{x\xi}{\Delta^*}(p_3+p_4)f_1(\xi)d\xi \tag{5.104}$$
$$+\int_0^x \frac{\xi - x}{p_2} f_2(\xi)d\xi + \int_0^l \frac{x}{\Delta^* p_2}\{\Delta^* - \xi[p_1(p_3+p_4)+p_3p_4]\}f_2(\xi)d\xi$$
$$+\int_0^l \frac{x}{\Delta^*}(lp_3+\xi p_4)f_3(\xi)d\xi + \int_0^l \frac{x\xi}{\Delta^*}p_3 f_4(\xi)d\xi,$$

$$u_3(x) = \int_0^l \frac{\xi}{\Delta^*}(lp_3+xp_4)f_1(\xi)d\xi + \int_0^l \frac{\xi}{\Delta^*}(lp_3+xp_4)f_2(\xi)d\xi \tag{5.105}$$
$$+\int_0^l \frac{1}{\Delta^* p_3}[l(p_1+p_2+p_3)-\xi(p_1+p_2)](lp_3+xp_4)f_3(\xi)d\xi$$
$$+\int_0^x \frac{\xi - x}{p_3} f_3(\xi)d\xi + \int_0^l \frac{\xi}{\Delta^*}[l(p_1+p_2+p_3)-x(p_1+p_2)]f_4(\xi)d\xi,$$

and

$$u_4(x) = \int_0^l \frac{x\xi}{\Delta^*}p_3 f_1(\xi)d\xi + \int_0^l \frac{x\xi}{\Delta^*}p_3 f_2(\xi)d\xi \tag{5.106}$$
$$+\int_0^l \frac{x}{\Delta^*}[l(p_1+p_2+p_3)-\xi(p_1+p_2)]f_3(\xi)d\xi$$
$$+\int_0^x \frac{\xi - x}{p_4} f_4(\xi)d\xi + \int_0^l \frac{x}{\Delta^* p_4}[\Delta^* - \xi p_3(p_1+p_2)]f_4(\xi)d\xi,$$

where $\Delta^* = \Delta/l$.

Since the solution to the five-point-posed boundary-value problem specified in (5.96)–(5.100) is expressed in the integral form of (5.90), the elements $g_{ij}(x,\xi)$ of the matrix of Green's type $G(x,\xi)$ for the corresponding homogeneous problem can be read off from the integral representations in (5.103) through (5.106). We find for example the elements $g_{i1}(x,\xi)$ of the first column of $G(x,\xi)$ as

$$g_{11}(x,\xi) = \frac{1}{\Delta^* p_1}\begin{cases} x\{\Delta^* - \xi[p_2(p_3+p_4)+p_3p_4]\}, & \text{for } x \le s \\ \xi\{\Delta^* - x[p_2(p_3+p_4)+p_3p_4]\}, & \text{for } x \ge s, \end{cases}$$

$$g_{21}(x,\xi) = \frac{x\xi}{\Delta^*}(p_3+p_4), \quad g_{31}(x,\xi) = \frac{\xi}{\Delta^*}(lp_3+xp_4),$$

$$g_{41}(x,\xi) = \frac{x\xi}{\Delta^*}p_3.$$

In physical terms, the above functions represent the response of the assembly of rods, depicted in Figure 5.6, to a unit point source acting at a source point ξ located arbitrarily within the rod l_1. The remaining elements of the matrix of Green's type $G(x,\xi)$, representing responses to a unit source acting at other rods, could, if required, also be obtained directly from the integral representations of (5.103) through (5.106).

5.4 Chapter Exercises

1. Determine whether the following multi-point-posed boundary-value problems have only the trivial solution:

   (a) The homogeneous problem corresponding to that in (5.7)–(5.10);

   (b) the homogeneous problem corresponding to that in (5.22)–(5.25);

   (c) the homogeneous problem corresponding to that in (5.32)–(5.34).

2. Construct the matrix of Green's type for the following multi-point-posed boundary-value problem:

   (a) $y_1''(x) = 0$ for $x \in (-a,0)$ and $y_2''(x) - k^2 y_2(x) = 0$ for $x \in (0,\infty)$ with $y_1(-a) = 0$, $|y_2(\infty)| < \infty$, $y_1(0) = y_2(0)$, $y_1'(0) = \lambda y_2'(0)$;

   (b) $y_1''(x) - k^2 y_1(x) = 0$, $x \in (-a,0)$ and $y_2''(x) - k^2 y_2(x) = 0$, $x \in (0,\infty)$ with $y_1(-a) = 0$, $|y_2(\infty)| < \infty$, $y_1(0) = y_2(0)$, $y_1'(0) = \lambda y_2'(0)$;

   (c) $y_1''(x) + k^2 y_1(x) = 0$ for $x \in (-a,0)$ and $y_2''(x) + k^2 y_2(x) = 0$ for $x \in (0,a)$ with $y_1(-a) = 0$, $y_2(a) = 0$, $y_1(0) = y_2(0)$, $y_1'(0) = \lambda y_2'(0)$;

   (d) $y_i''(x) - k^2 y_i(x) = 0$, $i = 1,2,3$ for $x \in (0,\infty)$ with $y_1(0) = y_2(0) = y_3(0)$, $h_1 y_1'(0) + h_2 y_2'(0) + h_3 y_3'(0) = 0$, $|y_i(\infty)| < \infty$, $i = 1,2,3$.

3. Solve the four-point-posed boundary-value problem in (5.32)–(5.34) for the right-hand side functions $f_1(x) \equiv \sin(\pi x)$ and $f_2(x) = f_3(x) \equiv 0$.

4. Determine whether the four-point-posed boundary-value problem introduced in (5.70)–(5.76) is well-posed.

5. Construct the influence matrix for a transverse unit point concentrated force for the double-span compound (EI_1 and EI_2) beam clamped at one endpoint, and simply-supported at another, having an intermediate simple support right in the middle, as depicted in Figure 5.7.

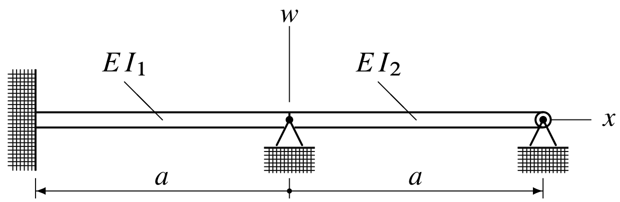

Figure 5.7. A beam with an intermediate support.

6. Construct the influence matrix for a transverse unit point concentrated force for the triple-span cantilever compound beam overhanging a simple support as depicted in Figure 5.8.

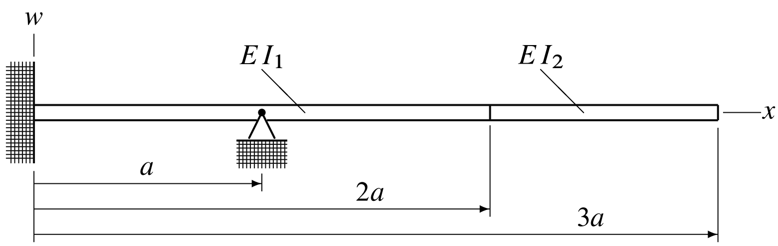

Figure 5.8. A compound cantilever beam overhanging one support.

7. Construct the influence matrix for a transverse unit point concentrated force for the triple-span beam of a uniform flexural rigidity EI, having two simple supports as depicted in Figure 5.9.

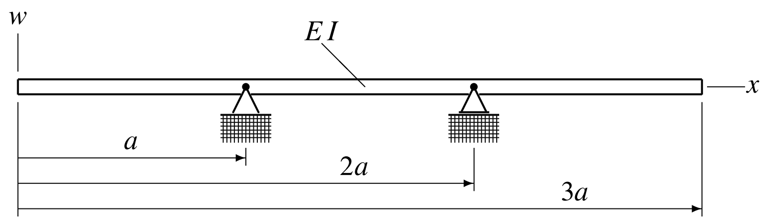

Figure 5.9. A triple-span beam having two supports.

8. Determine whether the five-point-posed boundary-value problem introduced in (5.96)–(5.100) is well-posed.
9. Justify the claim that the system of linear algebraic equations appearing in (5.102) is well-posed.

Chapter 6
PDE Matrices of Green's type

In the previous chapter, we discussed a special class of problems, and extended the classical concept of the Green's function to differential equations with discontinuous coefficients. As a result, the novel concept of *matrix of Green's type* was introduced for multi-point-posed boundary-value problems for linear ordinary differential equations.

The objective in the present chapter is to proceed with the above idea. We will extend the concept of matrix of Green's type to boundary-value problems for specific sets of elliptic partial differential equations with piecewise constant coefficients. The idea behind such an extension is to make the Green's function formalism a working instrument, applicable to two-dimensional problems of continuum mechanics in compound media, the properties of which vary discontinuously within the region under consideration.

We will obtain equivalents of Green's functions for sets of the Laplace and the static Klein–Gordon equations on regions filled with piecewise homogeneous isotropic conducting materials, which will prove to be practical for immediate computer implementation. We allow Dirichlet, Neumann and Robin conditions on the outer boundary of a simply-connected region, whilst conditions of ideal contact are assumed on the interface lines along which the material property looses its continuity.

6.1 Introductory Comments

To extend the concept of Green's function to problems specified in a medium of a compound structure, we consider a region $\Omega = \bigcup_{i=1}^{m} \Omega_i$, in two-dimensional Euclidean space (see Figure 6.1). Let the constants λ_i specify physical properties of the homogeneous materials (thermal, electric, or other conductivities) with which the fragments Ω_i of Ω are filled in. Let also each of the functions $u_i(P)$, defined in Ω_i, satisfy the non-homogeneous equation

$$\Lambda[u_i(P)] = -f_i(P), \quad P \in \Omega_i,\ i = \overline{1,m}, \tag{6.1}$$

and are subject to the boundary condition

$$B[u_i(P)] = 0, \quad P \in \Gamma_0,\ i = \overline{1,m}, \tag{6.2}$$

imposed on the contour Γ_0 of Ω, and to the conditions of ideal contact

$$u_k(P) = u_{k+1}(P), \qquad P \in \Gamma_k, k = \overline{1,m-1}, \tag{6.3}$$

$$\lambda_k \frac{\partial u_k(P)}{\partial n_k} = \lambda_{k+1} \frac{\partial u_{k+1}(P)}{\partial n_k}, \quad P \in \Gamma_k, k = \overline{1,m-1}, \tag{6.4}$$

imposed on the interfaces Γ_k, with n_k, representing the normal direction to Γ_k. Here Λ represents either the two-dimensional Laplace ∇^2 or the two-dimensional static Klein–Gordon $\nabla^2 - k^2$ operator, whilst B specifies one of the conventional boundary conditions for applications (either Dirichlet, or Neumann, or Robin).

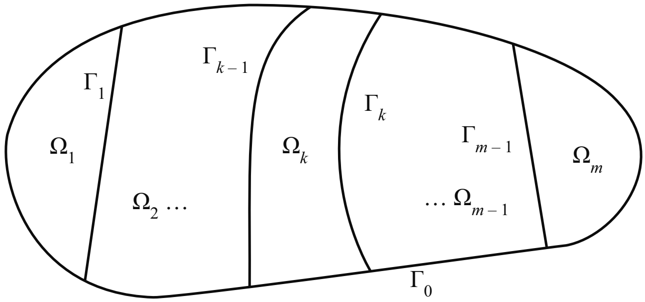

Figure 6.1. Compound, sandwich-like region.

We assume that the boundary-value problem in (6.1)–(6.4) is well-posed (in other words, it has a unique solution), implying that the corresponding homogeneous problem, with $f_i(P) \equiv 0$, $i = \overline{1,m}$, has only the trivial solution, such that $u_i(P) \equiv 0$, $i = \overline{1,m}$.

To compact the notation of our presentation, we introduce two vector-functions in the following fashion

$$\mathbf{U}(P) = (U_i(P))_{i=\overline{1,m}} \quad \text{and} \quad \mathbf{F}(P) = (F_i(P))_{i=\overline{1,m}} \tag{6.5}$$

with their components $U_i(P)$ and $F_i(P)$ defined in a piecewise manner as

$$U_i(P) = \begin{cases} u_i(P), & P \in \Omega_i, \\ 0, & P \in \Omega/\Omega_i, \end{cases}$$

and

$$F_i(P) = \begin{cases} f_i(P), & P \in \Omega_i, \\ 0, & P \in \Omega/\Omega_i. \end{cases}$$

With this problem setting in mind, specified as above , we are now ready to introduce a key concept in our presentation in the current chapter.

Definition. If, for any vector-function $\mathbf{F}(P)$, integrable on Ω, the vector-function $\mathbf{U}(P)$ is expressed in the form

$$\mathbf{U}(P) = \iint_{\Omega} \mathbf{G}(P, Q)\mathbf{F}(Q)d\Omega(Q) \tag{6.6}$$

then the kernel-matrix $\mathbf{G}(P, Q)$ in (6.6) represents the matrix of Green's type for the homogeneous problem corresponding to (6.1)–(6.4).

The term 'integrable' with respect to $\mathbf{F}(Q)$ implies

$$\iint_{\Omega} \mathbf{F}(Q) d\Omega(Q) < \infty.$$

Note that in our presentation, as is commonly accepted within the Green's function formalism, we will refer to P and Q, as the *field* (*observation*) *point* and the *source point*, respectively.

At this point, an important comment should be made: $\mathbf{G}(P, Q)$ represents an $m \times m$ matrix the elements $G_{i,j}(P, Q)$ of which are defined with the field point P belonging to the segment Ω_i, whereas the source point Q belongs to Ω_j. This implies that P and Q share their domain only for the diagonal elements $G_{i,i}(P, Q)$. Keeping this in mind, the elements $G_{i,j}(P, Q)$ of the matrix of Green's type meet the following properties:

1. The peripheral ($i \neq j$) elements $G_{i,j}(P, Q)$, as functions of the coordinates of the field point P on Ω_i, satisfy the homogeneous equation corresponding to (6.1), that is:

$$\Lambda[G_{i,j}(P, Q)] = 0, \quad \text{for } P \in \Omega_i.$$

2. The diagonal ($i = j$) elements $G_{i,i}(P, Q)$, as functions of the coordinates of the field point P in Ω_i, satisfy the homogeneous equation corresponding to (6.1) everywhere on Ω_i except for $P = Q$, that is:

$$\Lambda[G_{i,i}(P, Q)] = 0, \quad \text{for } P, Q \in \Omega_i \text{ and } P \neq Q.$$

3. For $P = Q$, the diagonal elements $G_{i,i}(P, Q)$ contain the logarithmic singularity:

$$G_{i,i}(P, Q) = \frac{1}{2\pi} \ln \frac{1}{|P - Q|}.$$

4. The elements $G_{i,j}(P, Q)$ satisfy all the relevant boundary and contact conditions imposed by (6.2)–(6.4).

It is evident that property 3 holds for both the Laplace and the Klein–Gordon equation, since the modified Bessel (or Macdonald) function of order zero of the second kind

$$\frac{1}{2\pi} K_0(k|P - Q|)$$

representing the fundamental solution for the static Klein–Gordon equation (see Chapter 3), contains the same logarithmic singularity as the fundamental solution of the Laplace equation in two dimensions.

In the following sections, we will show how matrices of Green's type can be constructed in practice, for a wide range of boundary-value problems, that model different phenomena in continuum mechanics.

6.2 Construction of Matrices of Green's Type

In this section, we will present a set of sample problems, in order to assist the reader in our technique. In each of those, several individual features of our approach will be highlighted and explained. We employed this methodology successfully, in our earlier chapters. We aim, at establishing a creative atmosphere, where the reader can gradually become familiar with the concept of matrix of Green's type. Our first example is perhaps the simplest of its brethren.

Example 6.1. Consider an infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$ of width b, and let it be composed of two semi-strips $\Omega_1 = \{-\infty < x < 0, 0 < y < b\}$ and $\Omega_2 = \{0 < x < \infty, 0 < y < b\}$, each filled with homogeneous isotropic conducting materials with conductivities λ_1 and λ_2, respectively. Let $u_1(x, y) \in \Omega_1$ and $u_2(x, y) \in \Omega_2$ satisfy the Poisson equations

$$\frac{\partial^2 u_i(x,y)}{\partial x^2} + \frac{\partial^2 u_i(x,y)}{\partial y^2} = -f_i(x,y), \quad (x,y) \in \Omega_i,\ i = 1,2, \tag{6.7}$$

with $f_1(x, y)$ and $f_2(x, y)$ being integrable on Ω_1 and Ω_2, respectively.

In addition, let the Dirichlet boundary conditions

$$u_i(x,0) = u_i(x,b) = 0, \quad i = 1,2, \tag{6.8}$$

be imposed on the outer contour of Ω, whilst we impose the contact conditions

$$u_1(0,y) = u_2(0,y), \quad \frac{\partial u_1(0,y)}{\partial x} = \lambda \frac{\partial u_2(0,y)}{\partial x} \tag{6.9}$$

on the interface $x = 0$, with λ defined as λ_2/λ_1. In addition, we require the boundedness conditions

$$\lim_{x\to-\infty} u_1(x,y) < \infty, \quad \lim_{x\to\infty} u_2(x,y) < \infty \tag{6.10}$$

for the problem to be well-posed.

Expand the functions $u_i(x, y)$ and $f_i(x, y)$, $i = 1, 2$, in the Fourier series

$$u_i(x,y) = \sum_{n=1}^{\infty} u_{i,n}(x) \sin \nu y, \quad f_i(x,y) = \sum_{n=1}^{\infty} f_{i,n}(x) \sin \nu y \tag{6.11}$$

with $\nu = n\pi/b$. After substituting these expansions in the original formulation of (6.7)–(6.10), we obtain the following set of three-point-posed boundary-value problems for the coefficients $u_{1,n}(x)$ and $u_{2,n}(x)$, $n = 1, 2, 3, \ldots$, of the first expansion in (6.11):

$$\frac{d^2u_{1,n}(x)}{dx^2} - \nu^2 u_{1,n}(x) = -f_{1,n}(x), \quad x \in (-\infty, 0), \tag{6.12}$$

$$\frac{d^2u_{2,n}(x)}{dx^2} - \nu^2 u_{2,n}(x) = -f_{2,n}(x), \quad x \in (0, \infty), \tag{6.13}$$

$$\lim_{x \to -\infty} u_{1,n}(x) < \infty, \qquad \lim_{x \to \infty} u_{2,n}(x) < \infty, \tag{6.14}$$

$$u_{1,n}(0) = u_{2,n}(0), \frac{du_{1,n}(0)}{dx} = \lambda \frac{du_{2,n}(0)}{dx}. \tag{6.15}$$

To find the solution to the above setting, we refer back to Chapter 5. Clearly, the fundamental sets of solutions for the homogeneous equations corresponding to (6.12) and (6.13) can be represented with, for example, the functions

$$\mathrm{e}^{\nu x} \quad \text{and} \quad \mathrm{e}^{-\nu x}.$$

Hence, in accordance with the standard procedure of the method of variation of parameters, we can write the solution to the boundary-value problem stated by (6.12)–(6.15) in the form

$$u_{i,n}(x) = C_{i,n}(x)\mathrm{e}^{\nu x} + D_{i,n}(x)\mathrm{e}^{-\nu x}, \quad i = 1, 2, \tag{6.16}$$

where $C_{i,n}(x)$ and $D_{i,n}(x)$ are to be determined. This yields the following well-posed system of linear algebraic equations

$$\begin{pmatrix} \mathrm{e}^{\nu x} & \mathrm{e}^{-\nu x} \\ \nu\mathrm{e}^{\nu x} & -\nu\mathrm{e}^{-\nu x} \end{pmatrix} \begin{pmatrix} C'_{i,n}(x) \\ D'_{i,n}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -f_{i,n}(x) \end{pmatrix}, \quad i = 1, 2,$$

for the derivatives of the coefficients $C_{i,n}(x)$ and $D_{i,n}(x)$ of (6.16). After solving the above system we obtain

$$C'_{i,n}(x) = -\frac{1}{2\nu}\mathrm{e}^{-\nu x} f_{i,n}(x), D'_{i,n}(x) = \frac{1}{2\nu}\mathrm{e}^{\nu x} f_{i,n}(x), \quad i = 1, 2.$$

Straightforward integration of the above yields

$$C_{1,n}(x) = -\frac{1}{2\nu}\int_{-\infty}^{x} \mathrm{e}^{-\nu\xi} f_{1,n}(\xi)d\xi + \gamma_1, \tag{6.17}$$

$$D_{1,n}(x) = \frac{1}{2\nu}\int_{-\infty}^{x} \mathrm{e}^{\nu\xi} f_{1,n}(\xi)d\xi + \delta_1, \tag{6.18}$$

$$C_{2,n}(x) = -\frac{1}{2\nu}\int_{0}^{x} \mathrm{e}^{-\nu\xi} f_{2,n}(\xi)d\xi + \gamma_2, \tag{6.19}$$

and

$$D_{2,n}(x) = \frac{1}{2\nu}\int_0^x e^{\nu\xi} f_{2,n}(\xi)d\xi + \delta_2. \tag{6.20}$$

After substituting these expressions in (6.16) and properly rearranging the integral terms, we obtain the general solutions to (6.12) and (6.13) of the form

$$u_{1,n}(x) = \gamma_1 e^{\nu x} + \delta_1 e^{-\nu x} + \frac{1}{2\nu}\int_{-\infty}^x [e^{-\nu(x-\xi)} - e^{\nu(x-\xi)}] f_{1,n}(\xi)d\xi \tag{6.21}$$

and

$$u_{2,n}(x) = \gamma_2 e^{\nu x} + \delta_2 e^{-\nu x} + \frac{1}{2\nu}\int_0^x [e^{-\nu(x-\xi)} - e^{\nu(x-\xi)}] f_{2,n}(\xi)d\xi. \tag{6.22}$$

We can obtain the constants of integration $\gamma_1, \gamma_2, \delta_1$, and δ_2 in $u_{1,n}(x)$ and $u_{2,n}(x)$ by imposing the boundary and contact conditions in (6.14) and (6.15). The first boundedness condition in (6.14), for example, yields $\delta_1 = 0$. The boundedness of $u_{2,n}(x)$ as x goes to infinity (see the second condition in (6.14)) allows us to find γ_2 directly. In order to do so, it is convenient to first transform the expression for $u_{2,n}(x)$ in (6.22) into an equivalent formula as

$$u_{2,n}(x) = \left[-\frac{1}{2\nu}\int_0^x e^{-\nu\xi} f_{2,n}(\xi)d\xi + \gamma_2\right]e^{\nu x} + \left[\frac{1}{2\nu}\int_0^x e^{\nu\xi} f_{2,n}(\xi)d\xi + \gamma_2\right]e^{-\nu x},$$

from which it follows immediately that since the exponential function $e^{\nu x}$ is unbounded at infinity, the factor of $e^{\nu x}$ must be zero for $u_{2,n}(x)$ to be bounded as x goes to infinity. This yields

$$\gamma_2 = \frac{1}{2\nu}\int_0^\infty e^{-\nu\xi} f_{2,n}(\xi)d\xi.$$

Now, we can use the contact conditions in (6.15) to get the following well-posed system of linear algebraic equations

$$\begin{pmatrix} 1 & -1 \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} \gamma_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix} \tag{6.23}$$

for the remaining constants of integration γ_1 and δ_2 in (6.17) and (6.20), with M and N defined as

$$M = \frac{1}{2\nu}\int_{-\infty}^0 (e^{-\nu\xi} - e^{\nu\xi}) f_{1,n}(\xi)d\xi + \frac{1}{2\nu}\int_0^\infty e^{-\nu\xi} f_{2,n}(\xi)d\xi,$$

$$N = \frac{1}{2\nu}\int_{-\infty}^0 (e^{\nu\xi} + e^{-\nu\xi}) f_{1,n}(\xi)d\xi + \frac{\lambda}{2\nu}\int_0^\infty e^{-\nu\xi} f_{2,n}(\xi)d\xi.$$

We then find the solution of the system in (6.23) as

$$\gamma_1 = \frac{1}{2\nu}\left\{\int_{-\infty}^{0}\left[e^{-\nu\xi} + \frac{1-\lambda}{1+\lambda}e^{\nu\xi}\right] f_{1,n}(\xi)d\xi + \int_{0}^{\infty}\frac{2\lambda}{1+\lambda}e^{-\nu\xi} f_{2,n}(\xi)d\xi\right\}$$

and

$$\delta_2 = \frac{1}{2\nu}\left\{\int_{-\infty}^{0}\frac{2}{1+\lambda}e^{\nu\xi} f_{1,n}(\xi)d\xi - \int_{0}^{\infty}\frac{1-\lambda}{1+\lambda}e^{-\nu\xi} f_{2,n}(\xi)d\xi\right\}.$$

Upon substituting $\gamma_1, \gamma_2, \delta_1$, and δ_2 into (6.17)–(6.20), and subsequent substitution of $C_{i,n}(x)$ and $D_{i,n}(x)$ into (6.16), the functions $u_{1,n}(x)$ and $u_{2,n}(x)$ can be written in compact form as

$$u_{1,n}(x) = \int_{-\infty}^{0} g_{11}^{n}(x,\xi) f_{1,n}(\xi)d\xi + \int_{0}^{\infty} g_{12}^{n}(x,\xi) f_{2,n}(\xi)d\xi \tag{6.24}$$

for $x \in (-\infty, 0]$, and

$$u_{2,n}(x) = \int_{-\infty}^{0} g_{21}^{n}(x,\xi) f_{1,n}(\xi)d\xi + \int_{0}^{\infty} g_{22}^{n}(x,\xi) f_{2,n}(\xi)d\xi \tag{6.25}$$

for $x \in [0, \infty)$.

The kernel-functions $g_{ij}^{n}(x,\xi)$ in the above integral representations are written as

$$g_{11}^{n}(x,\xi) = \frac{(1+\lambda)e^{-\nu|x-\xi|} + (1-\lambda)e^{\nu(x+\xi)}}{2\nu(1+\lambda)}, \quad -\infty < x, \xi \le 0,$$

$$g_{12}^{n}(x,\xi) = \frac{\lambda}{\nu(1+\lambda)}e^{\nu(x-\xi)}, \quad -\infty < x \le 0 \le \xi < \infty,$$

$$g_{21}^{n}(x,\xi) = \frac{1}{\nu(1+\lambda)}e^{\nu(\xi-x)}, \quad -\infty < \xi \le 0 \le x < \infty,$$

and

$$g_{22}^{n}(x,\xi) = \frac{(1+\lambda)e^{-\nu|x-\xi|} - (1-\lambda)e^{-\nu(x+\xi)}}{2\nu(1+\lambda)}, \quad 0 \le x, \xi < \infty.$$

Making use of the fundamental rule for Fourier coefficients (the Fourier–Euler formulas), we can write $f_{i,n}(x)$ in the second series in (6.11) as

$$f_{i,n}(\xi) = \frac{2}{b}\int_{0}^{b} f_i(\xi,\eta)\sin\nu\eta\, d\eta, \quad i = 1,2, \; n = 1,2,3,\ldots.$$

After substituting $f_{1,n}(\xi)$ and $f_{2,n}(\xi)$ into (6.24) and (6.25), and subsequently substituting $u_{1,n}(x)$ and $u_{2,n}(x)$ into the first series in (6.11), we finally obtain the solution of the boundary-value problem of (6.7)–(6.10) in integral form

$$u_1(x,y) = \int_0^b \int_{-\infty}^0 \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{11}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_1(\xi,\eta) d\xi d\eta$$

$$+ \int_0^b \int_0^{\infty} \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{12}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_2(\xi,\eta) d\xi d\eta, \quad (x,y) \in \Omega_1, \quad (6.26)$$

and

$$u_2(x,y) = \int_0^b \int_{-\infty}^0 \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{21}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_1(\xi,\eta) d\xi d\eta$$

$$+ \int_0^b \int_0^{\infty} \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{22}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_2(\xi,\eta) d\xi d\eta, \quad (x,y) \in \Omega_2. \quad (6.27)$$

Upon inspection of above integrals and recalling the relation in (6.6), where $u_i(x,y)$ and $f_i(x,y)$, with $i = 1,2$ in (6.26) and (6.27), defining the components of the vector-functions $\mathbf{U}(x,y)$ and $\mathbf{F}(x,y)$

$$U_i(x,y) = \begin{cases} u_i(x,y), & (x,y) \in \Omega_i, \\ 0, & (x,y) \notin \Omega_i, \end{cases} \quad i = 1,2,$$

and

$$F_i(x,y) = \begin{cases} f_i(x,y), & (x,y) \in \Omega_i, \\ 0, & (x,y) \notin \Omega_i, \end{cases} \quad i = 1,2,$$

we recognize that the kernel-functions

$$\frac{2}{b} \sum_{n=1}^{\infty} g_{ij}^n(x,\xi) \sin \nu y \sin \nu \eta, \quad i,j = 1,2, \quad (6.28)$$

in (6.26) and (6.27) represent the elements $G_{ij}(x,y;\xi,\eta)$ in the matrix of Green's type $\mathbf{G}(x,y;\xi,\eta)$ for the homogeneous boundary-value problem corresponding to (6.7)–(6.10).

Upon close analysis we find that the series expansions in (6.28), with the expressions for $g_{ij}^n(x,\xi)$ found earlier in this section, can be readily summed up with the aid of the standard summation formula

$$\sum_{n=1}^{\infty} \frac{q^n}{n} \cos n\alpha = -\frac{1}{2} \ln(1 - 2q\cos\alpha + q^2) \quad (6.29)$$

valid for $q \leq 1$ and $0 \leq \alpha < 2\pi$, and was utilized repeatedly, earlier in our book. Indeed, the expansion in (6.28) can be rewritten as

$$G_{ij}(x, y; \xi, \eta) = \frac{1}{b} \sum_{n=1}^{\infty} g_{ij}^{n}(x, \xi) \left[\cos \nu(y - \eta) - \cos \nu(y + \eta)\right], \quad i, j = 1, 2,$$

and the summation ultimately yields the following compact closed formulas

$$G_{11}(z, \zeta) = \frac{1}{2\pi} \left[\ln \frac{|1 - e^{\omega(z-\bar{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|} - \frac{1-\lambda}{1+\lambda} \ln \frac{|1 - e^{\omega(z+\bar{\zeta})}|}{|1 - e^{\omega(z+\zeta)}|} \right],$$

$$G_{12}(z, \zeta) = \frac{\lambda}{\pi(1+\lambda)} \ln \frac{|1 - e^{\omega(z-\bar{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|},$$

$$G_{21}(z, \zeta) = \frac{1}{\pi(1+\lambda)} \ln \frac{|1 - e^{\omega(z-\bar{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|}$$

and

$$G_{11}(z, \zeta) = \frac{1}{2\pi} \left[\ln \frac{|1 - e^{\omega(z-\bar{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|} + \frac{1-\lambda}{1+\lambda} \ln \frac{|1 - e^{\omega(z+\bar{\zeta})}|}{|1 - e^{\omega(z+\zeta)}|} \right]$$

for the elements of the matrix of Green's type $\mathbf{G}(x, y; \xi, \eta)$ for the homogeneous boundary-value problem, corresponding to that in (6.7)–(6.10). Here $\omega = \pi/b$. We introduced complex variable notations $z = x + iy$ and $\zeta = \xi + i\eta$ for the field point (x, y) and the source point (ξ, η), respectively, with the bar on ζ denoting its complex conjugate.

Clearly, the above expressions for the elements of the matrix of Green's type are computable immediately, since they represent real-valued functions, whilst the complex variables are only used for compactness.

It can be readily seen that, for λ equal to unity (that is, the materials occupying the fragments Ω_1 and Ω_2 of Ω are identical), the above expressions reduce to the well-known closed form (refer to, for example, Chapter 2)

$$G(z, \zeta) = \frac{1}{2\pi} \ln \frac{|1 - e^{\omega(z-\bar{\zeta})}|}{|1 - e^{\omega(z-\zeta)}|}$$

of the Green's function for the Dirichlet problem for the Laplace equation in the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$.

Example 6.2. We continue with a mixed boundary-value problem for two Klein–Gordon equations stated on the segments $\Omega_1 = \{-a < x < 0, 0 < y < b\}$ and $\Omega_2 = \{0 < x < \infty, 0 < y < b\}$ of the compound semi-infinite strip $\Omega = \{-a <$

$x < \infty, 0 < y < b\}$. Let Ω_1 and Ω_2 be filled with materials whose conductivities are defined by λ_1 and λ_2, respectively. Consider now the problem

$$\nabla^2 u_i(x,y) - k_i^2 u_i(x,y) = -f_i(x,y), \quad (x,y) \in \Omega_i,\ i = 1,2, \tag{6.30}$$

$$\frac{\partial u_1(-a,y)}{\partial x} - \beta u_1(-a,y) = 0, \quad \lim_{x\to\infty} u_2(x,y) < \infty, \tag{6.31}$$

$$u_i(x,0) = 0, \quad \frac{\partial u_i(x,b)}{\partial y} = 0, \quad i = 1,2, \tag{6.32}$$

and

$$u_1(0,y) = u_2(0,y), \quad \frac{\partial u_1(0,y)}{\partial x} = \lambda \frac{\partial u_2(0,y)}{\partial x}, \tag{6.33}$$

where $\beta \geq 0$ and $\lambda = \lambda_2/\lambda_1$.

Due to the form of the conditions in (6.32), we can expand the unknown functions $u_i(x,y)$ and the right-hand side terms $f_i(x,y)$, $i = 1,2$, from the above equation into the Fourier series

$$u_i(x,y) = \sum_{n=1}^{\infty} u_{i,n}(x) \sin \nu y, \quad \nu = \frac{(2n-1)\pi}{2b}, \tag{6.34}$$

and

$$f_i(x,y) = \sum_{n=1}^{\infty} f_{i,n}(x) \sin \nu y. \tag{6.35}$$

After substituting these expansions in the original formulation of (6.30)–(6.33), we obtain the following set ($n = 1,2,3,\ldots$) of three-point-posed boundary-value problems

$$\frac{d^2 u_{1,n}(x)}{dx^2} - (\nu^2 + k_1^2) u_{1,n}(x) = -f_{1,n}(x), \quad x \in (-a,0), \tag{6.36}$$

$$\frac{d^2 u_{2,n}(x)}{dx^2} - (\nu^2 + k_2^2) u_{2,n}(x) = -f_{2,n}(x), \quad x \in (0,\infty), \tag{6.37}$$

$$\frac{du_{1,n}(-a)}{dx} - \beta u_{1,n}(-a) = 0, \quad \lim_{x\to\infty} u_{2,n}(x) < \infty, \tag{6.38}$$

$$u_{1,n}(0) = u_{2,n}(0), \quad \frac{du_{1,n}(0)}{dx} = \lambda \frac{du_{2,n}(0)}{dx} \tag{6.39}$$

for the coefficients $u_{1,n}(x)$ and $u_{2,n}(x)$ of the expansion in (6.34).

Clearly, fundamental sets of solutions for the homogeneous equations corresponding to (6.36) and (6.37) can be composed of, for example, the functions

$$e^{h_i x} \quad \text{and} \quad e^{-h_i x}$$

with $h_i = \sqrt{\nu^2 + k_i^2}$, for $i = 1,2$.

Hence, in accordance with the standard procedure of the method of variation of parameters, the solution of the boundary-value problem in (6.36)–(6.39) can be written in the form

$$u_{i,n}(x) = C_{i,n}(x)e^{h_i x} + D_{i,n}(x)e^{-h_i x}, \quad i = 1, 2, \tag{6.40}$$

where the coefficients $C_{i,n}(x)$ and $D_{i,n}(x)$ are to be determined. This yields the following well-posed system of linear algebraic equations

$$\begin{pmatrix} e^{h_i x} & e^{-h_i x} \\ h_i e^{h_i x} & -h_i e^{-h_i x} \end{pmatrix} \begin{pmatrix} C'_{i,n}(x) \\ D'_{i,n}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -f_{i,n}(x) \end{pmatrix}, \quad i = 1, 2,$$

for the derivatives of $C_{i,n}(x)$ and $D_{i,n}(x)$ of (6.40). After solving the above system, we obtain

$$C'_{i,n}(x) = -\frac{1}{2h_i} e^{-h_i x} f_{i,n}(x), \quad D'_{i,n}(x) = \frac{1}{2h_i} e^{h_i x} f_{i,n}(x), \quad i = 1, 2.$$

Straightforward integration yields

$$C_{1,n}(x) = -\frac{1}{2h_1} \int_{-a}^{x} e^{-h_1 \xi} f_{1,n}(\xi) d\xi + \gamma_1, \tag{6.41}$$

$$D_{1,n}(x) = \frac{1}{2h_1} \int_{-a}^{x} e^{h_1 \xi} f_{1,n}(\xi) d\xi + \delta_1, \tag{6.42}$$

$$C_{2,n}(x) = -\frac{1}{2h_2} \int_{0}^{x} e^{-h_2 \xi} f_{2,n}(\xi) d\xi + \gamma_2 \tag{6.43}$$

and

$$D_{2,n}(x) = \frac{1}{2h_2} \int_{0}^{x} e^{h_2 \xi} f_{2,n}(\xi) d\xi + \delta_2. \tag{6.44}$$

Substituting these expressions in (6.40) and rearranging the integral terms appropriately, we obtain general solutions to the equations in (6.36) and (6.37) as

$$u_{1,n}(x) = \gamma_1 e^{h_1 x} + \delta_1 e^{-h_1 x} + \frac{1}{2h_1} \int_{-a}^{x} [e^{-h_1(x-\xi)} - e^{h_1(x-\xi)}] f_{1,n}(\xi) d\xi \tag{6.45}$$

and

$$u_{2,n}(x) = \gamma_2 e^{h_2 x} + \delta_2 e^{-h_2 x} + \frac{1}{2h_2} \int_{0}^{x} [e^{-h_2(x-\xi)} - e^{h_2(x-\xi)}] f_{2,n}(\xi) d\xi. \tag{6.46}$$

The constant coefficients $\gamma_1, \gamma_2, \delta_1$, and δ_2 in $u_{1,n}(x)$ and $u_{2,n}(x)$ as written above, can be obtained by imposing the boundary and contact conditions in (6.38) and (6.39). The boundedness of $u_{2,n}(x)$ as x goes to infinity allows one to directly find γ_2 (see the second condition in (6.38)). In order to do this, it is convenient to transform the expression for $u_{2,n}(x)$ in (6.46) into the equivalent form

$$u_{2,n}(x) = \left[-\frac{1}{2h_2}\int_0^x \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi + \gamma_2\right]\mathrm{e}^{h_2x} + \left[\frac{1}{2h_2}\int_0^x \mathrm{e}^{h_2\xi} f_{2,n}(\xi)d\xi + \delta_2\right]\mathrm{e}^{-h_2x}$$

from which it follows immediately that the factor of e^{h_2x} must be zero, since $u_{2,n}(x)$ is required to be bounded as x goes to infinity. This yields

$$\gamma_2 = \frac{1}{2h_2}\int_0^\infty \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi.$$

The remaining conditions in (6.38) and (6.39) lead us to the well-posed system of linear algebraic equations

$$\begin{pmatrix} (h_1-\beta)\mathrm{e}^{-h_1a} & -(h_1+\beta)\mathrm{e}^{h_1a} & 0 \\ 1 & 1 & -1 \\ h_1 & -h_1 & \lambda h_2 \end{pmatrix}\begin{pmatrix}\gamma_1\\ \delta_1\\ \delta_2\end{pmatrix} = \begin{pmatrix}0\\ M\\ N\end{pmatrix} \tag{6.47}$$

in γ_1, δ_1,and δ_2, with M and N in the right-hand side vector defined as

$$M = \frac{1}{2h_1}\int_{-a}^0 (\mathrm{e}^{-h_1\xi} - \mathrm{e}^{h_1\xi}) f_{1,n}(\xi)d\xi + \frac{1}{2h_2}\int_0^\infty \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi,$$

$$N = \frac{1}{2}\int_{-a}^0 (\mathrm{e}^{h_1\xi} + \mathrm{e}^{-h_1\xi}) f_{1,n}(\xi)d\xi + \frac{\lambda}{2}\int_0^\infty \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi.$$

The solution of the system in (6.47) now is found as

$$\gamma_1 = \frac{h_1+\beta}{2h_1\Delta}\left\{\int_{-a}^0 [(h_1+\lambda h_2)\mathrm{e}^{-h_1\xi} + (h_1-\lambda h_2)\mathrm{e}^{h_1\xi}] f_{1,n}(\xi)d\xi + 2h_1\lambda\int_0^\infty \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi\right\}\mathrm{e}^{2h_1a},$$

$$\delta_1 = \frac{h_1-\beta}{2h_1\Delta}\left\{\int_{-a}^0 [(h_1+\lambda h_2)\mathrm{e}^{-h_1\xi} + (h_1-\lambda h_2)\mathrm{e}^{h_1\xi}] f_{1,n}(\xi)d\xi + 2h_1\lambda\int_0^\infty \mathrm{e}^{-h_2\xi} f_{2,n}(\xi)d\xi\right\}$$

and

$$\delta_2 = \frac{1}{\Delta}\left\{\int_{-a}^0 [(h_1-\beta)\mathrm{e}^{h_1\xi} + (h_1+\beta)\mathrm{e}^{h_1(2a+\xi)}] f_{1,n}(\xi)d\xi - \int_0^\infty \frac{h_1-\lambda h_2}{2h_2}[(h_1+\beta)\mathrm{e}^{2h_1a-h_2\xi} + (h_1-\beta)\mathrm{e}^{-h_2\xi}] f_{2,n}(\xi)d\xi\right\},$$

where

$$\Delta = (h_1 + \lambda h_2)(h_1 + \beta)\mathrm{e}^{2h_1 a} - (h_1 - \lambda h_2)(h_1 - \beta).$$

Upon substituting $\gamma_1, \gamma_2, \delta_1$, and δ_2 in (6.41)–(6.44), the functions $u_{1,n}(x)$ and $u_{2,n}(x)$ can be rewritten in compact form as

$$u_{1,n}(x) = \int_{-a}^{0} g_{11}^{n}(x,\xi) f_{1,n}(\xi) d\xi + \int_{0}^{\infty} g_{12}^{n}(x,\xi) f_{2,n}(\xi) d\xi \qquad (6.48)$$

and

$$u_{2,n}(x) = \int_{-a}^{0} g_{21}^{n}(x,\xi) f_{1,n}(\xi) d\xi + \int_{0}^{\infty} g_{22}^{n}(x,\xi) f_{2,n}(\xi) d\xi \qquad (6.49)$$

with the kernel-functions $g_{ij}^{n}(x,\xi)$ written as

$$\begin{aligned} g_{11}^{n}(x,\xi) = \frac{1}{2h_1\Delta}\{&(h_1+\beta)[(h_1+\lambda h_2)\mathrm{e}^{-h_1|x-\xi|} + (h_1-\lambda h_2)\mathrm{e}^{h_1(x+\xi)}]\mathrm{e}^{2h_1 a} \\ &+ (h_1-\beta)[(h_1+\lambda h_2)\mathrm{e}^{-h_1(x+\xi)} + (h_1-\lambda h_2)\mathrm{e}^{h_1|x-\xi|}]\}, \\ g_{12}^{n}(x,\xi) = \frac{\lambda}{\Delta}&[(h_1+\beta)\mathrm{e}^{h_1(2a+x)} + (h_1-\beta)\mathrm{e}^{-h_1 x}]\mathrm{e}^{-h_2\xi}, \\ g_{21}^{n}(x,\xi) = \frac{1}{\Delta}&[(h_1-\beta)\mathrm{e}^{-h_1\xi} + (h_1+\beta)\mathrm{e}^{h_1(2a+\xi)}]\mathrm{e}^{-h_2 x}, \end{aligned}$$

and

$$\begin{aligned} g_{22}^{n}(x,\xi) = \frac{1}{2h_2\Delta}\{&(h_1+\beta)[(h_1+\lambda h_2)\mathrm{e}^{-h_2|x-\xi|} - (h_1-\lambda h_2)\mathrm{e}^{-h_2(x+\xi)}]\mathrm{e}^{2h_1 a} \\ &- (h_1-\beta)(h_1-\lambda h_2)[\mathrm{e}^{-h_2(x+\xi)} + \mathrm{e}^{-h_2|x-\xi|}]\}. \end{aligned}$$

Note that the variables x and ξ in $g_{11}^{n}(x,\xi)$ range between $-a \le x, \xi \le 0$, while for $g_{12}^{n}(x,\xi)$ we have $-a \le x \le 0 \le \xi < \infty$. For $g_{22}^{n}(x,\xi)$ the variables range between $0 \le x, \xi < \infty$, whilst for $g_{21}^{n}(x,\xi)$ we have $-a \le \xi \le 0 \le x < \infty$.

Recall that, in accordance with the fundamental rule for Fourier coefficients (Fourier–Euler formulas), the coefficients $f_{i,n}(x)$ of the series in (6.35) can be written as

$$f_{i,n}(\xi) = \frac{2}{b}\int_{0}^{b} f_i(\xi,\eta)\sin\nu\eta\, d\eta, \quad i = 1,2,\ n = 1,2,3,\ldots.$$

Upon substituting $f_{1,n}(\xi)$ and $f_{2,n}(\xi)$ in (6.48) and (6.49), and subsequently substituting $u_{1,n}(x)$ and $u_{2,n}(x)$ into the series in (6.34), we finally obtain the solution of the boundary-value problem of (6.30)–(6.33) in integral form

$$u_1(x,y) = \int_0^b \int_{-a}^0 \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{11}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_1(\xi,\eta) d\xi d\eta$$
$$+ \int_0^b \int_0^{\infty} \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{12}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_2(\xi,\eta) d\xi d\eta, \quad (x,y) \in \Omega_1,$$

and

$$u_2(x,y) = \int_0^b \int_{-a}^0 \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{21}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_1(\xi,\eta) d\xi d\eta$$
$$+ \int_0^b \int_0^{\infty} \left(\frac{2}{b} \sum_{n=1}^{\infty} g_{22}^n(x,\xi) \sin \nu y \sin \nu \eta \right) f_2(\xi,\eta) d\xi d\eta, \quad (x,y) \in \Omega_2.$$

Thus, in compliance with the definition given in the introductory section of this chapter, we conclude that the series

$$G_{ij}(x,y;\xi,\eta) = \frac{2}{b} \sum_{n=1}^{\infty} g_{ij}^n(x,\xi) \sin \nu y \sin \nu \eta \tag{6.50}$$
$$= \frac{1}{b} \sum_{n=1}^{\infty} g_{ij}^n(x,\xi) \left[\cos \nu(y-\eta) - \cos \nu(y+\eta)\right], \quad i,j = 1,2,$$

represents the elements $G_{ij}(x,y;\xi,\eta)$ of the matrix of Green's type $\mathbf{G}(x,y;\xi,\eta)$ for the homogeneous boundary-value problem corresponding to (6.30)–(6.33).

If we set $k_1 = 0$ and $k_2 = 0$ in (6.30) we obtain $h_1 = h_2 = \nu$. This transforms the coefficients $g_{ij}^n(x,\xi)$ of the series in (6.50) to

$$g_{11}^n(x,\xi) = \frac{1}{2\nu\Delta^*}[(\nu+\beta)((1+\lambda)e^{-\nu|x-\xi|} + (1-\lambda)e^{\nu(x+\xi)})e^{2\nu a}$$
$$+ (\nu-\beta)((1+\lambda)e^{-\nu(x+\xi)} + (1-\lambda)e^{\nu|x-\xi|})],$$
$$g_{12}^n(x,\xi) = \frac{\lambda}{\nu\Delta^*}[(\nu+\beta)e^{\nu(2a+x)} + (\nu-\beta)e^{-\nu x}]e^{-\nu\xi},$$
$$g_{21}^n(x,\xi) = \frac{1}{\nu\Delta^*}[(\nu-\beta)e^{-\nu\xi} + (\nu+\beta)e^{\nu(2a+\xi)}]e^{-\nu x},$$

and

$$g_{22}^n(x,\xi) = \frac{1}{2\nu\Delta^*}[(\nu+\beta)((1+\lambda)e^{-\nu|x-\xi|} - (1-\lambda)e^{-\nu(x+\xi)})e^{2\nu a}$$
$$+ \nu(\nu-\beta)(1-\lambda)(e^{-\nu(x+\xi)} + e^{-\nu|x-\xi|})]$$

with $\Delta^* = (1+\lambda)(\nu+\beta)e^{2\nu a} - (1-\lambda)(\nu-\beta)$.

Upon close analysis, it is revealed that the expansions in (6.50), with the coefficients $g_{ij}^n(x, \xi)$, as written above, represent a series of the type

$$\sum_{n=1}^{\infty} \frac{q^n}{n} \cos n\alpha, \quad \text{where } q \leq 1 \text{ and } 0 \leq \alpha < 2\pi,$$

which diverges for $q = 1$ and $\alpha = 0$. Hence, the series in (6.50) converges nonuniformly (contains the logarithmic singularity) for the elements $G_{11}(x, y; \xi, \eta)$ and $G_{22}(x, y; \xi, \eta)$.

In order to get an idea of the convergence of the series in (6.50), we have conducted a numerical experiment. The accuracy level that we can attain by truncating the series can be observed in Figure 6.2, where we depict a profile of the elements $G_{11}(x, y; \xi, \eta)$ and $G_{21}(x, y; \xi, \eta)$, with the series (6.50) truncated to its 10th partial sum. We have chosen $a = \pi$, $b = \pi$, $\lambda = 0.01$, and $\beta = 2.5$ and the source point is placed at $(-1.0, 1.5) \in \Omega_1$.

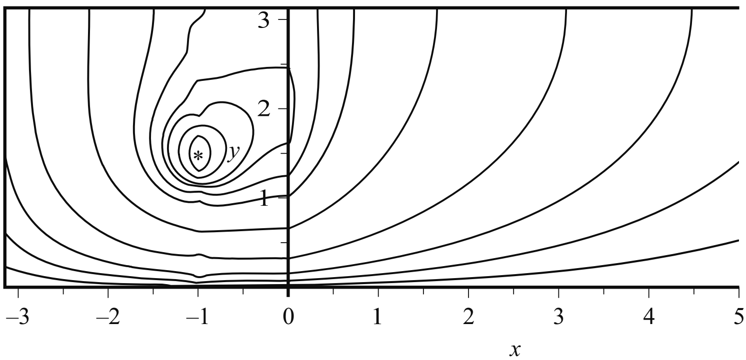

Figure 6.2. Convergence of the series in (6.50).

Based on the resulting numerical data, we can conclude that, if we require accurate values of $\mathbf{G}(x, y; \xi, \eta)$ in the entire region Ω, then a series-only representation of similar to (6.50) is not efficient and cannot be recommended. To enhance its computability, we will implement the approach originally proposed in [43]. In doing so, the coefficients $g_{11}^n(x, \xi)$ of $G_{11}(x, y; \xi, \eta)$ are transformed as

$$\begin{aligned} g_{11}^n(x, \xi) &= \frac{p(x, \xi)}{2\nu} \left[\frac{1}{\Delta^*} - \frac{e^{-2\nu a}}{\nu(1+\lambda)} + \frac{e^{-2\nu a}}{\nu(1+\lambda)} \right] \\ &= \frac{p(x, \xi)}{2\nu} \left[\frac{(1-\lambda)(\nu-\beta)e^{-2\nu a} - \beta(1+\lambda)}{\nu(1+\lambda)\Delta^*} + \frac{e^{-2\nu a}}{\nu(1+\lambda)} \right], \end{aligned}$$

where

$$p(x, \xi) = \nu p_1(x, \xi) + \beta p_2(x, \xi)$$

with

$$p_1(x,\xi) = [((1+\lambda)\mathrm{e}^{-\nu|x-\xi|} + (1-\lambda)\mathrm{e}^{\nu(x+\xi)})\mathrm{e}^{2\nu a} \\ + ((1+\lambda)\mathrm{e}^{-\nu(x+\xi)} + (1-\lambda)\mathrm{e}^{-\nu|x-\xi|})] \qquad (6.51)$$

and

$$p_2(x,\xi) = [((1+\lambda)\mathrm{e}^{-\nu|x-\xi|} + (1-\lambda)\mathrm{e}^{\nu(x+\xi)})\mathrm{e}^{2\nu a} \\ - ((1+\lambda)\mathrm{e}^{-\nu(x+\xi)} + (1-\lambda)\mathrm{e}^{-\nu|x-\xi|})].$$

This splits the series representation of $G_{11}(x,y;\xi,\eta)$ into two segments, the first of which

$$\frac{1}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{p(x,\xi)\left[(1-\lambda)(\nu-\beta)\mathrm{e}^{-2\nu a} - \beta(1+\lambda)\right]}{\nu^2\left((1+\lambda)(\nu+\beta)\mathrm{e}^{2\nu a} - (1-\lambda)(\nu-\beta)\right)} \\ \times [\cos\nu(y-\eta) - \cos\nu(y+\eta)] \qquad (6.52)$$

converges uniformly at the rate of

$$\sum_{n=1}^{\infty} \frac{q^n}{n^2} \cos n\alpha, \quad \text{where } q \le 1 \text{ and } 0 \le \alpha < 2\pi.$$

Hence it is already in a computer-friendly form. With regard to the second series in $G_{11}(x,y;\xi,\eta)$, written as

$$\frac{1}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{p(x,\xi)}{\nu^2\mathrm{e}^{2\nu a}} [\cos\nu(y-\eta) - \cos\nu(y+\eta)]$$

we can split it into

$$\frac{1}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{p_1(x,\xi)}{\nu\mathrm{e}^{2\nu a}} [\cos\nu(y-\eta) - \cos\nu(y+\eta)] \\ + \frac{\beta}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{p_2(x,\xi)}{\nu^2\mathrm{e}^{2\nu a}} [\cos\nu(y-\eta) - \cos\nu(y+\eta)], \quad (6.53)$$

where the second expansion converges uniformly at the same rate as (6.52). Hence, by combining the two, we obtain a uniformly convergent series component for $G_{11}(x,y;\xi,\eta)$ as

$$\frac{1}{2b(1+\lambda)} \sum_{n=1}^{\infty} \frac{(1-\lambda)(\nu-\beta)p_1(x,\xi)\mathrm{e}^{-2\nu a} - \beta(1+\lambda)p_3(x,\xi)}{\nu\left((1+\lambda)(\nu+\beta)\mathrm{e}^{2\nu a} - (1-\lambda)(\nu-\beta)\right)} \\ \times [\cos\nu(y-\eta) - \cos\nu(y+\eta)] \qquad (6.54)$$

where

$$p_3(x,\xi) = 2((1+\beta)e^{-\nu(x+\xi)} + (1-\beta)e^{\nu|x-\xi|}).$$

The non-uniform convergence of the first series in (6.53)

$$\frac{1}{2b(1+\lambda)}\sum_{n=1}^{\infty}\frac{p_1(x,\xi)}{\nu e^{2\nu a}}\left[\cos\nu(y-\eta)-\cos\nu(y+\eta)\right] \tag{6.55}$$

is evident. The good news is, however, that it is completely summable. To prepare for the summation, we substitute $p_1(x,\xi)$ from (6.51) in (6.55), and rewrite the latter as

$$\begin{aligned}\frac{1}{2b(1+\lambda)}\Bigg\{(1+\beta)\Bigg[&\sum_{n=1}^{\infty}\frac{e^{-\nu|x-\xi|}}{\nu}\left[\cos\nu(y-\eta)-\cos\nu(y+\eta)\right]\\ &+\sum_{n=1}^{\infty}\frac{e^{-\nu(2a+x+\xi)}}{\nu}\left[\cos\nu(y-\eta)-\cos\nu(y+\eta)\right]\Bigg]\\ +(1-\beta)\Bigg[&\sum_{n=1}^{\infty}\frac{e^{\nu(x+\xi)}}{\nu}\left[\cos\nu(y-\eta)-\cos\nu(y+\eta)\right]\\ &+\sum_{n=1}^{\infty}\frac{e^{-\nu(2a-|x-\xi|)}}{\nu}\left[\cos\nu(y-\eta)-\cos\nu(y+\eta)\right]\Bigg]\Bigg\}.\end{aligned} \tag{6.56}$$

Recalling the expression for $\nu = (2n-1)\pi/2b$, we sum all of the above series using the standard summation formula [1, 27]

$$\sum_{n=1}^{\infty}\frac{q^{2n-1}}{2n-1}\cos(2n-1)\alpha = \frac{1}{4}\ln\frac{1+2q\cos\alpha+q^2}{1-2q\cos\alpha+q^2}$$

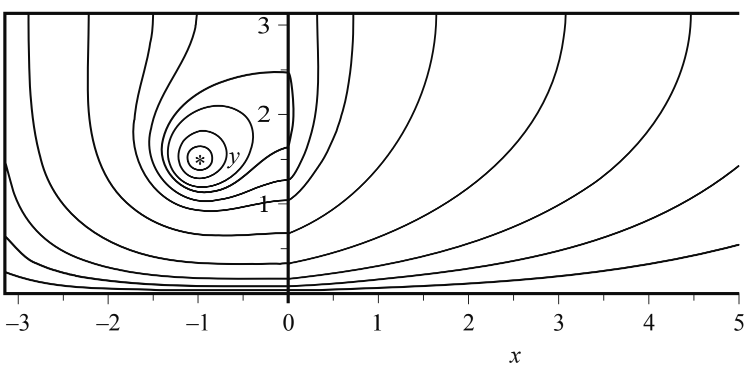

Figure 6.3. Improvement of the convergence.

yielding the following closed form for (6.56)

$$\frac{1}{2\pi(1+\lambda)}\left\{(1+\beta)\ln\left(\frac{1+2e^{\omega(x-\xi)}\cos(y-\eta)+e^{2\omega(x-\xi)}}{1-2e^{\omega(x-\xi)}\cos(y-\eta)+e^{2\omega(x-\xi)}}\right.\right. \tag{6.57}$$

$$\times\frac{1-2e^{\omega(x-\xi)}\cos(y+\eta)+e^{2\omega(x-\xi)}}{1+2e^{\omega(x-\xi)}\cos(y+\eta)+e^{2\omega(x-\xi)}}$$

$$\times\frac{1+2e^{\omega(2a+x+\xi)}\cos(y-\eta)+e^{2\omega(2a+x+\xi)}}{1-2e^{\omega(2a+x+\xi)}\cos(y-\eta)+e^{2\omega(2a+x+\xi)}}$$

$$\left.\times\frac{1-2e^{\omega(2a+x+\xi)}\cos(y+\eta)+e^{2\omega(2a+x+\xi)}}{1+2e^{\omega(2a+x+\xi)}\cos(y+\eta)+e^{2\omega(2a+x+\xi)}}\right)$$

$$+(1-\beta)\ln\left(\frac{1+2e^{\omega(x+\xi)}\cos(y-\eta)+e^{2\omega(x+\xi)}}{1-2e^{\omega(x+\xi)}\cos(y-\eta)+e^{2\omega(x+\xi)}}\right.$$

$$\times\frac{1-2e^{\omega(x-\xi)}\cos(y+\eta)+e^{2\omega(x-\xi)}}{1+2e^{\omega(x-\xi)}\cos(y+\eta)+e^{2\omega(x-\xi)}}$$

$$\times\frac{1+2e^{\omega(2a-x+\xi)}\cos(y-\eta)+e^{2\omega(2a-x+\xi)}}{1-2e^{\omega(2a-x+\xi)}\cos(y-\eta)+e^{2\omega(2a-x+\xi)}}$$

$$\left.\left.\times\frac{1-2e^{\omega(2a-x+\xi)}\cos(y+\eta)+e^{2\omega(2a-x+\xi)}}{1+2e^{\omega(2a-x+\xi)}\cos(y+\eta)+e^{2\omega(2a-x+\xi)}}\right)\right\},$$

where $\omega = \pi/2b$.

Hence, the sum of the uniformly convergent series in (6.54) and the expression in (6.57) provide a computer-friendly formula of $G_{11}(x, y; \xi, \eta)$ of $\mathbf{G}(x, y; \xi, \eta)$. The improvement of the series convergence for the rest of the elements can be accomplished in a similar manner.

To illustrate the accuracy improvement attained by the development we have just described, with regard to the convergence of the series representing elements of the matrix of Green's type, we depict, in Figure 6.3, the same profile of $\mathbf{G}(x, y; \xi, \eta)$ as in Figure 6.2. We employed the computer-friendly elements of $\mathbf{G}(x, y; \xi, \eta)$, with their series components truncated to the 10th partial sum.

Example 6.3. Consider yet another problem where its matrix of Green's type can be obtained in closed series-free form. Let the half-plane $\Omega_2 = \{a < r < \infty, 0 < \varphi < \pi\}$, reduced by a semi-circular cut-out of radius a, be filled with a conducting (λ_2) isotropic homogeneous material. Let also Ω_2 contain a semi-circular inclusion $\Omega_1 = \{0 < r < a, 0 < \varphi < \pi\}$ made out of a foreign conducting (λ_1) isotropic homogeneous material. To determine the potential field generated in $\Omega = \Omega_1 \cup \Omega_2$ by a unit point source located arbitrarily on Ω, we consider the boundary-value problem

$$\nabla^2 u_i(r,\varphi) = -f_i(r,\varphi), (r,\varphi) \in \Omega_i, i = 1,2, \tag{6.58}$$

$$u_i(r,0) = u_i(r,\pi) = 0, \quad i = 1,2, \tag{6.59}$$

$$\lim_{r\to 0} u_1(r,\varphi) > \infty, \quad \lim_{r\to\infty} u_2(r,\varphi) > \infty \tag{6.60}$$

and

$$u_1(a,\varphi) = u_2(a,\varphi), \quad \frac{\partial u_1(a,\varphi)}{\partial r} = \lambda \frac{\partial u_2(a,\varphi)}{\partial r}, \tag{6.61}$$

where ∇^2 represents the Laplace operator written in polar coordinates

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

and $\lambda = \lambda_2/\lambda_1$.

Note that in [45], we have derived a series expansion for elements of the matrix of Green's type for the Dirichlet problem in a compound semi-circle. The matrix of Green's type for the above problem can be obtained from the one in the reference, by taking a limit for the semi-circle's radius approaching infinity. However, in order to be consistent, we will derive it by following the procedure described in Example 6.2. That is, expanding the functions $u_i(r,\varphi)$ and the right-hand side terms $f_i(r,\varphi)$, $i = 1,2$, in (6.58)–(6.61) into the Fourier series

$$u_i(r,\varphi) = \sum_{n=1}^{\infty} u_{i,n}(r) \sin n\varphi \tag{6.62}$$

and

$$f_i(r,\varphi) = \sum_{n=1}^{\infty} f_{i,n}(r) \sin n\varphi \tag{6.63}$$

and substituting these expansions in the original problem, we obtain the following set ($n = 1,2,3,\ldots$) of three-point-posed boundary-value problems

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_{1,n}(r)}{dr}\right) - \frac{n^2}{r^2}u_{1,n}(r) = -f_{1,n}(r), \quad r \in (0,a), \tag{6.64}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_{2,n}(r)}{dr}\right) - \frac{n^2}{r^2}u_{2,n}(r) = -f_{2,n}(r), \quad r \in (a,\infty), \tag{6.65}$$

$$\lim_{r\to 0} u_{1,n}(r) > \infty, \quad \lim_{r\to\infty} u_{2,n}(r) > \infty, \quad i = 1,2, \tag{6.66}$$

$$u_{1,n}(a) = u_{2,n}(a), \quad \frac{du_{1,n}(a)}{dr} = \lambda \frac{du_{2,n}(a)}{dr} \tag{6.67}$$

for the coefficients $u_{1,n}(r)$ and $u_{2,n}(r)$ of the expansion in (6.62).

Following the procedure described in detail earlier, the solutions $u_{1,n}(r)$ and $u_{2,n}(r)$ to (6.64)–(6.67) are found in integral form

$$u_{1,n}(r) = \int_0^a g_{11}^n(r,\varrho) f_{1,n}(\varrho) d\varrho + \int_a^\infty g_{12}^n(r,\varrho) f_{2,n}(\varrho) d\varrho \tag{6.68}$$

and

$$u_{2,n}(r) = \int_0^a g_{21}^n(r,\varrho) f_{1,n}(\varrho) d\varrho + \int_a^\infty g_{22}^n(r,\varrho) f_{2,n}(\varrho) d\varrho \tag{6.69}$$

where the kernel-functions $g_{ij}^n(r,\varrho)$ are expressed as

$$g_{11}^n(r,\varrho) = \frac{1}{2n}\left[\left(\frac{r}{\varrho}\right)^n + \frac{1-\lambda}{1+\lambda}\left(\frac{r\varrho}{a^2}\right)^n\right], \quad 0 \le r \le \varrho \le a,$$
$$g_{12}^n(r,\varrho) = \frac{\lambda}{n(1+\lambda)}\left(\frac{r}{\varrho}\right)^n, \quad 0 \le r \le a \le \varrho < \infty,$$
$$g_{21}^n(r,\varrho) = \frac{1}{n(1+\lambda)}\left(\frac{\varrho}{r}\right)^n, \quad 0 \le \varrho \le a \le r < \infty,$$

and

$$g_{22}^n(r,\varrho) = \frac{1}{2n}\left[\left(\frac{r}{\varrho}\right)^n - \frac{1-\lambda}{1+\lambda}\left(\frac{a^2}{r\varrho}\right)^n\right], \quad a \le r \le \varrho < \infty.$$

Note that the main-diagonal elements $g_{11}^n(r,\varrho)$ and $g_{22}^n(r,\varrho)$, for $\varrho \le r$, can be obtained from the above ones by exchanging r and ϱ.

Upon proceeding with our routine, we ultimately express the solution to the boundary-value problem in (6.58)–(6.61) as

$$u_1(r,\varphi) = \int_0^\pi \int_0^a \left(\frac{2}{\pi}\sum_{n=1}^\infty g_{11}^n(r,\varrho)\sin n\varphi \sin n\psi\right) f_1(\varrho,\psi)\psi d\varrho d\psi$$
$$+ \int_0^\pi \int_a^\infty \left(\frac{2}{\pi}\sum_{n=1}^\infty g_{12}^n(r,\varrho)\sin n\varphi \sin n\psi\right) f_2(\varrho,\psi)\psi d\varrho d\psi, \quad (r,\varphi) \in \Omega_1,$$

and

$$u_2(r,\varphi) = \int_0^\pi \int_0^a \left(\frac{2}{\pi}\sum_{n=1}^\infty g_{21}^n(r,\varrho)\sin n\varphi \sin n\psi\right) f_1(\varrho,\psi)\psi d\varrho d\psi$$
$$+ \int_0^\pi \int_a^\infty \left(\frac{2}{\pi}\sum_{n=1}^\infty g_{22}^n(r,\varrho)\sin n\varphi \sin n\psi\right) f_2(\varrho,\psi)\psi d\varrho d\psi, \quad (r,\varphi) \in \Omega_2.$$

This implies that we can find the elements $G_{i,j}(r,\varphi;\varrho,\psi)$ of the sought-after matrix of Green's type $\mathbf{G}(r,\varphi;\varrho,\psi)$ as

$$G_{i,j}(r,\varphi;\varrho,\psi) = \frac{2}{\pi}\sum_{n=1}^{\infty} g_{i,j}^{n}(r,\varrho)\sin n\varphi \sin n\psi. \tag{6.70}$$

The series in (6.70) is completely summable. We will illustrate this for $G_{11}(r,\varphi;\varrho,\psi)$ which allows us the following transformation

$$\begin{aligned}
G_{11}(r,\varphi;\varrho,\psi) &= \frac{2}{\pi}\sum_{n=1}^{\infty} g_{11}^{n}(r,\varrho)\sin n\varphi \sin n\psi \\
&= \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\left(\frac{r}{\varrho}\right)^{n} + \frac{1-\lambda}{1+\lambda}\left(\frac{r\varrho}{a^2}\right)^{n}\right][\cos n(\varphi-\psi) - \cos n(\varphi+\psi)] \\
&= \frac{1}{2\pi}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{\varrho}\right)^{n}\cos n(\varphi-\psi) + \frac{1-\lambda}{1+\lambda}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r\varrho}{a^2}\right)^{n}\cos n(\varphi-\psi)\right. \\
&\quad \left. - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{\varrho}\right)^{n}\cos n(\varphi+\psi) - \frac{1-\lambda}{1+\lambda}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r\varrho}{a^2}\right)^{n}\cos n(\varphi+\psi)\right].
\end{aligned} \tag{6.71}$$

The four series in the above formula can be summed by making use of the standard formula in (6.29), yielding the following closed series-free form for $G_{11}(r,\varphi;\varrho,\psi)$

$$G_{11}(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left(\ln\frac{|z-\bar{\zeta}|}{|z-\zeta|} + \frac{1-\lambda}{1+\lambda}\ln\frac{|a^2-z\bar{\zeta}|}{|a^2-z\zeta|}\right),$$

where, for compactness, complex variable notation

$$z = r(\cos\varphi + i\sin\varphi) \quad \text{and} \quad \zeta = \varrho(\cos\psi + i\sin\psi)$$

is used for the observation and the source point, with the bar on ζ representing its complex conjugate.

The remaining elements of $\mathbf{G}(r,\varphi;\varrho,\psi)$ are found similarly as

$$G_{12}(r,\varphi;\varrho,\psi) = \frac{\lambda}{\pi(1+\lambda)}\ln\frac{|z-\bar{\zeta}|}{|z-\zeta|},$$

$$G_{21}(r,\varphi;\varrho,\psi) = \frac{1}{\pi(1+\lambda)}\ln\frac{|z-\bar{\zeta}|}{|z-\zeta|},$$

and

$$G_{22}(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left(\ln\frac{|z-\bar{\zeta}|}{|z-\zeta|} - \frac{1-\lambda}{1+\lambda}\ln\frac{|a^2-z\bar{\zeta}|}{|a^2-z\zeta|}\right).$$

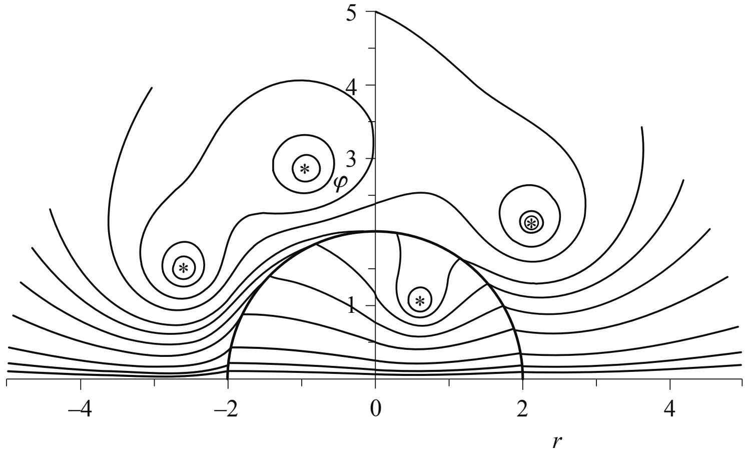

Figure 6.4. The field generated by multiple sources.

To present a practical implementation of the matrix of Green's type that we just constructed, we can apply the superposition principle for obtaining a potential field generated by multiple sources with different intensities, acting at a number of points on Ω. As an example, Figure 6.4 depicts the field generated by sources with intensities $K_1 = 10$, and $K_2 = 20$, $K_3 = 25$ and $K_4 = 30$, acting at: $(1.2, \pi/3)$ on Ω_1,and $(3, \pi/4)$, $(3, 3\pi/5)$, and $(3, 5\pi/6)$ on Ω_2, respectively. For the parameters in the problem, we chose $a = 2$ and $\lambda = 0.1$.

Example 6.4. To explore the technique for obtaining computer-friendly forms of matrices of Green's type further, we let the region $\Omega_2 = \{b < r < \infty, 0 < \varphi < \pi\}$, representing a half-plane reduced by a semi-circular cut-out with radius b, be filled with a conducting (λ_2) homogeneous isotropic material. Let also Ω_2 contain a semi-annular inclusion $\Omega_1 = \{a < r < b, 0 < \varphi < \pi\}$ made out of a foreign conducting (λ_1) homogeneous isotropic material. To determine the field of potential generated by an arbitrarily located unit point source in $\Omega = \Omega_1 \cup \Omega_2$ analytically, we construct the matrix of Green's type for the following boundary-value problem, written in polar coordinates

$$\nabla^2 u_i(r,\varphi) = -f_i(r,\varphi), \quad (r,\varphi) \in \Omega_i,\ i = 1,2, \tag{6.72}$$

$$u_i(r,0) = u_i(r,\pi) = 0, \quad i = 1,2, \tag{6.73}$$

$$u_1(a,\varphi) = 0, \quad \lim_{r\to\infty} u_2(r,\varphi) > \infty, \tag{6.74}$$

and

$$u_1(b,\varphi) = u_2(b,\varphi), \quad \frac{\partial u_1(b,\varphi)}{\partial r} = \lambda \frac{\partial u_2(b,\varphi)}{\partial r} \tag{6.75}$$

with $\lambda = \lambda_2/\lambda_1$.

Following the derivation procedure, that we described in detail in Example 6.3, we now express the elements of the matrix of Green's type $\mathbf{G}(r, \varphi; \varrho, \psi)$ for the problem in (6.72)–(6.75) in series form:

$$G_{ij}(r, \varphi; \varrho, \psi) = \frac{2}{\pi} \sum_{n=1}^{\infty} g_{ij}^{n}(r, \varrho) \sin n\varphi \sin n\psi, \quad i, j = 1, 2. \tag{6.76}$$

In our case, the coefficients $g_{ij}^{n}(r, \varrho)$ of the above series are found as

$$g_{11}^{n}(r, \varrho) = \frac{a^{2n} - r^{2n}}{2n\Delta^*} \left[(1-\lambda) \left(\frac{\varrho}{r}\right)^n + (1+\lambda) \left(\frac{b^2}{r\varrho}\right)^n \right], \quad r \le \varrho,$$

$$g_{12}^{n}(r, \varrho) = \frac{\lambda}{n\Delta^*}(a^{2n} - r^{2n}) \left(\frac{b^2}{r\varrho}\right)^n, \quad a \le r \le b \le \varrho < \infty,$$

$$g_{21}^{n}(r, \varrho) = \frac{1}{n\Delta^*}(a^{2n} - \varrho^{2n}) \left(\frac{b^2}{r\varrho}\right)^n, \quad a \le \varrho \le b \le r < \infty,$$

and

$$g_{22}^{n}(r, \varrho) = \frac{1}{2n} \left\{ \frac{1}{\Delta^*} \left(\frac{b^2}{r\varrho}\right)^n \left[(1+\lambda)a^{2n} + (1-\lambda)b^{2n}\right] + \left(\frac{r}{\varrho}\right)^n \right\}, \quad r \le \varrho,$$

where

$$\Delta^* = (\lambda - 1)a^{2n} - (\lambda + 1)b^{2n}.$$

Note that for $r \ge \varrho$ we can obtain the expressions for the diagonal elements $g_{11}^{n}(r, \varrho)$ and $g_{22}^{n}(r, \varrho)$ from those listed above by exchanging r and ϱ.

The series in (6.76) is non-uniformly convergent, but its convergence can be improved in the same way as we have shown before: we first consider the entry $G_{11}(r, \varphi; \varrho, \psi)$ and transform its coefficient to

$$\begin{aligned}
g_{11}^{n}(r, \varrho) &= \left[\frac{1}{\Delta^*} + \frac{1}{(\lambda+1)b^{2n}} - \frac{1}{(\lambda+1)b^{2n}} \right] \\
&\quad \times \frac{a^{2n} - r^{2n}}{2n} \left[(1-\lambda) \left(\frac{\varrho}{r}\right)^n + (1+\lambda) \left(\frac{b^2}{r\varrho}\right)^n \right] \\
&= \frac{(\lambda-1)a^{2n}}{(\lambda+1)b^{2n}} \frac{a^{2n} - r^{2n}}{2n\Delta^*} \left[(1-\lambda) \left(\frac{\varrho}{r}\right)^n + (1+\lambda) \left(\frac{b^2}{r\varrho}\right)^n \right] \\
&\quad - \frac{a^{2n} - r^{2n}}{2n(\lambda+1)b^{2n}} \left[(1-\lambda) \left(\frac{\varrho}{r}\right)^n + (1+\lambda) \left(\frac{b^2}{r\varrho}\right)^n \right], \quad r \le \varrho.
\end{aligned}$$

Now, we denote the first of the two additive terms in $g_{11}^n(r,\varrho)$ with $R_{11}^n(r,\varrho)$ and transform it to

$$R_{11}^n(r,\varrho) = \frac{(\lambda-1)a^{2n}}{(\lambda+1)b^{2n}}\frac{a^{2n}-r^{2n}}{2n\Delta^*}\left[(1-\lambda)\left(\frac{\varrho}{r}\right)^n + (1+\lambda)\left(\frac{b^2}{r\varrho}\right)^n\right]$$
$$= \frac{(\lambda-1)a^{2n}(a^{2n}-r^{2n})\left((\lambda+1)b^{2n}-(\lambda-1)\varrho^{2n}\right)}{2n(\lambda+1)\left(b^2r\varrho\right)^n\Delta^*}, \quad r \le \varrho.$$

Removing the brackets from the second additive term $g_{11}^n(r,\varrho)$, we can rewrite the latter as

$$g_{11}^n(r,\varrho) = R_{11}^n(r,\varrho) - \frac{1-\lambda}{2n(\lambda+1)}\left(\frac{a^2\varrho}{b^2r}\right)^n + \frac{1-\lambda}{2n(\lambda+1)}\left(\frac{r\varrho}{b^2}\right)^n$$
$$-\frac{1}{2n}\left(\frac{a^2}{r\varrho}\right)^n + \frac{1}{2n}\left(\frac{r}{\varrho}\right)^n.$$

Once $g_{11}^n(r,\varrho)$ is substituted into (6.76), the series can be partially summed. With complex variable notation, for the observation and the source point, a computer-friendly form of $G_{11}(r,\varphi;\varrho,\psi)$ reads

$$G_{11}(r,\varphi;\varrho,\psi) = \frac{2}{\pi}\sum_{n=1}^{\infty} R_{11}^n(r,\varrho)\sin n\varphi \sin n\psi$$
$$+\frac{1}{2\pi}\left(\ln\frac{|z-\overline{\zeta}|\,|a^2-z\zeta|}{|z-\zeta|\,|a^2-z\overline{\zeta}|} + \frac{\lambda-1}{\lambda+1}\ln\frac{|a^2z-b^2\zeta|\,|b^2-z\overline{\zeta}|}{|a^2z-b^2\overline{\zeta}|\,|b^2-z\zeta|}\right). \quad (6.77)$$

Note that for $r \ge \varrho$, the first logarithmic term in $G_{11}(r,\varphi;\varrho,\psi)$ is unchanged whereas the second one transforms to

$$\frac{\lambda-1}{\lambda+1}\ln\frac{|b^2z-a^2\zeta|\,|b^2-z\overline{\zeta}|}{|b^2z-a^2\overline{\zeta}|\,|b^2-z\zeta|}.$$

For the remaining elements of $\mathbf{G}(r,\varphi;\varrho,\psi)$for the setting in (6.72)–(6.75), we find the following

$$G_{12}(r,\varphi;\varrho,\psi) = \frac{2}{\pi}\sum_{n=1}^{\infty} R_{12}^n(r,\varrho)\sin n\varphi \sin n\psi + \frac{\lambda}{\pi(\lambda+1)}\ln\frac{|z-\overline{\zeta}|\,|a^2-z\zeta|}{|z-\zeta|\,|a^2-z\overline{\zeta}|} \quad (6.78)$$

with

$$R_{12}^n(r,\varrho) = \frac{\lambda(\lambda-1)a^{2n}(a^{2n}-r^{2n})}{n(\lambda+1)r^n\varrho^n\Delta^*},$$

$$G_{21}(r,\varphi;\varrho,\psi) = \frac{2}{\pi}\sum_{n=1}^{\infty} R_{21}^n(r,\varrho)\sin n\varphi \sin n\psi + \frac{1}{\pi(\lambda+1)}\ln\frac{|z-\overline{\zeta}|\,|a^2-z\zeta|}{|z-\zeta|\,|a^2-z\overline{\zeta}|} \quad (6.79)$$

with

$$R_{21}^{n}(r,\varrho) = \frac{(\lambda-1)a^{2n}(a^{2n}-\varrho^{2n})}{n(\lambda+1)r^{n}\varrho^{n}\Delta^{*}},$$

and

$$G_{22}(r,\varphi;\varrho,\psi) = \frac{2}{\pi}\sum_{n=1}^{\infty} R_{22}^{n}(r,\varrho)\sin n\varphi \sin n\psi$$
$$+\frac{1}{2\pi}\left(\frac{\lambda-1}{\lambda+1}\ln\frac{|b^2-z\overline{\zeta}|}{|b^2-z\zeta|}+\ln\frac{|z-\overline{\zeta}|\,|a^2-z\zeta|}{|z-\zeta|\,|a^2-z\overline{\zeta}|}\right) \tag{6.80}$$

with

$$R_{22}^{n}(r,\varrho) = \frac{(\lambda-1)a^{2n}\left[(\lambda+1)a^{2n}-(\lambda-1)b^{2n}\right]}{2n(\lambda+1)r^{n}\varrho^{n}\Delta^{*}}.$$

Clearly, the series components in (6.77)–(6.80) are uniformly convergent, making them suitable for immediate computer implementation.

Example 6.5. Let the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$ be composed of three segments: the semi-infinite strip $\Omega_1 = \{-\infty < x < -a, 0 < y < b\}$, the rectangle $\Omega_2 = \{-a < x < a, 0 < y < b\}$ and another semi-infinite strip $\Omega_3 = \{a < x < \infty, 0 < y < b\}$. Let in addition each of the three segments of Ω be filled with a homogeneous isotropic conducting (with conductivities Λ_1, Λ_2 and Λ_3) material. To obtain the potential field, generated in $\Omega = \bigcup_{i=1}^{3}\Omega_i$ by an arbitrarily located unit point source, analytically, we define the following mixed boundary-value problem

$$\frac{\partial^2 u_i(x,y)}{\partial x^2}+\frac{\partial^2 u_i(x,y)}{\partial y^2} = 0, \quad (x,y)\in\Omega_i,\ i=1,2,3, \tag{6.81}$$

$$u_i(x,0) = \frac{\partial u_i(x,b)}{\partial y} = 0, \quad i=1,2,3, \tag{6.82}$$

$$\lim_{x\to-\infty} u_1(x,y) > \infty, \quad \lim_{x\to\infty} u_3(x,y) > \infty, \tag{6.83}$$

$$u_1(-a,y) = u_2(-a,y), \quad \frac{\partial u_1(-a,y)}{\partial x} = \lambda_1\frac{\partial u_2(-a,y)}{\partial x}, \tag{6.84}$$

and

$$u_2(a,y) = u_3(a,y), \quad \frac{\partial u_2(a,y)}{\partial x} = \lambda_2\frac{\partial u_3(a,y)}{\partial x}, \tag{6.85}$$

where λ_1 and λ_2 in the contact conditions (6.84) and (6.85) are defined in terms of the conductivities of the materials with which the segments are filled as $\lambda_1 = \Lambda_2/\Lambda_1$ and $\lambda_2 = \Lambda_3/\Lambda_2$.

Following our familiar procedure, we expresses the elements $G_{i,j}(x, y; \xi, \eta)$ of the (3×3) matrix of Green's type $\mathbf{G}(x, y; \xi, \eta)$ of the problem in (6.81)–(6.85) in series form

$$G_{i,j}(x, y; \xi, \eta) = \frac{2}{b} \sum_{n=1}^{\infty} g_{i,j}^{n}(x, \xi) \sin \nu y \sin \nu \eta, \quad i, j = 1, 2, 3, \tag{6.86}$$

with $\nu = (2n - 1)\pi/2b$.

For compactness, we will only treat those elements of $\mathbf{G}(x, y; \xi, \eta)$ which model the potential field generated in each segment of Ω by a unit source acting at a point (ξ, η) belonging to Ω_1. These are $G_{i1}(x, y; \xi, \eta)$, $i = 1, 2, 3$, in the first column of $\mathbf{G}(x, y; \xi, \eta)$. Their coefficients $g_{i1}^{n}(x, \xi)$ are found as

$$g_{11}^{n}(x, \xi) = \frac{1}{2\nu\Delta^*}\{\mathrm{e}^{-\nu|x-\xi|}\Delta^* + [(1 + \lambda_1)(1 - \lambda_2) + (1 - \lambda_1)(1 + \lambda_2)\mathrm{e}^{4\nu a}]\mathrm{e}^{\nu(x+\xi+2a)}\},$$

$$g_{21}^{n}(x, \xi) = \frac{1}{\nu\Delta^*}[(1 - \lambda_2)\mathrm{e}^{\nu(x+a)} + (1 + \lambda_2)\mathrm{e}^{\nu(3a-x)}]\mathrm{e}^{\nu(\xi+a)}$$

and

$$g_{31}^{n}(x, \xi) = \frac{2}{\nu\Delta^*}\mathrm{e}^{\nu(4a-x+\xi)}$$

with

$$\Delta^* = (1 - \lambda_1)(1 - \lambda_2) + (1 + \lambda_1)(1 + \lambda_2)\mathrm{e}^{4\nu a}.$$

The series in (6.86) converges non-uniformly at the same rate as the series in (6.50). To describe the method for improving the series convergence, we focus on $G_{11}(x, y; \xi, \eta)$, and transform its coefficient $g_{11}^{n}(x, \xi)$ through subtraction and addition of the term

$$\frac{1}{2\nu}\mathrm{e}^{\nu(x+\xi+2a)}.$$

This reduces $g_{11}^{n}(x, \xi)$ to

$$g_{11}^{n}(x, \xi) = \frac{1}{2\nu}[\mathrm{e}^{\nu(x+\xi+2a)} + \mathrm{e}^{-\nu|x-\xi|}] + \frac{\lambda_1}{\nu\Delta^*}[(1 - \lambda_2) - (1 + \lambda_2)\mathrm{e}^{\nu(3a-x)}]\mathrm{e}^{\nu(x+\xi+2a)}.$$

Now, adding and subtracting

$$\frac{\lambda_1}{\nu(1 + \lambda_1)}\mathrm{e}^{\nu(x+\xi+2a)}$$

and performing some trivial algebra, we transform $g_{11}^n(x,\xi)$ to

$$g_{11}^n(x,\xi) = \frac{1}{2\nu}\left[\frac{4\lambda_1(1-\lambda_2)}{(1+\lambda_1)\Delta^*}e^{\nu(x+\xi+2a)} + \frac{1-\lambda_1}{1+\lambda_1}e^{\nu(x+\xi+2a)} + e^{-\nu|x-\xi|}\right].$$

With expression for $g_{11}^n(x,\xi)$, the entry $G_{11}(x,y;\xi,\eta)$ in (6.86) reduces to

$$G_{11}(x,y;\xi,\eta) = \frac{1}{b}\sum_{n=1}^{\infty}\frac{1}{\nu}\left[\frac{4\lambda_1(1-\lambda_2)}{(1+\lambda_1)\Delta^*}e^{\nu(x+\xi+2a)} + \frac{1-\lambda_1}{1+\lambda_1}e^{\nu(x+\xi+2a)} + e^{-\nu|x-\xi|}\right]\sin\nu y\sin\nu\eta. \quad (6.87)$$

Clearly, the series

$$\sum_{n=1}^{\infty}\frac{4\lambda_1(1-\lambda_2)}{\nu(1+\lambda_1)\Delta^*}e^{\nu(x+\xi+2a)}\sin\nu y\sin\nu\eta$$

is uniformly convergent, whereas the other series in (6.87)

$$\sum_{n=1}^{\infty}\frac{1}{\nu}\left[\frac{1-\lambda_1}{1+\lambda_1}e^{\nu(x+\xi+2a)} + e^{-\nu|x-\xi|}\right]\sin\nu y\sin\nu\eta$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{\nu}\left[\frac{1-\lambda_1}{1+\lambda_1}e^{\nu(x+\xi+2a)} + e^{-\nu|x-\xi|}\right][\cos\nu(y-\eta) - \cos\nu(y+\eta)]$$

is completely summable. Upon conducting the summation, we finally obtain the following computer-friendly formula for $G_{11}(x,y;\xi,\eta)$,

$$G_{11}(x,y;\xi,\eta) = \frac{8\lambda_1(1-\lambda_2)}{\pi(1+\lambda_1)}\sum_{n=1}^{\infty}\frac{e^{\nu(x+\xi+2a)}}{(2n-1)\Delta^*}\sin\nu y\sin\nu\eta$$
$$+ \frac{1-\lambda_1}{2\pi(1+\lambda_1)}\ln\frac{|1+e^{\omega(z+\overline{\zeta}+2a)}|\,|1-e^{\omega(z+\zeta+2a)}|}{|1-e^{\omega(z+\overline{\zeta}+2a)}|\,|1+e^{\omega(z+\zeta+2a)}|}$$
$$+ \frac{1}{2\pi}\ln\frac{|1+e^{\omega(z-\zeta)}|\,|1-e^{\omega(z-\overline{\zeta})}|}{|1-e^{\omega(z-\zeta)}|\,|1+e^{\omega(z-\overline{\zeta})}|}, \quad \omega = \frac{\pi}{2b}, \quad (6.88)$$

where we customarily employ complex variable notation for the field and the source points. Note that for $\lambda_1 = \lambda_2 = 1$, equation (6.88) reduces to the classical Green's function [45, 57]

$$G(x,y;\xi,\eta) = \frac{1}{2\pi}\ln\frac{|1+e^{\omega(z-\zeta)}|\,|1-e^{\omega(z-\overline{\zeta})}|}{|1-e^{\omega(z-\zeta)}|\,|1+e^{\omega(z-\overline{\zeta})}|}$$

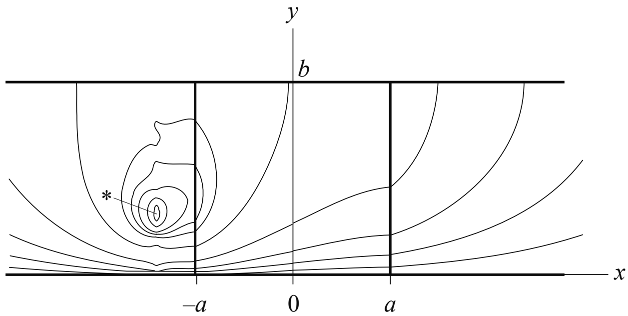

Figure 6.5. Profile of the matrix of Green's type computed by (6.86).

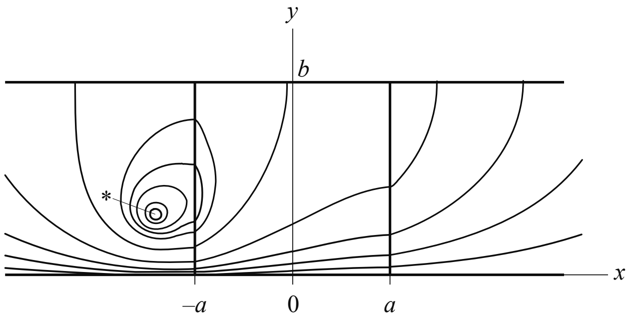

Figure 6.6. Profile of the matrix of Green's type computed by (6.88).

for the Dirichlet–Neumann problem for Laplace equation on the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$.

In Figures 6.5 and 6.6, we can observe a smoothing effect, achieved by the conversion of the series-only representation in (6.86) to a mixed, computer-friendly formula in (6.88). We depict a profile of $\mathbf{G}(x, y; \xi, \eta)$, with the 10th partial sum of the series components included.

In summary, we find the algorithm based on the eigenfunction expansion method and the variation of parameters to be successful in obtaining series representations of matrices of Green's type for a number of specific boundary-value problems for the two-dimensional Laplace and static Klein–Gordon equations, on regions filled with piecewise homogeneous isotropic materials.

The formulas for elements of the matrices of Green's type, that we obtained here, contain non-uniformly convergent series and are therefore not quite suitable for immediate computer implementation. We propose special analytical transformations to improve the series convergence radically, converting them into computer-friendly formulas.

6.3 Fields of Potential on Surfaces of Revolution

In this section, the reader will encounter a new class of boundary-value problems for specific sets of two-dimensional elliptic partial differential equations. We will construct a number of matrices of Green's type modeling potential fields, generated by point sources in assemblies of thin shells of revolution, where each single fragment is made out of a homogeneous isotropic conducting material.

Example 6.6. Let two thin-walled elements be joined together to form a cylindrical shell, closed at one edge, with a hemispherical cap, both shells are of unit radius as shown in Figure 6.7. Let the middle surface of the cap occupy the region $\Omega_1 = \{0 < \varphi < \pi/2, 0 \le \vartheta < 2\pi\}$ whilst the middle surface of the cylindrical element is $\Omega_2 = \{0 < x < a, 0 \le y < 2\pi\}$. In addition, let the materials, of which the elements are made, have conductivities of λ_1 and λ_2, and the interior and exterior surfaces of the assembly $\Omega = \Omega_1 \cup \Omega_2$ be insulated.

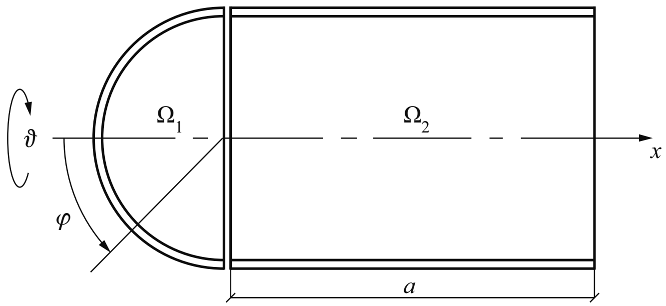

Figure 6.7. Cylindrical shell closed with spherical cap.

We consider the Poisson equation

$$\nabla^2 u_1(\varphi, \vartheta) = -f_1(\varphi, \vartheta), \quad (\varphi, \vartheta) \in \Omega_1, \tag{6.89}$$

with the Laplace operator

$$\nabla^2 \equiv \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial\vartheta^2}$$

written in geographical coordinates φ and ϑ on the unit spherical surface, and another Poisson equation

$$\frac{\partial^2 u_2(x, y)}{\partial x^2} + \frac{\partial^2 u_2(x, y)}{\partial y^2} = -f_2(x, y), \quad (x, y) \in \Omega_2, \tag{6.90}$$

with the Laplacian written in Cartesian coordinates x and y, representing geographical coordinates on the cylindrical surface. Let the functions $u_1(\varphi, \vartheta)$ and $u_2(x, y)$ be subjected to the boundary and contact conditions

$$\lim_{\varphi\to 0}|u_1(\varphi,\vartheta)|<\infty, \quad \alpha\frac{\partial u_2(a,y)}{\partial x}+\beta u_2(a,y)=0, \tag{6.91}$$

$$u_1(\pi/2,\vartheta)=u_2(0,y), \quad \frac{\partial u_1(\pi/2,\vartheta)}{\partial\varphi}=\lambda\frac{\partial u_2(0,y)}{\partial x}, \tag{6.92}$$

$$u_1(\varphi,0)=u_1(\varphi,2\pi), \quad \frac{\partial u_1(\varphi,0)}{\partial\vartheta}=\frac{\partial u_1(\varphi,2\pi)}{\partial x}, \tag{6.93}$$

$$u_2(x,0)=u_2(x,2\pi), \quad \frac{\partial u_2(x,0)}{\partial y}=\frac{\partial u_2(x,2\pi)}{\partial y} \tag{6.94}$$

with $\lambda=\lambda_2/\lambda_1$.

Note that we impose the first condition in (6.91) in order to ensure the boundedness of $u_1(\varphi,\vartheta)$ at the pole $\varphi=0$ of the sphere, whilst the second condition in (6.91) could mimic either Dirichlet type (if $\alpha=0$), or Neumann type (if $\beta=0$), or Robin type (if neither of them are zero). The conditions (6.92) are imposed to model ideal contact of the elements in the assembly. The conditions in (6.93) and (6.94) reflect the 2π-periodicity of $u_1(\varphi,\vartheta)$ and $u_2(x,y)$, with respect to the variables ϑ and y, respectively.

Clearly, the boundary-value problem in (6.89)–(6.94) models a potential field, generated in the assembly, by $f_1(\varphi,\vartheta)$ and $f_2(x,y)$.

Our approach to the boundary-value problem in (6.89)–(6.94) is driven by relation (6.6). In other words, we will intend to obtain its solution in integral form

$$u_1(\varphi,\vartheta)=\iint_{\Omega_1}G_{11}(\varphi,\vartheta;\psi,\tau)f_1(\psi,\tau)d\Omega_1(\psi,\tau)$$
$$+\iint_{\Omega_2}G_{12}(\varphi,\vartheta;\xi,\eta)f_2(\xi,\eta)d\Omega_2(\xi,\eta)$$

and

$$u_2(x,y)=\iint_{\Omega_1}G_{21}(x,y;\psi,\tau)f_1(\psi,\tau)d\Omega_1(\psi,\tau)$$
$$+\iint_{\Omega_2}G_{22}(x,y;\xi,\eta)f_2(\xi,\eta)d\Omega_2(\xi,\eta)$$

which will give us explicit expressions for the elements $G_{i,j}$ of the sought-after matrix of Green's type.

In view of the 2π-periodicity of the functions $u_1(\varphi,\vartheta)$ and $f_1(\varphi,\vartheta)$ with respect to ϑ, and of $u_2(x,y)$ and $f_2(x,y)$ with respect to y, we expand them into Fourier series as

$$u_1(\varphi,\vartheta) = \frac{1}{2}u_{10}(\varphi) + \sum_{n=1}^{\infty} u_{1n}^c(\varphi)\cos n\vartheta + u_{1n}^s(\varphi)\sin n\vartheta,$$

$$f_1(\varphi,\vartheta) = \frac{1}{2}f_{10}(\varphi) + \sum_{n=1}^{\infty} f_{1n}^c(\varphi)\cos n\vartheta + f_{1n}^s(\varphi)\sin n\vartheta \tag{6.95}$$

and

$$u_2(x,y) = \frac{1}{2}u_{20}(x) + \sum_{n=1}^{\infty} u_{2n}^c(x)\cos ny + u_{2n}^s(x)\sin ny,$$

$$f_2(x,y) = \frac{1}{2}f_{20}(x) + \sum_{n=1}^{\infty} f_{2n}^c(x)\cos ny + f_{2n}^s(x)\sin ny. \tag{6.96}$$

Upon substituting these expansions into (6.89)–(6.94), we arrive, for $n = 0$, at the following three-point-posed boundary-value problem

$$\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{du_{10}(\varphi)}{d\varphi}\right) = -f_{10}(\varphi), \quad 0 < \varphi < \pi/2, \tag{6.97}$$

$$\frac{d^2u_{20}(x)}{dx^2} = -f_{20}(x), \quad 0 < x < a, \tag{6.98}$$

$$\lim_{\varphi\to 0}|u_{10}(\varphi)| < \infty, \quad \alpha\frac{du_{20}(a)}{dx} + \beta u_{20}(a) = 0 \tag{6.99}$$

and

$$u_{10}(\pi/2) = u_{20}(0), \quad \frac{du_{10}(\pi/2)}{d\varphi} = \lambda\frac{du_{20}(0)}{dx} \tag{6.100}$$

whilst for $n = 1, 2, 3, \ldots$, we obtain

$$\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{du_{1n}(\varphi)}{d\varphi}\right) - \frac{n^2u_{1n}(\varphi)}{\sin^2\varphi} = -f_{1n}(\varphi), \quad 0 < \varphi < \pi/2, \tag{6.101}$$

$$\frac{d^2u_{2n}(x)}{dx^2} - n^2u_{2n}(x) = -f_{2n}(x), \quad 0 < x < a, \tag{6.102}$$

$$\lim_{\varphi\to 0}|u_{1n}(\varphi)| < \infty, \quad \alpha\frac{du_{2n}(a)}{dx} + \beta u_{2n}(a) = 0 \tag{6.103}$$

and

$$u_{1n}(\pi/2) = u_{2n}(0), \quad \frac{du_{1n}(\pi/2)}{d\varphi} = \lambda\frac{du_{2n}(0)}{dx}. \tag{6.104}$$

Note that in (6.101)–(6.104), we omit the superscripts 'c' and 's' on $u_{1n}(\varphi)$ and $u_{2n}(x)$. We will recall them at a later stage in the development, when they must be treated separately.

Consider first the case of $n = 0$. To construct matrix of Green's type for the homogeneous boundary-value problem corresponding to (6.97)–(6.100) we recall from Chapter 2 the fundamental set of solutions of (6.97)

$$u_{10}^{(1)}(\varphi) = 1 \quad \text{and} \quad u_{10}^{(2)}(\varphi) = \ln(\tan\varphi/2)$$

whilst for the trivial case of (6.98) we have

$$u_{20}^{(1)}(x) = 1 \quad \text{and} \quad u_{20}^{(2)}(x) = x.$$

Following the standard procedure of Lagrange's method of variation of parameters, we obtain the general solution of (6.97) as

$$u_{10}(\varphi) = -\int_0^{\varphi} \ln\left(\frac{\tan\varphi/2}{\tan\psi/2}\right) f_{10}(\psi)d\psi + C_1 + C_2\ln(\tan\varphi/2) \tag{6.105}$$

while for the general solution of (6.98), we find

$$u_{20}(x) = -\int_0^{x} (x-\xi) f_{20}(\xi)d\xi + D_1 + D_2 x. \tag{6.106}$$

We employ the boundary and contact conditions in (6.99) and (6.100) to obtain the constants of integration in (6.105) and (6.106). The first condition in (6.99) yields $C_2 = 0$. The remaining constants are obtained as

$$C_1 = \int_0^{\frac{\pi}{2}} \left[\frac{\alpha+\beta a}{\lambda\beta} - \ln\left(\tan\frac{\psi}{2}\right)\right] f_{10}(\psi)\sin\psi d\psi + \int_0^a \frac{\alpha+\beta(a-\xi)}{\beta} f_{20}(\xi)d\xi,$$

$$D_1 = \int_0^{\frac{\pi}{2}} \frac{\alpha+\beta a}{\lambda\beta} f_{10}(\psi)\sin\psi d\psi + \int_0^a \frac{\alpha+\beta(a-\xi)}{\beta} f_{20}(\xi)d\xi$$

and

$$D_2 = -\int_0^{\frac{\pi}{2}} \frac{\sin\psi}{\lambda} f_{10}(\psi)d\psi.$$

Substituting the constants of integration C_1, C_2, D_1 and D_2 into (6.105) and (6.106), we obtain

$$u_{10}(\varphi) = \int_0^{\frac{\pi}{2}} g_{11}^0(\varphi,\psi) f_{10}(\psi)\sin\psi d\psi + \int_0^a g_{12}^0(\varphi,\xi) f_{20}(\xi)d\xi \tag{6.107}$$

and

$$u_{20}(x) = \int_0^{\frac{\pi}{2}} g_{21}^0(x,\psi) f_{10}(\psi) \sin\psi d\psi + \int_0^a g_{22}^0(x,\xi) f_{20}(\xi) d\xi \qquad (6.108)$$

with the kernel-functions defined as

$$g_{11}^0(\varphi,\psi) = \frac{1}{\lambda\beta}\begin{cases}(\alpha+\beta a) - \lambda\beta\ln(\tan\psi/2), & 0 \le \varphi \le \psi \le \pi/2,\\ (\alpha+\beta a) - \lambda\beta\ln(\tan\varphi/2), & 0 \le \psi \le \varphi \le \pi/2,\end{cases}$$

$$g_{12}^0(\varphi,\xi) = \frac{\alpha+\beta(a-\xi)}{\beta}, \quad 0 \le \varphi \le \pi/2,\ 0 \le \xi \le a,$$

$$g_{21}^0(x,\psi) = \frac{\alpha+\beta(a-x)}{\lambda\beta}, \quad 0 \le x \le a,\ 0 \le \psi \le \pi/2,$$

and

$$g_{22}^0(x,\xi) = \frac{1}{\beta}\begin{cases}\alpha+\beta(a-\xi), & 0 \le x \le \xi \le a,\\ \alpha+\beta(a-x), & 0 \le \xi \le x \le a.\end{cases}$$

Hence, we have completed the analysis of the boundary-value problem in (6.97)–(6.100) and can now address equations (6.101)–(6.104). In order to construct their matrix of Green's type, we recall from Chapter 2 that the fundamental set of solutions for the homogeneous equation corresponding to (6.101) consists of the functions

$$u_{1n}^{(1)}(\varphi) = \tan^n\varphi/2 \quad \text{and} \quad u_{1n}^{(2)}(\varphi) = \cot^n\varphi/2$$

whereas the set for the homogeneous equation corresponding to (6.102) can be represented by

$$u_{2n}^{(1)}(x) = \mathrm{e}^{nx} \quad \text{and} \quad u_{2n}^{(2)}(x) = \mathrm{e}^{-nx}.$$

Following Lagrange's procedure, we find the general solution to (6.101) as

$$u_{1n}(\varphi) = \int_0^{\varphi} \frac{1}{2n}\left(\tan^n\frac{\psi}{2}\cot^n\frac{\varphi}{2} - \tan^n\frac{\varphi}{2}\cot^n\frac{\psi}{2}\right) f_{1n}(\psi)\sin\psi d\psi$$
$$+ C_1\tan^n\frac{\varphi}{2} + C_2\cot^n\frac{\varphi}{2} \qquad (6.109)$$

and

$$u_{2n}(x) = \int_0^x \frac{\sinh n(\xi - x)}{n} f_{2n}(\xi)d\xi + D_1\mathrm{e}^{nx} + D_2\mathrm{e}^{-nx}. \qquad (6.110)$$

The first condition in (6.103) yields $C_2 = 0$. Applying the remaining conditions in (6.103) and (6.104), we arrive at a well-posed system of linear algebraic equations for the remaining constants of integration C_1, D_1 and D_2. The determinant Δ^* of the coefficient matrix of this system is

$$\Delta^* = (1-\lambda)(\beta - \alpha n)\mathrm{e}^{-na} + (1+\lambda)(\beta + \alpha n)\mathrm{e}^{na}$$

and the constants are found as

$$\begin{aligned}
D_1 &= \frac{\lambda(\alpha n - \beta)\mathrm{e}^{-na}}{n\Delta^*}\int_0^{\frac{\pi}{2}} \tan^n \frac{\psi}{2} f_{1n}(\psi)\sin\psi d\psi \\
&\quad + \frac{1+\lambda}{n\Delta^*}\int_0^a [\alpha n \cosh n(\xi - a) - \beta \sinh n(\xi - a)]\, f_{2n}(\xi) d\xi, \\
D_2 &= \frac{\lambda(\alpha n + \beta)\mathrm{e}^{na}}{n\Delta^*}\int_0^{\frac{\pi}{2}} \tan^n \frac{\psi}{2} f_{1n}(\psi)\sin\psi d\psi \\
&\quad + \frac{1-\lambda}{n\Delta^*}\int_0^a [\alpha n \cosh n(\xi - a) - \beta \sinh n(\xi - a)]\, f_{2n}(\xi) d\xi,
\end{aligned}$$

and

$$\begin{aligned}
C_1 = \int_0^{\frac{\pi}{2}} &\left[\frac{\lambda\left[(\beta + \alpha n)\mathrm{e}^{na} - (\beta - \alpha n)\mathrm{e}^{-na}\right]}{n\Delta^*}\tan^n \frac{\psi}{2}\right. \\
&\left. - \frac{1}{2n}\left(\tan^n \frac{\psi}{2} - \cot^n \frac{\psi}{2}\right)\right] f_{1n}(\psi)\sin\psi d\psi \\
&+ \frac{2}{n\Delta^*}\int_0^a [\alpha n \cosh n(\xi - a) - \beta \sinh n(\xi - a)]\, f_{2n}(\xi) d\xi.
\end{aligned}$$

Upon substituting C_1, C_2, D_1 and D_2 into (6.109) and (6.110), we arrive at

$$u_{1n}(\varphi) = \int_0^{\frac{\pi}{2}} g_{11}^n(\varphi,\psi) f_{1n}(\psi)\sin\psi d\psi + \int_0^a g_{12}^n(\varphi,\xi) f_{2n}(\xi) d\xi \qquad (6.111)$$

and

$$u_{2n}(x) = \int_0^{\frac{\pi}{2}} g_{21}^n(x,\psi) f_{1n}(\psi)\sin\psi d\psi + \int_0^a g_{22}^n(x,\xi) f_{2n}(\xi) d\xi \qquad (6.112)$$

with the kernel-functions defined as

$$g_{11}^{n}(\varphi,\psi)=\frac{1}{2n\Delta^{*}}\begin{cases}(\alpha n+\beta)\mathrm{e}^{na}\tan^{n}\psi/2[(1+\lambda)\cot^{n}\varphi/2\\ \quad -(1-\lambda)\tan^{n}\varphi/2]-(\alpha n-\beta)\mathrm{e}^{-na}\tan^{n}\psi/2\\ \quad \times[(1-\lambda)\cot^{n}\varphi/2-(1+\lambda)\tan^{n}\varphi/2], & \psi\le\varphi,\\ (\alpha n+\beta)\mathrm{e}^{na}\tan^{n}\varphi/2[(1+\lambda)\cot^{n}\psi/2\\ \quad -(1-\lambda)\tan^{n}\psi/2]-(\alpha n-\beta)\mathrm{e}^{-na}\tan^{n}\varphi/2\\ \quad \times[(1-\lambda)\cot^{n}\psi/2-(1+\lambda)\tan^{n}\psi/2], & \varphi\le\psi,\end{cases}$$

$$g_{12}^{n}(\varphi,\xi)=2\,[\alpha n\cosh n(a-\xi)-\beta\sinh n(a-\xi)]\,/(n\Delta^{*})$$
$$\times\tan^{n}\varphi/2,\quad 0\le\varphi\le\pi/2,\ 0\le\xi\le a,$$
$$g_{21}^{n}(x,\psi)=2\,[\alpha n\cosh n(a-x)-\beta\sinh n(a-x)]\,/(n\Delta^{*})$$
$$\times\tan^{n}\psi/2,\quad 0\le x\le a,\ 0\le\psi\le\pi/2,$$

and

$$g_{22}^{n}(x,\xi)=\frac{1}{n\Delta^{*}}\begin{cases}[\alpha n\cosh n(x-a)-\beta\sinh n(x-a)]\\ \quad \times[(1+\lambda)\mathrm{e}^{n\xi}+(1-\lambda)\mathrm{e}^{-n\xi}], & \xi\le x,\\ [\alpha n\cosh n(\xi-a)-\beta\sinh n(\xi-a)]\\ \quad \times[(1+\lambda)\mathrm{e}^{nx}+(1-\lambda)\mathrm{e}^{-nx}], & x\le\xi.\end{cases}$$

At this point in our development, with the elements of the matrices of Green's type to the problems in (6.97)–(6.100) and (6.101)–(6.104) determined successfully, we return to the series expansions in (6.95) and (6.96), and express their coefficients $f_{1n}^{c}(\psi)$, $f_{1n}^{s}(\psi)$, $f_{2n}^{c}(\xi)$,and $f_{2n}^{s}(\xi)$ in terms of the functions $f_1(\psi,\tau)$ and $f_2(\xi,\eta)$ as

$$f_{1n}^{c}(\psi)=\frac{1}{\pi}\int_{0}^{2\pi}f_1(\psi,\tau)\cos n\tau d\tau,\quad n=0,1,2,\ldots,$$

$$f_{1n}^{s}(\psi)=\frac{1}{\pi}\int_{0}^{2\pi}f_1(\psi,\tau)\sin n\tau d\tau,\quad n=1,2,3,\ldots,$$

$$f_{2n}^{c}(\xi)=\frac{1}{\pi}\int_{0}^{2\pi}f_2(\xi,\eta)\cos n\eta d\eta,\quad n=0,1,2,\ldots,$$

and

$$f_{2n}^{s}(\xi)=\frac{1}{\pi}\int_{0}^{2\pi}f_2(\xi,\eta)\sin n\eta d\eta,\quad n=1,2,3,\ldots.$$

Substituting $f_{1n}^c(\psi)$ and $f_{2n}^c(\xi)$ into (6.107), (6.108), (6.111) and (6.112), we obtain

$$u_{1n}^c(\varphi) = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \frac{1}{\pi} g_{11}^n(\varphi, \psi) \cos n\tau f_1(\psi, \tau) \sin\psi d\tau d\psi + \int_0^a \int_0^{2\pi} \frac{1}{\pi} g_{12}^n(\varphi, \xi) \cos n\eta f_2(\xi, \eta) d\eta d\xi, \quad n = 0, 1, 2, \ldots,$$

and

$$u_{2n}^c(x) = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \frac{1}{\pi} g_{21}^n(x, \psi) \cos n\tau f_1(\psi, \tau) \sin\psi d\tau d\psi + \int_0^a \int_0^{2\pi} \frac{1}{\pi} g_{22}^n(x, \xi) \cos n\eta f_2(\xi, \eta) d\eta d\xi, \quad n = 0, 1, 2, \ldots,$$

whilst substituting $f_{1n}^s(\psi)$ and $f_{2n}^s(\xi)$ into (6.107), (6.108), (6.111) and (6.112), we get

$$u_{1n}^s(\varphi) = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \frac{1}{\pi} g_{11}^n(\varphi, \psi) \sin n\tau f_1(\psi, \tau) \sin\psi d\tau d\psi + \int_0^a \int_0^{2\pi} \frac{1}{\pi} g_{12}^n(\varphi, \xi) \sin n\eta f_2(\xi, \eta) d\eta d\xi, \quad n = 1, 2, 3, \ldots,$$

and

$$u_{2n}^s(x) = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \frac{1}{\pi} g_{21}^n(x, \psi) \sin n\tau f_1(\psi, \tau) \sin\psi d\tau d\psi + \int_0^a \int_0^{2\pi} \frac{1}{\pi} g_{22}^n(x, \xi) \sin n\eta f_2(\xi, \eta) d\eta d\xi, \quad n = 1, 2, 3, \ldots.$$

Continuing with the procedure, we now substitute the above expressions $u_{1n}^c(\varphi)$ and $u_{1n}^s(\varphi)$ into the expansion of $u_1(\varphi, \vartheta)$ from (6.95), yielding

$$u_1(\varphi, \vartheta) = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} G_{11}(\varphi, \vartheta; \psi, \tau) f_1(\psi, \tau) \sin\psi d\tau d\psi + \int_0^a \int_0^{2\pi} G_{12}(\varphi, \vartheta; \xi, \eta) f_2(\xi, \eta) d\eta d\xi, \quad (\varphi, \vartheta) \in \Omega_1, \tag{6.113}$$

which gives us the first two elements $G_{11}(\varphi, \vartheta; \psi, \tau)$ and $G_{12}(\varphi, \vartheta; \xi, \eta)$ of the matrix of Green's type **G** for the homogeneous boundary-value problem corresponding to (6.89)–(6.94). For $G_{11}(\varphi, \vartheta; \psi, \tau)$ we found the series expansion

$$G_{11}(\varphi, \vartheta; \psi, \tau) = \frac{1}{2\pi} g_{11}^0(\varphi, \psi) + \frac{1}{\pi} \sum_{n=1}^{\infty} g_{11}^n(\varphi, \psi) \cos n(\vartheta - \tau)$$

whose coefficients $g_{11}^{0}(\varphi, \psi)$ and $g_{11}^{n}(\varphi, \psi)$ have been derived earlier in this section. Substituting them into the above series, we transform it, for $\varphi \leq \psi$, to

$$G_{11}(\varphi, \vartheta; \psi, \tau) = \frac{1}{2\pi}\left[\frac{\lambda}{\beta}(\alpha + \beta a) - \ln\left(\tan\frac{\psi}{2}\right)\right] \tag{6.114}$$
$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n\Delta^*}\left\{(\alpha n + \beta)e^{na}\tan^n\frac{\varphi}{2}\left[(1+\lambda)\cot^n\frac{\psi}{2} - (1-\lambda)\tan^n\frac{\psi}{2}\right]\right.$$
$$\left. - (\alpha n - \beta)e^{-na}\tan^n\frac{\psi}{2}\left[(1-\lambda)\cot^n\frac{\psi}{2} - (1+\lambda)\tan^n\frac{\psi}{2}\right]\right\}\cos n(\vartheta - \tau).$$

In order to get an expression of $G_{11}(\varphi, \vartheta; \psi, \tau)$ valid for $\psi \leq \varphi$, we must exchange the variables φ and ψ in (6.114).

Accordingly, we find for the entry $G_{12}(\varphi, \vartheta; \xi, \eta)$ of the matrix of Green's type

$$G_{12}(\varphi, \vartheta; \xi, \eta) = \frac{1}{2\pi\beta}\left[\alpha + \beta(a - \xi)\right]$$
$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n\Delta^*}\left[(\beta + \alpha n)e^{n(a-\xi)} - (\beta - \alpha n)e^{n(\xi - a)}\right]\tan^n\frac{\varphi}{2}\cos n(\vartheta - \eta).$$

Analogously, upon substituting $u_{2n}^{c}(x)$ and $u_{2n}^{s}(x)$ into the expansion of $u_2(x, y)$ from (6.96) we arrive at

$$u_2(x, y) = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} G_{21}(x, y; \psi, \tau) f_1(\psi, \tau)\sin\psi \, d\tau \, d\psi$$
$$+ \int_0^{a}\int_0^{2\pi} G_{22}(x, y; \xi, \eta) f_2(\xi, \eta) \, d\eta \, d\xi, \quad (x, y) \in \Omega_2, \tag{6.115}$$

which discloses the other two elements of the matrix of Green's type **G**. We find an expression for $G_{21}(x, y; \psi, \tau)$ as

$$G_{21}(x, y; \psi, \tau) = \frac{\lambda}{2\pi\beta}\left[\alpha + \beta(a - x)\right]$$
$$+ \frac{\lambda}{\pi}\sum_{n=1}^{\infty}\frac{1}{n\Delta^*}\left[(\beta + \alpha n)e^{n(a-x)} - (\beta - \alpha n)e^{n(x-a)}\right]\tan^n\frac{\psi}{2}\cos n(y - \tau),$$

whilst for $G_{22}(x, y; \xi, \eta)$, valid for $x \leq \xi$, we have

$$G_{22}(x, y; \xi, \eta) = \frac{\alpha + \beta(a - \xi)}{2\pi\beta} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(1-\lambda)e^{-nx} + (1+\lambda)e^{nx}}{n\Delta^*}$$
$$\times\left[(\beta + \alpha n)e^{n(a-\xi)} - (\beta - \alpha n)e^{n(\xi - a)}\right]\cos n(y - \eta).$$

To obtain an expression for $G_{22}(x, y; \xi, \eta)$ valid for $\xi \le x$, we have to exchange the variables x and ξ in the above.

Notice that the series components in the peripheral elements $G_{12}(\varphi, \vartheta; \xi, \eta)$ and $G_{21}(x, y; \psi, \tau)$ of $\mathbf{G}$ converge uniformly, whereas the series in the diagonal elements $G_{11}(\varphi, \vartheta; \psi, \tau)$ and $G_{22}(x, y; \xi, \eta)$ are not uniformly convergent and diverge for $(\varphi, \vartheta) = (\psi, \tau)$ and $(x, y) = (\xi, \eta)$, respectively. This reveals the logarithmic singularity of the diagonal elements of $\mathbf{G}$. To improve the series convergence, we can split off the terms responsible for the singularity similar to what we used repeatedly earlier in this manual.

It is worth noting that there is a particular instance of the problem in (6.89)–(6.94), for which all series in the elements of the matrix of Green's type that we just found allow a complete summation: (i) if both elements in the assembly under consideration are made out of the same material ($\lambda_1 = \lambda_2$ and, subsequently, $\lambda = 1$), and (ii) if the Dirichlet boundary condition is imposed at $x = a$ in (6.91) (implying $\alpha = 0$ and $\beta = 1$), then the elements of $\mathbf{G}$ reduce to

$$G_{11}(\varphi, \vartheta; \psi, \tau) = \frac{1}{4\pi} \ln \frac{e^{2a} - 2\Phi\Psi\cos(\vartheta - \tau) + e^{-2a}\Phi^2\Psi^2}{\Phi^2 - 2\Phi\Psi\cos(\vartheta - \tau) + \Psi^2},$$

$$G_{12}(\varphi, \vartheta; \xi, \eta) = \frac{1}{4\pi} \ln \frac{e^{2a} - 2e^{\xi}\Phi\cos(\vartheta - \eta) + e^{2(\xi - a)}\Phi^2}{e^{2\xi} - 2e^{\xi}\Phi\cos(\vartheta - \eta) + \Phi^2},$$

$$G_{21}(x, y; \psi, \tau) = \frac{1}{4\pi} \ln \frac{e^{2a} - 2e^{x}\Psi\cos(y - \tau) + e^{2(x - a)}\Psi^2}{e^{2x} - 2e^{x}\Psi\cos(y - \tau) + \Psi^2},$$

and

$$G_{22}(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{e^{2(a - x)} - 2e^{(\xi - x)}\cos(y - \eta) + e^{2(\xi - a)}}{1 - 2e^{(\xi - x)}\cos(y - \eta) + e^{2(\xi - x)}},$$

where

$$\Phi = \tan\varphi/2 \quad \text{and} \quad \Psi = \tan\psi/2.$$

In one of this Chapter Exercises, we challenge the reader to derive the above compact formulas.

Example 6.7. Let a thin hemispherical shell of unit radius, whose middle surface is $\Omega = \{0 < \varphi < \pi, 0 < \vartheta < \pi\}$, consist of two congruent segments $\Omega_1 = \{0 < \varphi < \pi/2, 0 < \vartheta < \pi\}$ and $\Omega_2 = \{\pi/2 < \varphi < \pi, 0 < \vartheta < \pi\}$. Let the segments be made out of different homogeneous isotropic materials whose conductivities are defined by λ_1 and λ_2, respectively.

Consider the following well-posed boundary-value problem defined in spherical coordinates

$$\nabla^2 u_i(\varphi, \vartheta) = -f_i(\varphi, \vartheta), \quad (\varphi, \vartheta) \in \Omega_i, \ i = 1, 2, \tag{6.116}$$

$$u_1(\varphi, 0) = u_1(\varphi, \pi), \quad u_2(\varphi, 0) = u_2(\varphi, \pi), \tag{6.117}$$

$$\lim_{\varphi \to 0} |u_1(\varphi, \vartheta)| < \infty, \quad \lim_{\varphi \to \pi} |u_2(\varphi, \vartheta)| < \infty, \tag{6.118}$$

and

$$u_1(\pi/2, \vartheta) = u_2(\pi/2, \vartheta), \quad \frac{\partial u_1(\pi/2, \vartheta)}{\partial \varphi} = \lambda \frac{\partial u_2(\pi/2, \vartheta)}{\partial \varphi}, \tag{6.119}$$

where ∇^2 is the Laplace operator and $\lambda = \lambda_2/\lambda_1$.

If $u_i(\varphi, \vartheta)$ and $f_i(\varphi, \vartheta)$, for $i = 1, 2$, are expanded in the Fourier sine-series

$$u_i(\varphi, \vartheta) = \sum_{n=1}^{\infty} u_{in}(\varphi) \sin n\vartheta \quad \text{and} \quad f_i(\varphi, \vartheta) = \sum_{n=1}^{\infty} f_{in}(\varphi) \sin n\vartheta, \tag{6.120}$$

then the coefficients $u_{1n}(\varphi)$ and $u_{2n}(\varphi)$ in the first of them are defined through the following three-point-posed boundary-value problem

$$\frac{1}{\sin\varphi} \frac{d}{d\varphi} \left(\sin\varphi \frac{du_{1n}(\varphi)}{d\varphi} \right) - \frac{n^2 u_{1n}(\varphi)}{\sin^2\varphi} = -f_{1n}(\varphi), \quad \varphi \in (0, \pi/2), \tag{6.121}$$

$$\frac{1}{\sin\varphi} \frac{d}{d\varphi} \left(\sin\varphi \frac{du_{2n}(\varphi)}{d\varphi} \right) - \frac{n^2 u_{2n}(\varphi)}{\sin^2\varphi} = -f_{2n}(\varphi), \quad \varphi \in (\pi/2, \pi), \tag{6.122}$$

$$\lim_{\varphi \to 0} |u_{1n}(\varphi)| < \infty, \quad \lim_{\varphi \to \pi} |u_{2n}(\varphi)| < \infty, \tag{6.123}$$

and

$$u_{1n}(\pi/2) = u_{2n}(\pi/2), \quad \frac{du_1(\pi/2)}{d\varphi} = \lambda \frac{du_2(\pi/2)}{d\varphi}. \tag{6.124}$$

As we recalled in Example 6.6, a fundamental set of solutions to the homogeneous equation corresponding to (6.121) can be represented by the functions $\tan^n \varphi/2$ and $\cot^n \varphi/2$. Selecting the latter, the general solutions to the equations in (6.121) and (6.122) are obtained as

$$u_{1n}(\varphi) = \int_0^{\varphi} \frac{1}{2n} \left(\tan^n \frac{\psi}{2} \cot^n \frac{\varphi}{2} - \tan^n \frac{\varphi}{2} \cot^n \frac{\psi}{2} \right) f_{1n}(\psi) \sin\psi d\psi$$
$$+ C_1 \tan^n \frac{\varphi}{2} + D_1 \cot^n \frac{\varphi}{2} \tag{6.125}$$

and

$$u_{2n}(\varphi) = \int_{\frac{\pi}{2}}^{\varphi} \frac{1}{2n} \left(\tan^n \frac{\psi}{2} \cot^n \frac{\varphi}{2} - \tan^n \frac{\varphi}{2} \cot^n \frac{\psi}{2} \right) f_{1n}(\psi) \sin \psi d\psi$$
$$+ C_2 \tan^n \frac{\varphi}{2} + D_2 \cot^n \frac{\varphi}{2}. \tag{6.126}$$

The constants of integration in the above expressions for $u_{1n}(\varphi)$ and $u_{2n}(\varphi)$ can be obtained by imposing the boundary and contact conditions (6.123) and (6.124), yielding

$$D_1 = 0, \quad C_2 = \frac{1}{2n} \int_{\frac{\pi}{2}}^{\pi} \cot^n \frac{\psi}{2} f_{2n}(\psi) \sin \psi d\psi,$$
$$C_1 = \frac{1}{2n} \int_0^{\frac{\pi}{2}} \left(\cot^n \frac{\psi}{2} + \frac{1-\lambda}{1+\lambda} \tan^n \frac{\psi}{2} \right) f_{1n}(\psi) \sin \psi d\psi$$
$$+ \frac{\lambda}{n(1+\lambda)} \int_{\frac{\pi}{2}}^{\pi} \cot^n \frac{\psi}{2} f_{2n}(\psi) \sin \psi d\psi$$

and

$$D_2 = \frac{1}{n(1+\lambda)} \int_0^{\frac{\pi}{2}} \tan^n \frac{\psi}{2} f_{1n}(\psi) \sin \psi d\psi$$
$$- \frac{1-\lambda}{2n(1+\lambda)} \int_{\frac{\pi}{2}}^{\pi} \cot^n \frac{\psi}{2} f_{2n}(\psi) \sin \psi d\psi.$$

After substituting C_1 and D_1 into (6.125), the latter transforms to

$$u_{1n}(\varphi) = \frac{1}{2n} \int_0^{\varphi} \left(\tan^n \frac{\psi}{2} \cot^n \frac{\varphi}{2} - \tan^n \frac{\varphi}{2} \cot^n \frac{\psi}{2} \right) f_{1n}(\psi) \sin \psi d\psi$$
$$+ \frac{1}{2n} \int_0^{\frac{\pi}{2}} \left(\cot^n \frac{\psi}{2} + \frac{1-\lambda}{1+\lambda} \tan^n \frac{\psi}{2} \right) \tan^n \frac{\varphi}{2} f_{1n}(\psi) \sin \psi d\psi$$
$$+ \frac{\lambda}{n(1+\lambda)} \int_{\frac{\pi}{2}}^{\pi} \cot^n \frac{\psi}{2} \tan^n \frac{\varphi}{2} f_{2n}(\psi) \sin \psi d\psi \tag{6.127}$$

whilst, after substituting C_2 and D_2 into (6.126), we obtain

$$u_{2n}(\varphi) = \frac{1}{2n} \int_{\frac{\pi}{2}}^{\varphi} \left(\tan^n \frac{\psi}{2} \cot^n \frac{\varphi}{2} - \tan^n \frac{\varphi}{2} \cot^n \frac{\psi}{2} \right) f_{2n}(\psi) \sin \psi d\psi$$
$$+ \frac{1}{2n} \int_{\frac{\pi}{2}}^{\pi} \left(\tan^n \frac{\varphi}{2} - \frac{1-\lambda}{1+\lambda} \cot^n \frac{\varphi}{2} \right) \cot^n \frac{\psi}{2} f_{2n}(\psi) \sin \psi d\psi$$
$$+ \frac{\lambda}{n(1+\lambda)} \int_0^{\frac{\pi}{2}} \tan^n \frac{\psi}{2} \cot^n \frac{\varphi}{2} f_{1n}(\psi) \sin \psi d\psi. \tag{6.128}$$

Hence, the solution of the boundary-value problem in (6.121)–(6.124) is expressed in integral form in (6.127) and (6.128), providing matrix of Green's type for the corresponding homogeneous problem. Elements of its first row appear as

$$g_{11}^n(\varphi,\psi)=\frac{1}{2n}\left(\cot^n\frac{\psi}{2}+\frac{1-\lambda}{1+\lambda}\tan^n\frac{\psi}{2}\right)\tan^n\frac{\varphi}{2},\quad 0\le\varphi\le\psi\le\frac{\pi}{2},$$

and

$$g_{12}^n(\varphi,\psi)=\frac{\lambda}{n(1+\lambda)}\cot^n\frac{\psi}{2}\tan^n\frac{\varphi}{2},\quad 0\le\varphi\le\frac{\pi}{2}\le\psi\le\pi,$$

whilst for elements of the second row, we have

$$g_{21}^n(\varphi,\psi)=\frac{1}{n(1+\lambda)}\cot^n\frac{\varphi}{2}\tan^n\frac{\psi}{2},\quad 0\le\psi\le\frac{\pi}{2}\le\varphi\le\pi,$$

and

$$g_{22}^n(\varphi,\psi)=\frac{1}{2n}\left(\tan^n\frac{\varphi}{2}-\frac{1-\lambda}{1+\lambda}\cot^n\frac{\varphi}{2}\right)\cot^n\frac{\psi}{2},\quad \frac{\pi}{2}\le\varphi\le\psi\le\pi.$$

To complete our derivation procedure, we follow the pattern developed in Example 6.6. We express the right-hand side functions $f_{1n}(\psi)$ and $f_{2n}(\psi)$ in (6.127) and (6.128) in terms of the functions $f_1(\varphi,\vartheta)$ and $f_2(\varphi,\vartheta)$ of (6.116) by making use of the standard Fourier–Euler formulas. Subsequently, we substitute $u_{1n}(\varphi)$ and $u_{2n}(\varphi)$ into the first series in (6.120). This reduces the solution of equations (6.116)–(6.119) to

$$u_1(\varphi,\vartheta)=\int_0^{\frac{\pi}{2}}\int_0^{\pi}G_{11}(\varphi,\vartheta;\psi,\tau)f_1(\psi,\tau)\sin\psi\,d\tau\,d\psi \tag{6.129}$$
$$+\int_{\frac{\pi}{2}}^{\pi}\int_0^{\pi}G_{12}(\varphi,\vartheta;\psi,\tau)f_2(\psi,\tau)\sin\psi\,d\tau\,d\psi,\quad(\varphi,\vartheta)\in\Omega_1,$$

and

$$u_2(\varphi,\vartheta)=\int_0^{\frac{\pi}{2}}\int_0^{\pi}G_{21}(\varphi,\vartheta;\psi,\tau)f_1(\psi,\tau)\sin\psi\,d\tau\,d\psi \tag{6.130}$$
$$+\int_{\frac{\pi}{2}}^{\pi}\int_0^{\pi}G_{22}(\varphi,\vartheta;\psi,\tau)f_2(\psi,\tau)\sin\psi\,d\tau\,d\psi,\quad(\varphi,\vartheta)\in\Omega_2,$$

which explicitly gives us the elements

$$G_{i,j}(\varphi,\vartheta;\psi,\tau)=\frac{2}{\pi}\sum_{n=1}^{\infty}g_{i,j}^n(\varphi,\psi)\sin n\vartheta\sin n\tau,\quad i,j=1,2, \tag{6.131}$$

of the matrix of Green's type for the homogeneous boundary-value problem corresponding to (6.116)–(6.119).

The series in (6.131) are completely summable, so that we arrive at

$$G_{11}(\varphi,\vartheta;\psi,\tau) = \frac{1}{4\pi}\left(H_0(\Phi,\vartheta;\Psi,\tau) + \frac{1-\lambda}{1+\lambda}H_1(\Phi,\vartheta;\Psi,\tau)\right), \qquad (6.132)$$

$$G_{12}(\varphi,\vartheta;\psi,\tau) = \frac{\lambda}{2\pi(1+\lambda)}H_0(\Phi,\vartheta;\Psi,\tau), \qquad (6.133)$$

$$G_{21}(\varphi,\vartheta;\psi,\tau) = \frac{1}{2\pi(1+\lambda)}H_0(\Phi,\vartheta;\Psi,\tau), \qquad (6.134)$$

and

$$G_{22}(\varphi,\vartheta;\psi,\tau) = \frac{1}{4\pi}\left(H_0(\Phi,\vartheta;\Psi,\tau) - \frac{1-\lambda}{1+\lambda}H_1(\Phi,\vartheta;\Psi,\tau)\right) \qquad (6.135)$$

with

$$H_0(\Phi,\vartheta;\Psi,\tau) = \ln\frac{\Phi^2 - 2\Phi\Psi\cos(\vartheta+\tau) + \Psi^2}{\Phi^2 - 2\Phi\Psi\cos(\vartheta-\tau) + \Psi^2}$$

and

$$H_1(\Phi,\vartheta;\Psi,\tau) = \ln\frac{1 - 2\Phi\Psi\cos(\vartheta+\tau) + \Phi^2\Psi^2}{1 - 2\Phi\Psi\cos(\vartheta-\tau) + \Phi^2\Psi^2}$$

with Φ and Ψ, as introduced earlier in Example 6.6, being

$$\Phi = \tan\varphi/2 \quad \text{and} \quad \Psi = \tan\psi/2.$$

In one of the Chapter Exercises, we challenge the reader to perform the actual summation of the series in (6.131) in order to obtain a closed analytical formulas for the elements $G_{i,j}(\varphi,\vartheta;\psi,\tau)$ exhibited in (6.132)–(6.135).

Example 6.8. Consider an assembly of two thin-walled elements, as depicted in Figure 6.8, consisting of a circular plate with radius R, joined to a segment of a toroidal shell, having a circular meridian cross-section. Let the plate's middle plane and the shell's middle surface occupy the regions $\Omega_1 = \{0 < r < R, 0 \le \vartheta < 2\pi\}$ and $\Omega_2 = \{\pi/2 < \varphi < \varphi_0, 0 \le \vartheta < 2\pi\}$, respectively, with $0 < \varphi_0 < \pi$. Let the shell's meridian cross-section represent a unit circle while the plate's radius R is greater than one. Assume also that both elements are made out of the same isotropic homogeneous material.

If the interior and exterior surfaces are insulated, the potential field in such an assembly can be modeled by the equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_1(r,\vartheta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_1(r,\vartheta)}{\partial\vartheta^2} = -f_1(r,\vartheta), \quad (r,\vartheta) \in \Omega_1, \qquad (6.136)$$

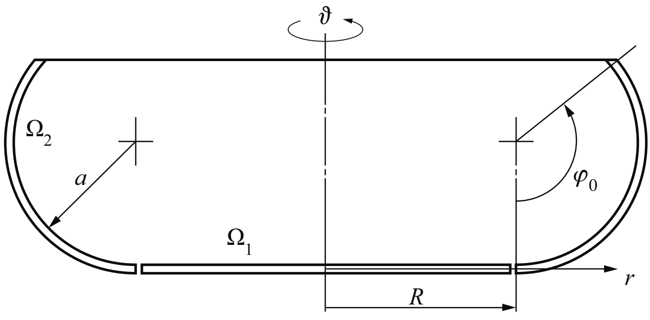

Figure 6.8. Circular plate joint with toroidal shell.

and

$$\frac{1}{D}\frac{\partial}{\partial\varphi}\left(D\frac{\partial u_2(\varphi,\vartheta)}{\partial\varphi}\right)+\frac{1}{D^2}\frac{\partial^2 u_2(\varphi,\vartheta)}{\partial\vartheta^2}=-f_2(\varphi,\vartheta),\quad (\varphi,\vartheta)\in\Omega_2,\tag{6.137}$$

with D representing a function of φ, defined as $D(\varphi)=R+\sin\varphi$.

Let equations (6.136) and (6.137) be subject to the boundary and contact conditions

$$\lim_{\varphi\to 0}|u_1(\varphi,\vartheta)|<\infty,\quad u_2(\varphi_0,\vartheta)=0\tag{6.138}$$

and

$$u_1(R,\vartheta)=u_2(0,\vartheta),\qquad \frac{\partial u_1(R,\vartheta)}{\partial r}=\frac{\partial u_2(0,\vartheta)}{\partial\varphi}.\tag{6.139}$$

Clearly, the above problem assumes 2π-periodicity with respect to the angular coordinate ϑ. With this in mind, we expand $u_1(r,\vartheta)$ and $u_2(\varphi,\vartheta)$ as well as the right-hand side functions $f_1(r,\vartheta)$ and $f_2(\varphi,\vartheta)$in (6.136) and (6.137) in the Fourier series

$$u_1(r,\vartheta)=\frac{1}{2}u_{10}(r)+\sum_{n=1}^{\infty}u_{1n}^c(r)\cos n\vartheta+u_{1n}^s(r)\sin n\vartheta,$$

$$f_1(r,\vartheta)=\frac{1}{2}f_{10}(r)+\sum_{n=1}^{\infty}f_{1n}^c(r)\cos n\vartheta+f_{1n}^s(r)\sin n\vartheta\tag{6.140}$$

and

$$u_2(\varphi,\vartheta)=\frac{1}{2}u_{20}(\varphi)+\sum_{n=1}^{\infty}u_{2n}^c(\varphi)\cos n\vartheta+u_{2n}^s(\varphi)\sin n\vartheta,$$

$$f_2(\varphi,\vartheta)=\frac{1}{2}f_{20}(\varphi)+\sum_{n=1}^{\infty}f_{2n}^c(\varphi)\cos n\vartheta+f_{2n}^s(\varphi)\sin n\vartheta.\tag{6.141}$$

This results in a set ($n = 0, 1, 2, \dots$) of three-point-posed boundary-value problems for the coefficients of the above expansions $u_{1n}(r)$ and $u_{2n}(\varphi)$. We have to consider $n = 0$ individually. Evidently, we arrive at the following problem formulation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_{10}(r)}{dr}\right) = -f_{10}(r), \quad 0 < r < R, \tag{6.142}$$

$$\frac{1}{D(\varphi)}\frac{d}{d\varphi}\left(D(\varphi)\frac{du_{20}(\varphi)}{d\varphi}\right) = -f_{20}(\varphi), \quad 0 < \varphi < \varphi_0, \tag{6.143}$$

$$\lim_{\varphi\to 0} |u_{10}(\varphi)| < \infty, \quad u_{20}(\varphi_0) = 0, \tag{6.144}$$

$$u_{10}(R) = u_{20}(0), \frac{du_{10}(R)}{dr} = \frac{du_{20}(0)}{d\varphi}. \tag{6.145}$$

As we recalled repeatedly, earlier in this book, a fundamental set of solutions for the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_{10}(r)}{dr}\right) = 0$$

can be composed of the functions

$$u_{10}^{(1)}(r) = 1 \quad \text{and} \quad u_{10}^{(2)}(r) = \ln r.$$

With regard to a fundamental set of solutions for the equation

$$\frac{1}{D(\varphi)}\frac{d}{d\varphi}\left(D(\varphi)\frac{du_{20}(\varphi)}{d\varphi}\right) = 0,$$

we note that one of its components is trivial, $u_{20}^{(1)}(\varphi) = 1$, whilst for the second component $u_{20}^{(2)}(\varphi)$ we arrive, after straightforward integration, at

$$u_{20}^{(2)}(\varphi) = \frac{2}{\sqrt{R^2-1}} \arctan \frac{1 + R\tan\varphi/2}{\sqrt{R^2-1}}. \tag{6.146}$$

Using the fundamental sets of solutions just displayed and following our routine procedure, we obtain the matrix of Green's type for the homogeneous problem corresponding to (6.142)–(6.145) the elements of which appear as

$$g_{11}^0(r,\varrho) = \begin{cases} T(\varphi_0, 0) - \ln\varrho/R, & r \le \varrho, \\ T(\varphi_0, 0) - \ln r/R, & r \ge \varrho, \end{cases} \quad g_{12}^0(r,\psi) = T(\varphi_0,\psi)$$

$$g_{21}^0(\varphi,\varrho) = T(\varphi_0,\varphi), \quad g_{22}^0(\varphi,\psi) = \begin{cases} T(\varphi_0,\psi), & \varphi \le \psi, \\ T(\varphi_0,\varphi), & \varphi \ge \psi, \end{cases}$$

with the function $T(\varphi, \psi)$ being defined in terms of the component $u_{20}^{(2)}(\varphi)$ as

$$T(\varphi, \psi) = u_{20}^{(2)}(\varphi) - u_{20}^{(2)}(\psi)$$

which, after some elementary algebra, appears in the form

$$T(\varphi, \psi) = \frac{2}{\sqrt{R^2-1}} \arctan\left(\frac{\sqrt{R^2-1}\sin(\varphi-\psi)/2}{R\cos(\varphi-\psi)/2 + \sin(\varphi-\psi)/2}\right).$$

Returning now to the case of $n = 1, 2, 3, \ldots$, we arrive at the three-point-posed boundary-value problem

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_{1n}(r)}{dr}\right) - \frac{n^2}{r^2}u_{1n}(r) = -f_{1n}(r), \quad 0 < r < R, \tag{6.147}$$

$$\frac{1}{D}\frac{d}{d\varphi}\left(D\frac{du_{2n}(\varphi)}{d\varphi}\right) - \frac{n^2}{D^2}u_{2n}(\varphi) = -f_{2n}(\varphi), \quad 0 < \varphi < \varphi_0, \tag{6.148}$$

$$\lim_{\varphi\to 0} |u_{1n}(\varphi)| < \infty, \quad u_{2n}(\varphi_0) = 0, \tag{6.149}$$

$$u_{1n}(R) = u_{2n}(0), \quad \frac{du_{1n}(R)}{dr} = \frac{du_{2n}(0)}{d\varphi} \tag{6.150}$$

for the coefficients $u_{1n}(r)$ and $u_{2n}(\varphi)$ of the series in (6.140) and (6.141).

Recall the standard fundamental set of solutions

$$u_{1n}^{(1)}(r) = r^n \quad \text{and} \quad u_{1n}^{(2)}(r) = r^{-n}$$

for the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du_{1n}(r)}{dr}\right) - \frac{n^2}{r^2}u_{1n}(r) = 0.$$

The fundamental set of solutions for the equation

$$\frac{1}{D}\frac{d}{d\varphi}\left(D\frac{du_{2n}(\varphi)}{d\varphi}\right) - \frac{n^2}{D^2}u_{2n}(\varphi) = 0$$

can be composed of the functions [33]

$$S_1(\varphi) = \exp\left(\frac{2n}{\sqrt{R^2-1}} \arctan\frac{1 + R\tan\varphi/2}{\sqrt{R^2-1}}\right)$$

and

$$S_2(\varphi) = \exp\left(-\frac{2n}{\sqrt{R^2-1}} \arctan\frac{1 + R\tan\varphi/2}{\sqrt{R^2-1}}\right)$$

With the fundamental sets of solutions mentioned above, the elements of the matrix of Green's type for the homogeneous problem corresponding to (6.147)–(6.150) are founds as

$$g_{11}^n(r,\varrho)=\frac{1}{2n}\left[\left(\frac{r}{\varrho}\right)^n-\frac{S_1(0)S_2(\varphi_0)}{S_1(\varphi_0)S_2(0)}\left(\frac{r\varrho}{R^2}\right)^n\right],\quad r\le\varrho,$$

$$g_{12}^n(r,\psi)=\frac{1}{2n}\left(\frac{r}{R}\right)^n\frac{S_1(0)}{S_1(\varphi_0)}[S_1(\varphi_0)S_2(\psi)-S_1(\psi)S_2(\varphi_0)],$$

$$g_{21}^n(\varphi,\varrho)=\frac{1}{2n}\left(\frac{\varrho}{R}\right)^n\frac{[S_1(\varphi_0)S_2(\psi)-S_1(\psi)S_2(\varphi_0)]}{S_1(\varphi_0)S_2(0)},$$

and

$$g_{22}^n(\varphi,\psi)=\frac{1}{2n}\frac{S_1(\varphi)}{S_1(\varphi_0)}[S_1(\varphi_0)S_2(\psi)-S_1(\psi)S_2(\varphi_0)],\quad \varphi\le\psi.$$

Note that, in order to obtain $g_{11}^n(r,\varrho)$ for $r\ge\varrho$, and $g_{22}^n(\varphi,\psi)$ for $\varphi\ge\psi$, we exchange the variables in the above formulas.

Proceeding with our derivation procedure, we obtain the elements of the matrix of Green's type for the homogeneous problem corresponding to (6.136)–(6.139) in series form

$$G_{11}(r,\vartheta;\varrho,\tau)=\frac{1}{2\pi}g_{11}^0(r,\varrho)+\frac{1}{\pi}\sum_{n=1}^{\infty}g_{11}^n(r,\varrho)\cos n(\vartheta-\tau),\tag{6.151}$$

$$G_{12}(r,\vartheta;\psi,\tau)=\frac{1}{2\pi}g_{12}^0(r,\psi)+\frac{1}{\pi}\sum_{n=1}^{\infty}g_{12}^n(r,\psi)\cos n(\vartheta-\tau),\tag{6.152}$$

$$G_{21}(\varphi,\vartheta;\varrho,\tau)=\frac{1}{2\pi}g_{21}^0(\varphi,\varrho)+\frac{1}{\pi}\sum_{n=1}^{\infty}g_{21}^n(\varphi,\varrho)\cos n(\vartheta-\tau),\tag{6.153}$$

and

$$G_{22}(\varphi,\vartheta;\psi,\tau)=\frac{1}{2\pi}g_{22}^0(\varphi,\psi)+\frac{1}{\pi}\sum_{n=1}^{\infty}g_{22}^n(\varphi,\psi)\cos n(\vartheta-\tau).\tag{6.154}$$

The series in the above equations are completely summable. After performing some elementary but quite cumbersome algebra, we obtain

$$\sum_{n=1}^{\infty}g_{11}^n(r,\varrho)\cos n(\vartheta-\tau)=\frac{1}{4}\ln\frac{R^4-2\alpha R^2r\varrho\cos(\vartheta-\tau)+(\alpha r\varrho)^2}{R^2(r^2-2r\varrho\cos(\vartheta-\tau)+\varrho^2)},$$

$$\sum_{n=1}^{\infty}g_{12}^n(r,\psi)\cos n(\vartheta-\tau)=\frac{1}{4}\ln\frac{R^2-2\gamma Rr\cos(\vartheta-\tau)+(\gamma r)^2}{R^2-2\beta Rr\cos(\vartheta-\tau)+(\beta r)^2},$$

$$\sum_{n=1}^{\infty}g_{21}^n(\varphi,\varrho)\cos n(\vartheta-\tau)=\frac{1}{4}\ln\frac{R^2-2\eta R\varrho\cos(\vartheta-\tau)+(\eta\varrho)^2}{R^2-2\delta R\varrho\cos(\vartheta-\tau)+(\delta\varrho)^2},$$

and

$$\sum_{n=1}^{\infty} g_{22}^{n}(\varphi, \psi) \cos n(\vartheta - \tau) = \frac{1}{4} \ln \frac{1 - 2\omega \cos(\vartheta - \tau) + \omega^2}{1 - 2\sigma \cos(\vartheta - \tau) + \sigma^2},$$

where the parameters $\alpha, \beta, \gamma, \delta, \eta, \omega$, and σ are introduced as

$$\alpha = \sqrt[n]{\frac{S_1(0)S_2(\varphi_0)}{S_1(\varphi_0)S_2(0)}}, \quad \gamma = \sqrt[n]{\frac{S_1(0)S_2(\varphi_0)S_1(\psi)}{S_1(\varphi_0)}},$$

$$\beta = \sqrt[n]{S_1(0)S_2(\psi)}, \quad \delta = \sqrt[n]{\frac{S_2(\varphi)}{S_2(0)}}, \quad \eta = \sqrt[n]{\frac{S_1(\varphi)S_2(\varphi_0)}{S_1(\varphi_0)S_2(0)}},$$

and

$$\omega = \sqrt[n]{\frac{S_1(\varphi)S_2(\varphi_0)S_1(\psi)}{S_1(\varphi_0)}}, \quad \sigma = \sqrt[n]{S_1(\varphi)S_2(\psi)}.$$

Hence, the expressions in (6.151)–(6.154) are computer-friendly and are ready for immediate practical implementation.

Example 6.9. Let a thin hemispherical shell of radius a, with middle surface $\Omega_1 = \{0 < \varphi < \pi/2, 0 < \vartheta < 2\pi\}$, be joined to another congruent hemispherical shell, with middle surface $\Omega_3 = \{\pi/2 < \varphi < \pi, 0 < \vartheta < 2\pi\}$, and to a thin annular plate occupying the region $\Omega_2 = \{a < r < b, 0 < \vartheta < 2\pi\}$, forming a 'Saturn' type assembly. Let each of the elements be made out of an isotropic homogeneous conducting material. In addition, the thickness of the elements is assumed to be negligibly small compared to the shells' radius a and to the width of the plate $w = b - a$.

Assuming that all surfaces of the assembly are insulated, the two-dimensional potential field in the assembly can be modeled by the following system of equations

$$\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{u_1(\varphi,\vartheta)}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 u_1(\varphi,\vartheta)}{\partial\vartheta^2} = 0, \quad (\varphi,\vartheta) \in \Omega_1, \tag{6.155}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_2(r,\vartheta)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_2(r,\vartheta)}{\partial\vartheta^2} = 0, \quad (r,\vartheta) \in \Omega_2, \tag{6.156}$$

$$\frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{u_3(\varphi,\vartheta)}{\partial\varphi}\right) + \frac{1}{\sin^2\varphi}\frac{\partial^2 u_3(\varphi,\vartheta)}{\partial\vartheta^2} = 0, \quad (\varphi,\vartheta) \in \Omega_3, \tag{6.157}$$

subject to the boundary and contact conditions

$$\lim_{\varphi\to 0} |u_1(\varphi,\vartheta)| < \infty, \quad u_2(b,\vartheta) = 0, \quad \lim_{\varphi\to\pi} |u_3(\varphi,\vartheta)| < \infty, \tag{6.158}$$

$$u_1(\pi/2,\vartheta) = u_2(a,\vartheta) = u_3(\pi/2,\vartheta) \tag{6.159}$$

and

$$\lambda_1 \frac{\partial u_1(\pi/2, \vartheta)}{\partial \varphi} - \lambda_2 \frac{\partial u_2(a, \vartheta)}{\partial r} - \lambda_3 \frac{\partial u_3(\pi/2, \vartheta)}{\partial \varphi} = 0, \tag{6.160}$$

where $\lambda_i, (i = 1, 2, 3)$ represent conductivities of the materials out of which the assembly elements are made.

Note that the above problem, in contrast to nearly all the others that we considered so far, is defined for the homogeneous equations in (6.155)–(6.157). We will not describe the derivation procedure in detail, which would involve expressing the solution to the problem in integral form, with the product of the sought-after matrix of Green's type and the vector of the right-hand side functions in the set of governing equations. Instead, we will only display ultimate expressions for the elements of the matrix of Green's type.

Since the setting in (6.155)–(6.160) is 2π-periodic with respect to ϑ, our customary procedure allows us to express the elements of its matrix of Green's type

$$G(x, \vartheta; \xi, \tau) = (G_{i,j}(x, \vartheta; \xi, \tau))_{i,j=\overline{1,3}}$$

in trigonometric series form

$$G_{i,j}(x, \vartheta; \xi, \tau) = \frac{1}{\pi}\left[\frac{1}{2} g^0_{i,j}(x, \xi) + \sum_{n=1}^{\infty} g^n_{i,j}(x, \xi) \cos n(\vartheta - \tau)\right] \tag{6.161}$$

with x and ξ conditional notations for the first coordinate of the observation and the source point, respectively.

The coefficients $g^n_{i,j}(x, \xi), (n = 0, 1, 2, \ldots)$ of the series in (6.161) represent elements of the matrix of Green's type for a four-point-posed boundary-value problem for the set of ordinary differential equations

$$\frac{1}{\sin\varphi} \frac{d}{d\varphi}\left(\sin\varphi \frac{u_{1n}(\varphi)}{d\varphi}\right) - \frac{n^2}{\sin^2\varphi} u_{1n}(\varphi) = 0, \quad \varphi \in (0, \pi/2), \tag{6.162}$$

$$\frac{1}{r} \frac{d}{dr}\left(r \frac{du_{2n}(r)}{dr}\right) - \frac{n^2}{r^2} u_{2n}(r) = 0, \quad r \in (a, b), \tag{6.163}$$

$$\frac{1}{\sin\varphi} \frac{d}{d\varphi}\left(\sin\varphi \frac{u_{3n}(\varphi)}{d\varphi}\right) - \frac{n^2}{\sin^2\varphi} u_{3n}(\varphi) = 0, \quad \varphi \in (\pi/2, \pi), \tag{6.164}$$

on the edges of the graph depicted in Figure 6.9. Recall that we developed the background for such a graph-based statement earlier, in Chapter 5.

The set of equations (6.162)–(6.164) is subject to the boundary conditions

$$\lim_{\varphi \to 0} |u_{1n}(\varphi)| < \infty, \quad u_{2n}(b) = 0, \quad \lim_{\varphi \to \pi} |u_{3n}(\varphi)| < \infty \tag{6.165}$$

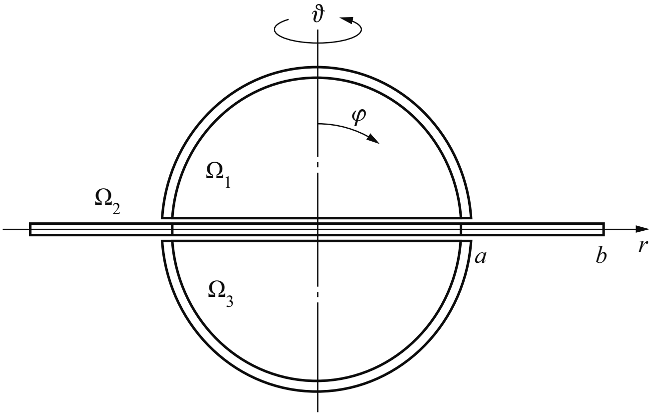

Figure 6.9. 'Saturn' type thin-walled assembly.

imposed at the end-points E_1, E_2,and E_3 of the graph, and to the ideal contact conditions

$$u_{1n}(\pi/2) = u_{2n}(a) = u_{3n}(\pi/2), \tag{6.166}$$

$$\lambda_1 \frac{du_{1n}(\pi/2)}{d\varphi} - \lambda_2 \frac{du_{2n}(a)}{dr} - \lambda_3 \frac{du_{3n}(\pi/2)}{d\varphi} = 0 \tag{6.167}$$

imposed at the graph's vertex V.

For the sake of simplicity, we assume in the following development, that each element of the assembly is made out of the same material, implying $\lambda_1 = \lambda_2 = \lambda_3$. In addition, we assume the unit radius $a = 1$ for the hemispherical shells occupying Ω_1 and Ω_3.

As usual, we need to treat the case of $n = 0$ individually, since its fundamental set of solutions is different from the one we found for $n = 1, 2, 3, \ldots$. The components $g^0_{i,j}(x, \xi)$ of the elements $G_{i,j}(x, \vartheta; \xi, \tau)$ in (6.161) are found to be

$$g^0_{11}(\varphi, \psi) = \ln\left(b \cot\frac{\psi}{2}\right), \ \varphi \le \psi; \quad g^0_{12}(\varphi, \varrho) = \ln\frac{b}{r}; \quad g^0_{13}(\varphi, \psi) = \ln b;$$

$$g^0_{21}(r, \psi) = \ln\frac{b}{r}; \quad g^0_{22}(r, \varrho) = \ln\frac{b}{\varrho}, \ r \le \varrho; \quad g^0_{23}(r, \psi) = \ln\frac{b}{r};$$

$$g^0_{31}(\varphi, \psi) = \ln b; \quad g^0_{32}(\varphi, \varrho) = \ln\frac{b}{\varrho}; \quad g^0_{33}(\varphi, \psi) = \ln\left(b \tan\frac{\varphi}{2}\right), \ \varphi \le \psi.$$

Note that to obtain the diagonal elements $g^0_{11}(\varphi, \psi)$ and $g^0_{33}(\varphi, \psi)$ for $\varphi \ge \psi$, and the element $g^0_{22}(r, \varrho)$ for $r \ge \varrho$, we exchange the variables in the above equations.

For $n = 1, 2, 3, \ldots$, the components $g^n_{1,j}(\varphi, \xi)$ (with the second variable ξ conditionally denoting the first coordinate of the source point) in the first row elements

$G_{1,j}(\varphi, \vartheta; \xi, \tau)$ of (6.161) are expressed as

$$g_{11}^n(\varphi, \psi) = \frac{1}{2n\Delta^*} \begin{cases} \tan^n \frac{\varphi}{2}[\Delta^* \cot^n \frac{\psi}{2} - (b^n + b^{-n}) \tan^n \frac{\psi}{2}], & \varphi \leq \psi, \\ \tan^n \frac{\psi}{2}[\Delta^* \cot^n \frac{\varphi}{2} - (b^n + b^{-n}) \tan^n \frac{\varphi}{2}], & \varphi \geq \psi, \end{cases}$$

$$g_{12}^n(\varphi, \varrho) = \frac{1}{n\Delta^*} \left[\left(\frac{b}{\varrho} \right)^n - \left(\frac{\varrho}{b} \right)^n \right] \tan^n \frac{\varphi}{2},$$

and

$$g_{13}^n(\varphi, \psi) = \frac{b^n - b^{-n}}{n\Delta^*} \tan^n \frac{\varphi}{2} \cot^n \frac{\psi}{2}.$$

For the components of the elements in the second row, we get

$$g_{21}^n(r, \psi) = \frac{1}{n\Delta^*} \left[\left(\frac{b}{r} \right)^n - \left(\frac{r}{b} \right)^n \right] \tan^n \frac{\psi}{2},$$

$$g_{22}^n(r, \varrho) = \frac{1}{2n\Delta^*} \begin{cases} (3r^n - r^{-n}) \left[\left(\frac{b}{\varrho} \right)^n - \left(\frac{\varrho}{b} \right)^n \right], & r \leq \varrho, \\ (3\varrho^n - \varrho^{-n}) \left[\left(\frac{b}{r} \right)^n - \left(\frac{r}{b} \right)^n \right], & r \geq \varrho, \end{cases}$$

and

$$g_{23}^n(r, \psi) = \frac{1}{n\Delta^*} \left[\left(\frac{b}{r} \right)^n - \left(\frac{r}{b} \right)^n \right] \cot^n \frac{\psi}{2}.$$

The components of the elements in the last row appear as

$$g_{31}^n(\varphi, \psi) = \frac{b^n - b^{-n}}{n\Delta^*} \cot^n \frac{\varphi}{2} \tan^n \frac{\psi}{2},$$

$$g_{32}^n(\varphi, \varrho) = \frac{1}{n\Delta^*} \left[\left(\frac{b}{\varrho} \right)^n - \left(\frac{\varrho}{b} \right)^n \right] \cot^n \frac{\varphi}{2},$$

and

$$g_{33}^n(\varphi, \psi) = \frac{1}{2n\Delta^*} \begin{cases} \cot^n \frac{\psi}{2}[\Delta^* \cot^n \frac{\varphi}{2} - (b^n + b^{-n}) \cot^n \frac{\varphi}{2}], & \varphi \leq \psi, \\ \cot^n \frac{\varphi}{2}[\Delta^* \cot^n \frac{\psi}{2} - (b^n + b^{-n}) \cot^n \frac{\psi}{2}], & \varphi \geq \psi, \end{cases}$$

where, in all components above, $\Delta^* = 3b^n - b^{-n}$.

Note that the series in the diagonal elements $G_{i,i}(x, \vartheta; \xi, \tau)$ of the matrix $G(x, \vartheta; \xi, \tau)$ in (6.161), with coefficients g_{ij}^n as just found, do not (and cannot) converge uniformly. This is an unavoidable consequence of the logarithmic singularity and decreases the practicality of the expansions in (6.161). However, the situation can be notably improved by using a partial summation of the series, transforming the components responsible for the singularity into elementary functions, which significantly raises the rate of convergence of the series.

To illustrate this improvement, we display final expressions for several of the elements of $G(x, \vartheta; \xi, \tau)$, after their series representations have been partially summed. In this case, the elements of the first column reduce to

$$G_{11}(\varphi, \vartheta; \psi, \tau) = -\frac{1}{12\pi}\left\{2\sum_{n=1}^{\infty}\frac{b^n + b^{-n}}{nb^{2n}\Delta^*}\Phi^n\Psi^n\cos n(\vartheta - \tau) - \ln\frac{b^2(1 - 2\Phi\Psi\Theta + \Phi^2\Psi^2)(b^4 - 2b^2\Phi\Psi\Theta + \Phi^2\Psi^2)}{(\Phi^2 - 2\Phi\Psi\Theta + \Psi^2)^3}\right\},$$

$$G_{21}(r, \vartheta; \psi, \tau) = \frac{1}{6\pi}\left\{2\sum_{n=1}^{\infty}\frac{b^{2n} - r^{2n}}{nb^{3n}r^n\Delta^*}\Psi^n\cos n(\vartheta - \tau) + \ln\frac{b^4 - 2rb^2\Psi\Theta + r^2\Psi^2}{br(r^2 - 2r\Psi\Theta + \Psi^2)}\right\},$$

and

$$G_{31}(\varphi, \vartheta; \psi, \tau) = \frac{1}{6\pi}\left\{2\sum_{n=1}^{\infty}\frac{b^n - b^{-n}}{nb^{2n}\Delta^*}\Phi^{-n}\Psi^n\cos n(\vartheta - \tau) + \ln\frac{\Phi^2 b^4 - 2b^2\Phi\Psi\Theta + \Psi^2}{b(\Phi^2 - 2\Phi\Psi\Theta + \Psi^2)}\right\},$$

where we introduce

$$\Phi = \tan\frac{\varphi}{2}, \quad \Psi = \tan\frac{\psi}{2}, \quad \text{and} \quad \Theta = \cos(\vartheta - \tau)$$

for compactness.

Similarly, we can derive a computer-friendly formula for the remaining elements of the matrix of Green's type for the problem in (6.155)–(6.160). In one of the Chapter Exercises, we recommend the reader to perform this derivation.

6.4 Chapter Exercises

1. Construct a matrix of Green's type for a problem modeling the potential field on the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$, comprised of the two semi-strips: $\Omega_1 = \{-\infty < x < 0, 0 < y < b\}$ and $\Omega_2 = \{0 < x < \infty, 0 < y < b\}$, each filled with a homogeneous isotropic conducting material, whose conductivities are λ_1 and λ_2, respectively. Assume ideal contact between the fragments and let the boundary conditions be imposed as

$$u_1(x, 0) = u_2(x, 0) = \frac{\partial u_1(x, b)}{\partial y} = \frac{\partial u_2(x, b)}{\partial y} = 0.$$

2. Construct a matrix of Green's type for the Dirichlet problem modeling the potential field on the rectangle $\Omega = \{-a < x < a, 0 < y < b\}$, comprised of two other rectangles, which are in ideal contact: $\Omega_1 = \{-a < x < 0, 0 < y < b\}$ and $\Omega_2 = \{0 < x < a, 0 < y < b\}$, each filled with a homogeneous isotropic conducting material whose conductivities are λ_1 and λ_2, respectively.
3. Construct a matrix of Green's type for a problem modeling the potential field on the compound half-disk $\Omega = \{0 < r < b, 0 < \varphi < \pi\}$, consisting of two segments, in ideal contact: $\Omega_1 = \{0 < r < a, 0 < \varphi < \pi\}$ and $\Omega_2 = \{a < r < b, 0 < \varphi < \pi\}$, each filled with a homogeneous isotropic conducting material whose conductivities are λ_1 and λ_2, respectively. Let the boundary conditions be

$$u_1(r,0) = u_2(r,0) = u_1(r,\pi) = u_2(r,\pi) = \frac{\partial u_2(b,\varphi)}{\partial r} = 0.$$

4. Determine the potential field generated by a unit point source, arbitrarily placed within a thin-walled structure composed of two segments, one of which is a hemispherical shell with radius R_1, whose middle surface occupies the region $\Omega_1 = \{(\varphi_1, \vartheta_1) | 0 < \varphi_1 < \pi/2, 0 \le \vartheta_1 < 2\pi\}$, whilst the second segment is the spherical belt $\Omega_2 = \{(\varphi_2, \vartheta_2) | \varphi_0 < \varphi_2 < \pi/2, 0 \le \vartheta_2 < 2\pi\}$ with radius R_2 greater than R_1. Note that the contact line between the segments is the equator $\varphi_1 = \pi/2$ of the hemispherical segment, whereas for the second segment it is the parallel $\varphi_2 = \varphi_0$. This predetermines the parameter φ_0 as $\arcsin(R_1/R_2)$. Assume that the inner and the outer surfaces of the structure are insulated, with ideal contact taking place between the fragments. Let λ_1 and λ_2 represent the conductivities of the materials out of which the fragments are made and, finally, let the Dirichlet boundary condition $u_2(\pi/2, \vartheta_2) = 0$ be imposed on the free edge of the structure.
5. Derive the compact expressions for the elements of matrix of Green's type, as displayed in Example 6.6.
6. Perform the summation of the series in (6.131) in Example 6.7 in order to obtain closed analytical formulas for elements of the matrix of Green's type shown in (6.132)–(6.135).
7. Derive a computer-friendly formula for all the elements of the matrix of Green's type, where the series representations were obtained in Example 6.9.

Chapter 7
Diffusion Equation

In the last two chapters, we will focus on parabolic partial differential equations. We will develop and use, in this chapter in particular, a special technique for obtaining Green's functions for a number of problems for the classical diffusion (heat) equation which has multiple applications in engineering and natural sciences. The next chapter also deals with parabolic equations. We will turn the reader's attention to the so-called Black–Scholes equation, which had relatively recently emerged as a powerful driving force in financial mathematics (see [8, 52, 55, 65, 75]). There, we will construct Green's functions for a number of terminal-boundary-value problems for the Black–Scholes equation.

The discussion that we will initiate in the present chapter, touches upon techniques for obtaining accurately computable formulas for Green's functions for a wide range of problems for the diffusion equation in one and two spatial dimensions. The approach based on the Green's function formalism has traditionally been widely used for problem solving in the field (e.g. [9, 13, 17, 29, 45, 46]), providing the user with a reasonably rigorous background for numeric work.

We intend to show that the experience we gained earlier in this book, in dealing with elliptic equations, can as well be used successfully for at least several particular problem settings for the diffusion equation. We will present a brief discussion on the potential of our approach, and we will provide evidence that it allows us to obtain several alternative formulas for Green's functions which are, in many cases, more economical computationally, compared to those already available in the literature. What is especially important, with the aid of our approach to the diffusion equation, many new expressions for Green's functions can be constructed.

In Section 7.1, we will briefly address basic concepts of the Laplace integral transform, which represents an important part of our approach to the construction of Green's functions for parabolic equations. We will review only those concepts from the subject, that are essential to our discussion later in both Chapter 7 and Chapter 8. The next two sections of the current chapter will describe methods that can be used to obtain compact formulas for Green's functions and matrices of Green's type, for the diffusion equation in one and two spatial variables.

7.1 Basics of the Laplace Transform

In those areas of engineering and natural sciences where processes and phenomena are modeled by partial differential equations, various integral transforms have traditionally been used. One of them is the Laplace transform, which turns out to be especially efficient in the diffusion (heat) equation. In this section, we will recall only the very basics of the Laplace transform. Since topics related to this transform have traditionally been included in standard undergraduate courses on differential equations, it seems reasonable to assume that our reader has at least a rudimentary background in this field.

Our objective in this section is not to deliver a detailed presentation of the theory of Laplace transform and its applications. This would simply be impossible within the scope of a single brief section. We target a more modest goal: in reviewing the subject, we restrict our presentation to the elementary aspects of the Laplace transform, that are required to proceed with the discussion in this and the next chapter. For a more detailed analysis of this subject, we refer the reader to more specialized sources like [13, 17, 29, 35, 37, 63].

Definition. Let $f(t)$ represent a function of the real variable t and let it be defined for $t \geq 0$. Now, the function $F(p)$, defined as the improper integral

$$F(p) = \int_0^{\infty} \mathrm{e}^{-pt} f(t)dt \tag{7.1}$$

is called the *Laplace transform* of $f(t)$, provided that the foregoing improper integral converges.

In conventional sources related to the method of the Laplace transform, the function $f(t)$ is customarily referred to as the *original*.

We will be using the standard short-hand notation

$$F(p) = \mathbf{L}\{f(t)\}$$

for the relation in (7.1) between the original $f(t)$ and its Laplace transform $F(p)$.

It can be readily shown that the existence of the Laplace transform $F(p)$ (that is, the convergence of the improper integral in (7.1)) is guaranteed, if the original function $f(t)$ is piecewise continuous on the interval $[0, \infty)$ and also is *of exponential order*. A function $f(t)$ is said to be of exponential order, if there exist nonnegative numbers σ, M and t_0 such that

$$|f(t)| \leq M\mathrm{e}^{\sigma t}$$

for $t > t_0$. This condition is sufficient for the Laplace transform $F(p)$ to exist for all $p > \sigma$.

The argument p of the Laplace transform $F(p)$ in (7.1) represents a complex variable and the original function $f(t)$ can be found in terms of $F(p)$ as the following *inverse transform*

$$f(t) = \lim_{R\to\infty} \int_{\sigma-iR}^{\sigma+iR} F(p)\mathrm{e}^{pt}\,dp. \tag{7.2}$$

In our notation of the relation in (7.2), we use the conventional short-hand

$$\mathbf{L}^{-1}\{F(p)\} = f(t).$$

Applications of the Laplace transform method to a problem in applied mathematics involves: (i) finding the direct Laplace transform (that is, converting $f(t)$ onto $F(p)$ by taking the integral in (7.1)) of the original problem (if the latter is somewhat difficult to tackle otherwise), (ii) solving the problem that results from stage (i) (these are, in most cases, much easier to solve), and (iii) finding the inverse Laplace transform by taking the integral in (7.2).

Laplace transforms can readily be obtained for most specific types of originals. This is a matter the straightforward integration in (7.1): it is not too difficult to integrate the product of the exponential function with most elementary and even with special functions. Subsequently, rather extensive tables of the $f(t) \Leftrightarrow F(p)$ relationship are available in nearly every existing textbook and handbook on engineering and science which discusses the topic (see, for example, [13, 29, 37, 63]).

From the appearance of the inverse Laplace transform in (7.2), it follows that we expect finding the original $f(t)$, given its transform $F(p)$, to be more complicated. However, we should note that direct computation of the integral in (7.2) can in many cases be avoided since, given a transform $F(p)$, it is often possible to find the corresponding original in the existing tables of integral transforms. However, this might not be the case in reality. If so, then actual integration in the plane of complex variable becomes the only method of computing the original. In cases when the analytical integration in (7.2) fails, numerical methods are unavoidable.

In the following, we review several important properties of the Laplace transform and will present a number of specific $f(t) \Leftrightarrow F(p)$ relations, which are critical to our developments in this and the next chapter.

We begin with the *L*inearity Property, which states that the Laplace transform of a linear combination of functions equals a linear combination of the transforms of each of the functions. That is

$$\begin{aligned}\mathbf{L}\left\{\sum_{i=1}^{n} C_i f_i(t)\right\} &= \int_0^{\infty} \mathrm{e}^{-pt} \sum_{i=1}^{n} C_i f_i(t)dt \\ &= \sum_{i=1}^{n} C_i \int_0^{\infty} \mathrm{e}^{-pt} f_i(t)dt. \end{aligned} \tag{7.3}$$

It is evident that this property is a direct corollary of the linearity property of a definite (improper) integral.

*D*ifferentiation of the original is another important property of the Laplace transform, which is widely usable when implemented to solve differential equations. It states that the transform of the derivative of $f(t)$ can be found in terms of the transform $F(p)$ of $f(t)$ itself as

$$\mathbf{L}\left\{\frac{df(t)}{dt}\right\} = \int_0^\infty \mathrm{e}^{-pt}\left(\frac{df(t)}{dt}\right)dt = pF(p) - f(0).$$

This property can be readily justified performing integration by parts:

$$\int_0^\infty \mathrm{e}^{-pt}\left(\frac{df(t)}{dt}\right)dt$$

writing the integrand as

$$\mathrm{e}^{-pt} = u, \quad du = -p\mathrm{e}^{-pt}dt, \quad \frac{df(t)}{dt}dt = dv, \quad v = f(t)$$

yielding

$$\int_0^\infty \mathrm{e}^{-pt}\left(\frac{df(t)}{dt}\right)dt = \mathrm{e}^{-pt}f(t)\Big|_0^\infty + p\int_0^\infty \mathrm{e}^{-pt}f(t)dt = pF(p) - f(0),$$

which justifies the property.

*I*ntegration of the original is another widely usable property of the Laplace transform. However, before going to its formulation and justification, it is convenient to prove an important statement, which is usually referred to as the *C*onvolution Theorem for the Laplace transform: if the functions $f(t)$ and $\varphi(t)$ are piecewise continuous on the interval $[0, \infty)$, are also of exponential order, and if $F(p)$ and $\Phi(p)$ represent the transforms of $f(t)$ and $\varphi(t)$, respectively, then

$$\mathbf{L}\left\{\int_0^t f(\tau)\varphi(t-\tau)d\tau\right\} = \int_0^\infty \mathrm{e}^{-pt}\left[\int_0^t f(\tau)\varphi(t-\tau)d\tau\right]dt = F(p)\Phi(p).$$

To proceed with the proof of the above statement, let

$$F(p) = \int_0^\infty \mathrm{e}^{-p\tau}f(\tau)d\tau \quad \text{and} \quad \Phi(p) = \int_0^\infty \mathrm{e}^{-p\xi}\varphi(\xi)d\xi$$

represent the Laplace transforms of $f(t)$ and $\varphi(t)$, respectively.

After multiplying $F(p)$ and $\Phi(p)$, we get

$$\begin{aligned} F(p)\Phi(p) &= \int_0^\infty \mathrm{e}^{-p\tau}f(\tau)d\tau \int_0^\infty \mathrm{e}^{-p\xi}f(\xi)d\xi \\ &= \int_0^\infty \int_0^\infty \mathrm{e}^{-p(\tau+\xi)}f(\tau)\varphi(\xi)d\tau d\xi \\ &= \int_0^\infty f(\tau)d\tau \int_0^\infty \mathrm{e}^{-p(\tau+\xi)}\varphi(\xi)d\xi. \end{aligned}$$

By making τ fixed and after introducing a new variable $t = \tau + \xi$, which yields $t - \tau = \xi$ and $dt = d\xi$, we can rewrite the above expression for $F(p)\Phi(p)$ in terms of τ and t as

$$F(p)\Phi(p) = \int_0^\infty f(\tau)d\tau \int_\tau^\infty \mathrm{e}^{-pt}\varphi(t-\tau)dt.$$

Since both the originals, $f(t)$ and $\varphi(t)$ satisfy the conditions sufficient for the existence for the Laplace transform, the order of integration in the above expression is not important. Hence, we exchange the order of integration

$$F(p)\Phi(p) = \int_0^\infty \mathrm{e}^{-pt}dt \int_0^t f(\tau)\varphi(t-\tau)d\tau$$

which can be rewritten as

$$\int_0^\infty \mathrm{e}^{-pt}\left[\int_0^t f(\tau)\varphi(t-\tau)d\tau\right]dt$$
$$= \mathbf{L}\left\{\int_0^t f(\tau)\varphi(t-\tau)d\tau\right\} = F(p)\Phi(p) \tag{7.4}$$

completing the proof of the convolution theorem. We are now ready to formulate the integration property for the original. That is, to express the Laplace transform of the integral

$$\int_0^t f(\tau)d\tau$$

in terms of the Laplace transform $F(p)$ of the original $f(t)$, we first obtain the Laplace transform of $\varphi(t) = 1$. That is

$$\mathbf{L}\{1\} = \int_0^\infty \mathrm{e}^{-pt}\cdot 1dt = \frac{1}{p}.$$

In other words, $\Phi(p) = 1/p$. Subsequently, if $\varphi(t) = 1$, the relation derived in (7.4) implies

$$\mathbf{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^\infty \mathrm{e}^{-pt}\left[\int_0^t f(\tau)d\tau\right]dt = \frac{F(p)}{p} \tag{7.5}$$

which gives us a very simple expression for the transform of the integral of the original function, in terms of the transform of the original itself.

*T*ranslation Theorem specifies another property of the Laplace transform, which is critical for our developments in this chapter. It states that if $F(p)$ represents the Laplace transform of $f(t)$, and if β is a real number, then

$$\mathbf{L}\{\mathrm{e}^{\beta t}f(t)\} = \int_0^\infty \mathrm{e}^{-pt}[\mathrm{e}^{\beta t}f(t)]dt = F(p-\beta).$$

The proof of this statement follows immediately from the definition of the Laplace transform. To proceed with this, we rewrite the transform of the product $e^{\beta t} f(t)$ as

$$\int_0^\infty e^{-pt}\left[e^{\beta t} f(t)\right] dt = \int_0^\infty e^{-(p-\beta)t} f(t) dt.$$

As mentioned earlier in this section, our objective is to provide the reader with those fundamentals of the Laplace transform method, which are crucial to the derivations in the current chapter and the next one. In order to have all of those listed, within the scope of the current section, provide several additional relations below, the first two of which are the inverse transforms of an elementary and a special function, namely

$$\mathbf{L}^{-1}\left\{\frac{1}{\sqrt{p}} e^{-a\sqrt{p}}\right\} = \frac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{4t}}, \quad a \geq 0, \tag{7.6}$$

and

$$\mathbf{L}^{-1}\left\{K_0(a\sqrt{p})\right\} = \frac{1}{2t} e^{-\frac{a^2}{4t}}, \tag{7.7}$$

where $K_0(a\sqrt{p})$ is the modified Bessel (or Macdonald) function of order zero of the second kind, which we have already encountered in Chapter 5 when we discussed the static Klein–Gordon equation.

Another important relation is the inverse transform of an elementary function, representing a special function. That is,

$$\mathbf{L}^{-1}\left\{\frac{e^{-a\sqrt{p}}}{\sqrt{p}(\sqrt{p}+b)}\right\} = e^{b(bt+a)} \operatorname{erfc}\left(b\sqrt{t} + \frac{a}{2\sqrt{t}}\right), \quad a \geq 0, \tag{7.8}$$

with $\operatorname{erfc}(\xi)$ a special function which is conventionally called the *complementary error function* (see [1, 3, 13, 27, 37]), and is defined as

$$\operatorname{erfc}(\xi) = \frac{2}{\sqrt{\pi}} \int_\xi^\infty e^{-\eta^2} d\eta.$$

The relations in (7.6)–(7.8), as well as many others, can be found in standard tables of integral transforms, available in numerous textbooks and handbooks within the field. Here we recommend, for example, [37] and [63].

7.2 Green's Functions

To introduce the notion of Green's function for the diffusion equation, we consider a simply connected region Ω belonging to the two-dimensional Euclidean space. Let Ω be bounded with a piecewise smooth contour $\Gamma = \bigcup_{i=1}^m \Gamma_i$.

Consider the following initial-boundary-value problem, where the homogeneous diffusion equation

$$\frac{\partial u(P,t)}{\partial t} = \varkappa \nabla^2 u(P,t) - \beta u(P,t), \quad P \in \Omega, \quad t > 0. \tag{7.9}$$

is subject to the initial condition

$$u(P,0) = f(P), \quad P \in \Omega, \tag{7.10}$$

and the boundary condition

$$\alpha_i(P)\frac{\partial u(P,t)}{\partial n_i} + \gamma_i(P)u(P,t) = 0, \quad i = \overline{1,m}, \quad P \in \Gamma_i. \tag{7.11}$$

In the above setting, ∇^2 represents the Laplace operator in the coordinates of P, $\varkappa > 0$ is a constant coefficient corresponding to, in physical settings, the *thermal diffusivity* of the material filling the region, and β is the *convection coefficient*. The functions $\alpha_i(P)$ and $\gamma_i(P)$ in (7.11) are defined on Γ in such a way that at least one of them is nonzero for every segment Γ_i of Γ, whilst n_i represents the direction normal to Γ_i in point P.

Assume that the initial-boundary-value problem specified in (7.9)–(7.11) is well-posed (has a unique solution).

Definition. If the solution $u(P,t)$ of the initial-boundary-value problem in (7.9)–(7.11) is expressed in integral form

$$u(P,t) = \iint_\Omega g(P,t;Q)f(Q)dQ, \quad P \in \Omega, \quad t > 0, \tag{7.12}$$

then the kernel $g(P,t;Q)$ of the above equation is called the *Green's function* of the homogeneous (with $f(P) \equiv 0$) problem corresponding to (7.9)–(7.11).

We believe that at this point in our presentation, it is appropriate to refer to a physical problem that can be modeled with the setting in (7.9)–(7.11), and recall a notion that interprets the Green's function $g(P,t;Q)$ in physical terms.

In the theory of heat conduction [13], the problem in (7.9)–(7.11) models the heat flow in a thin plate occupying the region Ω. The plate is made of an isotropic homogeneous conductive material with thermal diffusivity $\varkappa$. Through the plate's lateral surfaces, Newtonian cooling takes place, with the parameter β being directly proportional to the heat transfer coefficient. In particular, if $\beta = 0$ then the surfaces of the plate are insulated. The boundary condition in (7.11) specifies the heat exchange of the plate with the surrounding media, with temperature at zero degrees, through the edge Γ. In this physical context, the function $u(P,t)$ is referred to as the temperature at a point P in the plate at time t.

With the above interpretation, the Green's function $g(P,t;Q)$ of the homogeneous problem corresponding to that in (7.9)–(7.11) is called the influence function of a point source and is interpreted as the temperature response at an observation point P at time t, caused by an instant point source of heat, located in Q, released at time $t = 0$. In addition, it is understood that $g(P,t;Q)$ is zero for $t < 0$.

Note that the definition we just introduced turns out to be somewhat instructional with respect to the search of efficient procedures for the construction of Green's functions. To elaborate on this, we point out that if we aim to obtain the Green's function to the problem in (7.9)–(7.11), then it looks natural to try and find the solution to the latter. The key part of such a strategy is that we aim not just to solve the problem, but to find the solution in integral form, as in (7.12). With this achieved, the kernel of (7.12) provides an explicit expression for the sought-after Green's function.

The strategy that we just sketched will be a driving force behind our work in this chapter. Keeping in mind the key idea, that we mentioned in the previous paragraph, we have chosen a specific routine to construct the Green's function, which turns out to be effective in a vast number of particular problems. In the following series of examples, we will provide a convincing justification of our assertion that a combination of the Laplace transform method and the method of variation of parameters method is quite efficient for our purposes.

7.2.1 Problems with one Spatial Variable

Example 7.1. We try to find the Green's function for the diffusion equation, for all of two-dimensional space $\Omega = \{-\infty < x < \infty, t > 0\}$. This is usually referred to as the *fundamental solution* of the diffusion equation in one spatial variable [3, 13, 18, 25, 53, 57, 66].

To obtain this Green's function, we consider the homogeneous diffusion equation in one spatial variable

$$\frac{\partial u(x,t)}{\partial t} = \varkappa \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (-\infty, \infty), \quad t > 0, \tag{7.13}$$

subject to the initial condition

$$u(x,0) = f(x) \tag{7.14}$$

and the 'boundary' conditions

$$\lim_{x \to -\infty} |u(x,t)| < \infty, \quad \lim_{x \to \infty} |u(x,t)| < \infty. \tag{7.15}$$

In the process of finding the Green's function for the homogeneous problem corresponding to (7.13)–(7.15), we will establish a pattern for tackling a significant number of other problems.

We begin by taking the Laplace transform $U(x, p)$ of the function $u(x,t)$ with respect to the time variable t. That is,

$$U(x, p) = \int_0^\infty \mathrm{e}^{-pt} u(x,t)dt.$$

Using the properties of linearity and differentiation of the original of the Laplace transform, we arrive at the following self-adjoint boundary-value problem

$$\frac{d^2U(x,p)}{dx^2} - \frac{p}{\varkappa}U(x,p) = -\frac{1}{\varkappa}f(x), \tag{7.16}$$

$$\lim_{x\to-\infty} |U(x,p)| < \infty, \quad \lim_{x\to\infty} |U(x,p)| < \infty \tag{7.17}$$

for the transform $U(x, p)$ of $u(x,t)$.

We approach the above problem with the method of variation of parameters, as developed in Chapter 1. In doing so, the general solution to the equation in (7.16) is found in the form

$$U(x,p) = \int_{-\infty}^{x} \frac{1}{2\varkappa\sqrt{q}}[\mathrm{e}^{\sqrt{q}(\xi-x)} - \mathrm{e}^{\sqrt{q}(x-\xi)}]f(\xi)d\xi \\ + D_1\mathrm{e}^{\sqrt{q}x} + D_2\mathrm{e}^{-\sqrt{q}x}, \tag{7.18}$$

where the parameter q is introduced for compactness for $q = p/\varkappa$.

Before imposing the boundary conditions in (7.17) and deriving with expressions for the constants of integration D_1 and D_2, we regroup the terms in the above equation and rewrite it as

$$U(x,p) = \left(D_1 - \frac{1}{2\varkappa\sqrt{q}}\int_{-\infty}^{x} \mathrm{e}^{-\sqrt{q}\xi} f(\xi)d\xi\right)\mathrm{e}^{\sqrt{q}x} \\ + \left(D_2 + \frac{1}{2\varkappa\sqrt{q}}\int_{-\infty}^{x} \mathrm{e}^{\sqrt{q}\xi} f(\xi)d\xi\right)\mathrm{e}^{-\sqrt{q}x}.$$

We now turn to the first of the conditions in (7.17). Noting that, since the function $\mathrm{e}^{-\sqrt{q}x}$ is unbounded as x goes to negative infinity, we see that its factor in the above expression must be zero. This yields $D_2 = 0$. With similar reasoning, we conclude that the second condition in (7.17) implies that the factor of $\mathrm{e}^{\sqrt{q}x}$ must also be zero, resulting in

$$D_1 = \frac{1}{2\varkappa\sqrt{q}}\int_{-\infty}^{\infty} \mathrm{e}^{-\sqrt{q}\xi} f(\xi)d\xi.$$

After substituting the expressions for D_1 and D_2, that we just found, into (7.18) and rearranging like integral terms, we arrive at the integral representation

$$U(x,p) = \frac{1}{2\varkappa\sqrt{q}}\left\{\int_{-\infty}^{\infty} \mathrm{e}^{\sqrt{q}(x-\xi)} f(\xi)d\xi \right.$$
$$\left. + \int_{-\infty}^{x} [\mathrm{e}^{\sqrt{q}(\xi-x)} - \mathrm{e}^{\sqrt{q}(x-\xi)}] f(\xi)d\xi \right\}$$

for the solution to the problem in (7.16) and (7.17).

In accordance with our recommendations developed in Chapter 1 regarding the use of the method of variation of parameters, the above integral representation for $U(x,p)$ can be rewritten in a compact single-integral form as

$$U(x,p) = \int_{-\infty}^{\infty} G(x,p;\xi) f(\xi)d\xi \tag{7.19}$$

with the kernel-function defined in two pieces as

$$G(x,p;\xi) = \frac{1}{2\varkappa\sqrt{q}} \begin{cases} \mathrm{e}^{\sqrt{q}(x-\xi)}, & -\infty < x \le \xi < \infty, \\ \mathrm{e}^{\sqrt{q}(\xi-x)}, & \infty > x \ge \xi > -\infty. \end{cases}$$

The appearance of $G(x,p;\xi)$ suggests that it can be written in single-piece form

$$G(x,p;\xi) = \frac{1}{2\varkappa\sqrt{q}} \mathrm{e}^{-\sqrt{q}|x-\xi|}.$$

Theorem 1.4 of Chapter 1 suggests that the above represents the Green's function of the homogeneous boundary-value problem, corresponding to that in (7.16) and (7.17). Recalling the relation $q = p/\varkappa$ that we introduced earlier, we rewrite $G(x,p;\xi)$ as

$$G(x,p;\xi) = \frac{1}{2\sqrt{\varkappa p}} \mathrm{e}^{-\frac{|x-\xi|}{\sqrt{\varkappa}}\sqrt{p}}. \tag{7.20}$$

This means, that we now have the Laplace transform $U(x,p)$ of the solution $u(x,t)$ of the initial-boundary-value problem in (7.13)–(7.15) available (see (7.19)). We now proceed to look for $u(x,t)$ itself, representing the inverse transform of $U(x,p)$. That is,

$$u(x,t) = \mathbf{L}^{-1}\left\{\int_{-\infty}^{\infty} G(x,p;\xi) f(\xi)d\xi\right\}$$
$$= \int_{-\infty}^{\infty} \mathbf{L}^{-1}\{G(x,p;\xi)\} f(\xi)d\xi. \tag{7.21}$$

Based on the definition that we introduced earlier in this section, we conclude that the kernel-function of the integral in (7.21) represents the Green's function $g(x,t;\xi)$

of the homogeneous problem, corresponding to (7.13)–(7.15), which implies that, in order to find it, we take the inverse Laplace transform

$$g(x,t;\xi) = \mathbf{L^{-1}}\{G(x,p;\xi)\}$$

of the Green's function $G(x,p;\xi)$ of the homogeneous boundary-value problem, corresponding to (7.16) and (7.17). Now, after recalling the relation in (7.6) and applying it to (7.20), we finally find $g(x,t;\xi)$ to be

$$g(x,t;\xi) = \frac{1}{2\sqrt{\varkappa\pi t}} e^{-\frac{(x-\xi)^2}{4\varkappa t}} \tag{7.22}$$

which is readily recognized as the classical fundamental solution of the one-dimensional diffusion (heat) equation.

Hence, the approach based on the combination of the Laplace transform and the method of variation of parameters turns out to be very fruitful. In the following example we will use it to derive another classical Green's function for the diffusion equation.

Example 7.2. In this example, we aim to derive the Green's function for the initial-boundary-value problem corresponding to

$$\frac{\partial u(x,t)}{\partial t} = \varkappa \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0,\infty), \quad t > 0, \tag{7.23}$$

$$u(x,0) = f(x), \tag{7.24}$$

$$u(0,t) = 0, \quad \lim_{x\to\infty} |u(x,t)| < \infty \tag{7.25}$$

stated on $\Omega = \{0 < x < \infty, t > 0\}$.

Following the procedure that we developed, and applied successfully, in Example 7.1, we define the self-adjoint boundary-value problem

$$\frac{d^2U(x,p)}{dx^2} - \frac{p}{\varkappa}U(x,p) = -\frac{1}{\varkappa}f(x) \tag{7.26}$$

$$U(0,p) = 0, \quad \lim_{x\to\infty} |U(x,p)| < \infty \tag{7.27}$$

for the Laplace transform $U(x,p)$ of the solution $u(x,t)$ of the problem in (7.23)–(7.25).

In continuing the method of variation of parameters, we find the general solution of equation (7.26) as

$$U(x,p) = \frac{1}{2\sqrt{\varkappa p}} \int_0^x [e^{\frac{\xi-x}{\sqrt{\varkappa}}\sqrt{p}} - e^{\frac{x-\xi}{\sqrt{\varkappa}}\sqrt{p}}] f(\xi) d\xi$$
$$+ D_1 e^{\sqrt{\frac{p}{\varkappa}}x} + D_2 e^{-\sqrt{\frac{p}{\varkappa}}x}. \tag{7.28}$$

To find the constants of integration, D_1 and D_2, we recall the boundary conditions stated in (7.27), the first of which implies

$$D_1 + D_2 = 0,$$

whilst the second condition provides us with

$$D_1 = \frac{1}{2\sqrt{\varkappa p}} \int_0^{\infty} \mathrm{e}^{-\sqrt{\frac{p}{\varkappa}}\xi} f(\xi) d\xi,$$

yielding

$$D_2 = -\frac{1}{2\sqrt{\varkappa p}} \int_0^{\infty} \mathrm{e}^{-\sqrt{\frac{p}{\varkappa}}\xi} f(\xi) d\xi.$$

Substituting D_1 and D_2 into (7.28), we obtain the following compact expression for $U(x, p)$

$$U(x, p) = \frac{1}{2\sqrt{\varkappa p}} \int_0^{\infty} [\mathrm{e}^{-\frac{|x-\xi|}{\sqrt{\varkappa}}\sqrt{p}} - \mathrm{e}^{-\frac{(x+\xi)}{\sqrt{\varkappa}}\sqrt{p}}] f(\xi) d\xi.$$

So, the Green's function for the homogeneous boundary-value problem corresponding to that in (7.26) and (7.27) is found to be

$$G(x, p; \xi) = \frac{1}{2\sqrt{\varkappa p}} [\mathrm{e}^{-\frac{|x-\xi|}{\sqrt{\varkappa}}\sqrt{p}} - \mathrm{e}^{-\frac{(x+\xi)}{\sqrt{\varkappa}}\sqrt{p}}],$$

the inverse Laplace transform of which

$$g(x, t; \xi) = \mathbf{L}^{-1}\{G(x, p; \xi)\}$$

is again found with the aid of relation (7.6) as

$$g(x, t; \xi) = \frac{1}{2\sqrt{\pi \varkappa t}} [\mathrm{e}^{-\frac{(x-\xi)^2}{4\varkappa t}} - \mathrm{e}^{-\frac{(x+\xi)^2}{4\varkappa t}}] \tag{7.29}$$

representing the classical expression for the Green's function for the homogeneous initial-boundary-value problem corresponding to (7.23)–(7.25).

Note that with the fundamental solution (see (7.22)) of the diffusion equation available, the method of images can readily be used to derive the Green's function in (7.29): the latter can be read as the sum of the field generated by unit heat source in (7.22), placed in the point ξ with that generated by the unit sink given by

$$g(x, t; \xi) = -\frac{1}{2\sqrt{\varkappa \pi t}} \mathrm{e}^{-\frac{(x+\xi)^2}{4\varkappa t}}$$

placed in the point $-\xi$, symmetric with respect to ξ about the origin.

Example 7.3. We will again exploit our approach, to derive another classical Green's function for the diffusion equation. We consider an initial-boundary-value problem on $\Omega = \{0 < x < \infty, t > 0\}$ for the diffusion equation in (7.23), subject to the initial condition in (7.24) and the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} - \beta u(0,t) = 0, \quad \lim_{x\to\infty} |u(x,t)| < \infty, \quad \beta \geq 0, \tag{7.30}$$

imposed for all times $t > 0$.

Proceeding in accordance with our approach, we reduce the initial-boundary-value problem appearing in (7.23), (7.24), and (7.30) to the self-adjoint boundary-value problem

$$\frac{d^2U(x,p)}{dx^2} - \frac{p}{\varkappa}U(x,p) = -\frac{1}{\varkappa}f(x) \tag{7.31}$$

$$\frac{dU(0,p)}{dx} - \beta U(0,p) = 0, \quad \lim_{x\to\infty} |U(x,p)| < \infty \tag{7.32}$$

for the Laplace transform $U(x,p)$ of $u(x,t)$.

Recalling the general solution of (7.31), which has already been displayed in (7.28), we impose the boundary conditions in (7.32). The second one, as we found in the previous example, yields

$$D_1 = \frac{1}{2\varkappa\sqrt{q}} \int_0^\infty e^{-\sqrt{q}\xi} f(\xi)d\xi.$$

Proceeding once more, we differentiate $U(x,p)$ in (7.28), yielding

$$\frac{dU(x,p)}{dx} = -\int_0^x \frac{1}{2\varkappa}[e^{\sqrt{q}(\xi-x)} + e^{\sqrt{q}(x-\xi)}]f(\xi)d\xi$$
$$+ D_1\sqrt{q}e^{\sqrt{q}x} - D_2\sqrt{q}e^{-\sqrt{q}x},$$

where again $q = p/\varkappa$, implying that the first boundary condition in (7.32) gives us

$$D_1(\sqrt{q} - \beta) - D_2(\sqrt{q} + \beta) = 0$$

leading, in light of the expression for D_1 that we found above, to

$$D_2 = \frac{\sqrt{q} - \beta}{2\varkappa\sqrt{q}(\sqrt{q} + \beta)} \int_0^\infty e^{-\sqrt{q}\xi} f(\xi)d\xi.$$

Once D_1 and D_2 are substituted into (7.28), we arrive at

$$U(x,p) = \frac{1}{2\varkappa\sqrt{q}}\left\{\int_0^x [e^{\sqrt{q}(\xi-x)} - e^{\sqrt{q}(x-\xi)}]f(\xi)d\xi\right.$$
$$\left. + \int_0^\infty \left[e^{\sqrt{q}(x-\xi)} + \frac{\sqrt{q}-\beta}{\sqrt{q}+\beta}e^{-\sqrt{q}(x+\xi)}\right]f(\xi)d\xi\right\}$$

for the solution to (7.31) and (7.32).

A shorthand expression for the above becomes the single integral

$$U(x,p)=\frac{1}{2\varkappa\sqrt{q}}\int_0^\infty\left[e^{-\sqrt{q}|x-\xi|}+\frac{\sqrt{q}-\beta}{\sqrt{q}+\beta}e^{-\sqrt{q}(x+\xi)}\right]f(\xi)d\xi,$$

the kernel-function of which

$$\begin{aligned}G(x,p;\xi)&=\frac{1}{2\varkappa\sqrt{q}}\left[e^{-\sqrt{q}|x-\xi|}+\frac{\sqrt{q}-\beta}{\sqrt{q}+\beta}e^{-\sqrt{q}(x+\xi)}\right]\\&=\frac{1}{2\sqrt{\varkappa p}}\left[e^{-\frac{|x-\xi|}{\sqrt{\varkappa}}\sqrt{p}}+\frac{\sqrt{p}-\sqrt{\varkappa}\beta}{\sqrt{p}+\sqrt{\varkappa}\beta}e^{-\frac{x+\xi}{\sqrt{\varkappa}}\sqrt{p}}\right]\end{aligned}\tag{7.33}$$

represents the Green's function for the homogeneous boundary-value problem corresponding to (7.31) and (7.32).

To ease performing the inverse Laplace transform of $G(x,p;\xi)$ in (7.33), we transform it to the equivalent formula

$$G(x,p;\xi)=\frac{1}{2\sqrt{\varkappa p}}\left[e^{-\frac{|x-\xi|}{\sqrt{\varkappa}}\sqrt{p}}+e^{-\frac{x+\xi}{\sqrt{\varkappa}}\sqrt{p}}-\frac{2\sqrt{\varkappa}\beta}{\sqrt{p}+\sqrt{\varkappa}\beta}e^{-\frac{x+\xi}{\sqrt{\varkappa}}\sqrt{p}}\right].$$

The inverse Laplace transform of $G(x,p;\xi)$, which represents the Green's function $g(x,t;\xi)$ of the homogeneous initial-boundary-value problem corresponding to (7.23), (7.24), and (7.30) is ultimately found with the aid of relations (7.6) and (7.7) as

$$\begin{aligned}g(x,t;\xi)=&\frac{1}{2\sqrt{\pi\varkappa t}}[e^{-\frac{(x-\xi)^2}{4\varkappa t}}+e^{-\frac{(x+\xi)^2}{4\varkappa t}}]\\&-\beta e^{\beta(x+\xi)}e^{\beta^2\varkappa t}\operatorname{erfc}\left(\beta\sqrt{\varkappa t}+\frac{x+\xi}{2\sqrt{\varkappa t}}\right).\end{aligned}\tag{7.34}$$

Note that for $\beta=0$ in (7.34), the latter reduces to the classical [13] Green's function

$$g(x,t;\xi)=\frac{1}{2\sqrt{\pi\varkappa t}}[e^{-\frac{(x-\xi)^2}{4\varkappa t}}+e^{-\frac{(x+\xi)^2}{4\varkappa t}}]\tag{7.35}$$

for the problem stated in (7.23), (7.24), with boundary conditions imposed as

$$\frac{\partial u(0,t)}{\partial x}=0,\quad \lim_{x\to\infty}|u(x,t)|<\infty.\tag{7.36}$$

In the Chapter Exercises, we advise the reader to use our approach for the direct derivation of the Green's function shown in (7.35).

It is worth noting that, analogous to the case of the Green's function appearing in (7.29), the method of images can help to derive the Green's function in (7.35). In contrast to (7.29), in order to satisfy the first condition of (7.36), we must add another unit source whose field strength is given by

$$g(x,t;\xi) = \frac{1}{2\sqrt{\varkappa\pi t}}\mathrm{e}^{-\frac{(x+\xi)^2}{4\varkappa t}}$$

to the fundamental solution of the diffusion equation shown in (7.22).

Summarizing the work we presented in the opening part of this section, we outline the following stages of our procedure for the construction of Green's functions for the diffusion equation:

(i) upon applying the Laplace transform, an original initial-boundary-value problem for the diffusion equation reduces to a boundary-value problem for a linear ordinary differential equation;

(ii) the Green's function for the boundary-value problem appearing in stage (i) is constructed by means of the method of variation of parameters;

(iii) taking the inverse Laplace transform of the Green's function obtained in stage (ii), we ultimately derive the desired Green's function for the diffusion equation.

Whilst considering another classical problem in our next example, we discuss two alternative approaches to the construction of its Green's function. Along with the Laplace transform and the method of variation of parameters, we discuss an alternative way, which is also helpful in several situations with the diffusion equation.

Example 7.4. We derive the Green's function for the homogeneous initial-boundary-value problem corresponding to

$$\frac{\partial u(x,t)}{\partial t} = \varkappa\frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0,a), \quad t > 0, \tag{7.37}$$

$$u(x,0) = f(x) \tag{7.38}$$

and

$$u(0,t) = 0, \quad u(a,t) = 0 \tag{7.39}$$

Taking the Laplace transform, the above reduces to the self-adjoint boundary-value problem

$$\frac{d^2U(x,p)}{dx^2} - \frac{p}{\varkappa}U(x,p) = -\frac{1}{\varkappa}f(x), \tag{7.40}$$

$$U(0,p) = 0, \quad U(a,p) = 0. \tag{7.41}$$

for the transform $U(x, p)$ of $u(x, t)$. We now find an expression for the Green's function for the problem in (7.40) and (7.41), valid for $x \leq \xi$:

$$G(x, p; \xi) = \frac{(e^{\sqrt{q}x} - e^{-\sqrt{q}x})(e^{\sqrt{q}(a-\xi)} - e^{\sqrt{q}(\xi-a)})}{2\sqrt{\varkappa p}(e^{\sqrt{q}a} - e^{-\sqrt{q}a})} \tag{7.42}$$

with $q = p/\varkappa$. Due to the problem in (7.40) and (7.41) being self-adjoint, an expression for $G(x, p; \xi)$, valid for $x \geq \xi$, can be obtained from that in (7.42) by exchanging x and ξ.

To ease performing the inverse Laplace transform of $G(x, p; \xi)$, representing the sought-after Green's function, we rewrite one of the factors in (7.42) as

$$\frac{1}{(e^{\sqrt{q}a} - e^{-\sqrt{q}a})} = \frac{e^{-\sqrt{q}a}}{1 - e^{-2\sqrt{q}a}}$$

and write it as the sum of a geometric progression:

$$\frac{1}{(e^{\sqrt{q}a} - e^{-\sqrt{q}a})} = e^{-\sqrt{q}a} \sum_{n=0}^{\infty} e^{-2\sqrt{q}na}.$$

With this in mind, $G(x, p; \xi)$ from (7.42) reads

$$G(x, p; \xi) = \frac{1}{2\sqrt{\varkappa p}} \sum_{n=0}^{\infty} [e^{-\sqrt{\frac{p}{\varkappa}}(\xi-x+2na)} - e^{-\sqrt{\frac{p}{\varkappa}}(\xi+x+2na)}$$
$$- e^{-\sqrt{\frac{p}{\varkappa}}(2(n+1)a-x-\xi)} + e^{-\sqrt{\frac{p}{\varkappa}}(2(n+1)a+x-\xi)}].$$

With the aid of the relation in (7.6), the Green's function $g(x, t; \xi)$ of the homogeneous initial-boundary-value problem corresponding to (7.37)–(7.39) is found as

$$g(x, t; \xi) = \mathbf{L}^{-1}\{G(x, p; \xi)\}$$
$$= \frac{1}{2\sqrt{\varkappa \pi t}} \sum_{n=0}^{\infty} [e^{-\frac{(\xi-x+2na)^2}{4\varkappa t}} - e^{-\frac{(\xi+x+2na)^2}{4\varkappa t}}$$
$$- e^{-\frac{(\xi+x-2(n+1)a)^2}{4\varkappa t}} + e^{-\frac{(\xi-x-2(n+1)a)^2}{4\varkappa t}}].$$

We can rewrite this in a more compact formula by appropriately rearranging the series components, yielding

$$g(x, t; \xi) = \frac{1}{2\sqrt{\varkappa \pi t}} \sum_{m=-\infty}^{\infty} [e^{-\frac{(x-\xi+2ma)^2}{4\varkappa t}} - e^{-\frac{(x+\xi+2ma)^2}{4\varkappa t}}] \tag{7.43}$$

which represents one of the two standard expressions for the Green's function to the problem in (7.37)–(7.39) known and used in the field [13, 17, 57].

To obtain an expression for the Green's function $g(x,t;\xi)$ alternative to (7.43), we turn to a technique rooted in the classical method of eigenfunction expansion [29, 66], and consider again the initial boundary-value problem in (7.37)–(7.39). By expanding the functions $u(x,t)$ and $f(x)$ into a Fourier series, with respect to the spatial variable x

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \nu x, \quad f(x) = \sum_{n=1}^{\infty} f_n \sin \nu x, \quad \nu = \frac{n\pi}{a}, \tag{7.44}$$

and substituting these into (7.37)–(7.39), we arrive at the trivial initial value problem

$$\frac{du_n(t)}{dt} + \varkappa \nu^2 u_n(t) = 0,$$
$$u_n(0) = f_n$$

for the coefficients of the first series of (7.44), the solution of which is

$$u_n(t) = f_n \mathrm{e}^{-\varkappa \nu^2 t}$$

Expressing f_n by means of the fundamental rule for the Fourier coefficients (the Fourier-Euler formula), we rewrite $u_n(t)$ as

$$u_n(t) = \left(\frac{2}{a} \int_0^a f(\xi) \sin \nu \xi d\xi \right) \mathrm{e}^{-\varkappa \nu^2 t}.$$

After substituting this into the first of the series in (7.44), and exchanging the order of the summation and integration, we obtain the solution of the problem in (7.37)–(7.39) in the formula

$$u(x,t) = \int_0^a \frac{2}{a} \sum_{n=1}^{\infty} \mathrm{e}^{-\varkappa \nu^2 t} \sin \nu x \sin \nu \xi f(\xi) d\xi.$$

Hence, since the solution of the initial boundary-value problem in (7.37)–(7.39) is found as an integral over $[0,a]$, we conclude that the kernel-function

$$g(x,t;\xi) = \frac{2}{a} \sum_{n=1}^{\infty} \mathrm{e}^{-\varkappa \nu^2 t} \sin \nu x \sin \nu \xi \tag{7.45}$$

of that integral represents the sought-after Green's function.

Hence, the standard algorithm of eigenfunction expansion has led us to a formula for the Green's function under consideration, alternative to (7.43). The formula in (7.45) is itself classical and can be found in every standard textbook in the field. Note that having these two different formulas is highly convenient for numerical implementations, because the formula in (7.45) converges at a high rate for *large* values of the time variable t whereas the one in (7.43) does so for *small* values of t.

We will not analyze convergence of the series in (7.45). We can justify this, at least formally, by noting that, in the following, its sum will be expressed in terms of a special function, which can be tabulated in order to make the problem of convergence a matter of software development rather than a subject that needs to be discussed in this context.

It turns out that a number of Green's functions for the diffusion equation, which we will derive later in this chapter, can be expressed in terms of the *Jacobi Theta function of the third kind* $\vartheta_3(\alpha,\beta)$. This is a two variable function whose series representation

$$\vartheta_3(\alpha,\beta) = 1 + 2\sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2\beta n^2} \cos 2n\pi\alpha \tag{7.46}$$

is available in [1, 27, 37, 66].

Consider, for example, the Green's function $g(x,t;\xi)$, that we derived in (7.45). Its Theta function form can be readily obtained if the product of sines in (7.45) transforms into the difference of cosines as

$$\begin{aligned} g(x,t;\xi) &= \frac{2}{a}\sum_{n=1}^{\infty} \mathrm{e}^{-\varkappa\nu^2 t} \sin\nu x \sin\nu\xi \\ &= \frac{1}{a}\sum_{n=1}^{\infty} \mathrm{e}^{-\varkappa\nu^2 t}[\cos\nu(x-\xi) - \cos\nu(x+\xi)] \\ &= \frac{1}{a}\left[\sum_{n=1}^{\infty} \mathrm{e}^{-\varkappa\nu^2 t}\cos\nu(x-\xi) - \sum_{n=1}^{\infty} \mathrm{e}^{-\varkappa\nu^2 t}\cos\nu(x+\xi)\right], \end{aligned}$$

which immediately converts the above into

$$\begin{aligned} g(x,t;\xi) &= \frac{1}{2a}\left[\vartheta_3\left(\frac{x-\xi}{2a},\frac{\varkappa t}{a^2}\right) - 1\right] - \frac{1}{2a}\left[\vartheta_3\left(\frac{x+\xi}{2a},\frac{\varkappa t}{a^2}\right) - 1\right] \\ &= \frac{1}{2a}\left[\vartheta_3\left(\frac{x-\xi}{2a},\frac{\varkappa t}{a^2}\right) - \vartheta_3\left(\frac{x+\xi}{2a},\frac{\varkappa t}{a^2}\right)\right]. \end{aligned} \tag{7.47}$$

7.2.2 Problems in Two Spatial Variables

In the following, we present a series of examples concerning problems for the diffusion equation in two spatial variables. A combination of the Laplace transform and the eigenfunction expansion methods turns out to be efficient for the construction of their Green's functions. The example below deals with probably one of the simplest problems of its kind.

Example 7.5. Let $u = u(x, y, t)$ be a solution of the diffusion equation

$$\frac{\partial u}{\partial t} = \varkappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \beta u, \quad (x, y) \in \Omega, \quad t > 0, \tag{7.48}$$

with a convectional term βu. Let Ω be the infinite strip $\{-\infty < x < \infty, 0 < y < b\}$, and assume the initial condition to be

$$u(x, y, 0) = f(x, y) \tag{7.49}$$

whilst boundary conditions are as

$$u(x, 0, t) = u(x, b, t) = 0, \qquad \lim_{x \to \pm\infty} |u(x, y, t)| < \infty. \tag{7.50}$$

Our objective is to obtain the Green's function $G(x, y, t; \xi, \eta)$ of the homogeneous initial-boundary-value problem corresponding to that in (7.48)–(7.50). In pursuing this objective, we recall the definition, introduced earlier in Section 7.2, which covers our approach to search for the solution to the above problem in integral form

$$u(x, y, t) = \iint_{\Omega} G(x, y, t; \xi, \eta) f(\xi, \eta) d\Omega(\xi, \eta) \tag{7.51}$$

providing an explicit expression of the sought-after Green's function.

Taking the Laplace transform of $u(x, y, t)$ with respect to the time variable t

$$U(x, y, p) = \int_0^{\infty} \mathrm{e}^{-pt} u(x, y, t) dt$$

we obtain a self-adjoint boundary-value problem for the following static Klein–Gordon equation

$$\frac{\partial^2 U(x, y, p)}{\partial x^2} + \frac{\partial^2 U(x, y, p)}{\partial y^2} - \frac{p + \beta}{\varkappa} U(x, y, p) = -\frac{1}{\varkappa} f(x, y)$$

subject to the boundary conditions

$$U(x, 0, p) = U(x, b, p) = 0, \qquad \lim_{x \to \pm\infty} |U(x, y, p)| < \infty.$$

It is evident that the set

$$\sin \frac{n\pi y}{b}, \quad n = 1, 2, 3, \ldots,$$

represents eigenfunctions of the above boundary-value problem. Hence, to solve it in accordance with the method of eigenfunction expansion, we expand $U(x, y, t)$ and the right-hand side function $f(x, y)$ in the following Fourier series with respect to y

$$U(x, y, p) = \sum_{n=1}^{\infty} U_n(x, p) \sin \nu y, \quad f(x, y) = \sum_{n=1}^{\infty} f_n(x) \sin \nu y, \tag{7.52}$$

where $\nu = n\pi/b$.

This allows us to impose the boundary conditions on the segments $y = 0$ and $y = b$ of Ω, yielding the boundary-value problem

$$\frac{d^2U_n(x,p)}{dx^2} - \omega^2 U_n(x,p) = -\frac{1}{\varkappa} f_n(x) \tag{7.53}$$

$$\lim_{x\to\pm\infty} |U_n(x,p)| < \infty \tag{7.54}$$

for the coefficient $U_n(x,p)$ of the first series of (7.52). As short-hand notation, we have introduced the parameter ω as

$$\omega = \sqrt{\frac{\varkappa\nu^2 + p + \beta}{\varkappa}}. \tag{7.55}$$

The problem in (7.53) and (7.54) has already been considered earlier in this section (see equations (7.16) and (7.17)), and its Green's function was displayed in (7.20). In our current notation, it reads

$$G_n(x,p;\xi) = \frac{1}{2\varkappa\omega} \mathrm{e}^{-\omega|x-\xi|}$$

leading to the solution of the problem in (7.53) and (7.54)

$$U_n(x,p) = \int_{-\infty}^{\infty} \frac{1}{2\varkappa\omega} \mathrm{e}^{-\omega|x-\xi|} f_n(\xi)d\xi.$$

Using the Fourier–Euler formula for the coefficients $f_n(x)$ of the second series in (7.52), and subsequently substituting $U_n(x,p)$ into the first series in (7.52), the inverse Laplace transform $U(x,y;p)$ of $u(x,y,t)$ is written as

$$U(x,y,p) = \int_0^b \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\omega|x-\xi|}}{b\varkappa\omega} \sin\nu y \sin\nu\eta f(\xi,\eta)d\xi d\eta$$

so that, after recalling the expression for ω from (7.55), we can write $u(x,y,t)$ itself as

$$\begin{aligned} u(x,y,t) &= \int_0^b \int_{-\infty}^{\infty} \frac{1}{b\sqrt{\varkappa}} \sum_{n=1}^{\infty} \mathbf{L}^{-1} \left\{ \frac{\mathrm{e}^{-\frac{|x-\xi|}{\sqrt{\varkappa}}\sqrt{p+\beta+\varkappa\nu^2}}}{\sqrt{p+\beta+\varkappa\nu^2}} \right\} \\ &\qquad \times \sin\nu y \sin\nu\eta f(\xi,\eta)d\xi d\eta \\ &= \int_0^b \int_{-\infty}^{\infty} \frac{1}{2b\sqrt{\varkappa}} \sum_{n=1}^{\infty} \mathbf{L}^{-1} \left\{ \frac{\mathrm{e}^{-\frac{|x-\xi|}{\sqrt{\varkappa}}\sqrt{p+\beta+\varkappa\nu^2}}}{\sqrt{p+\beta+\varkappa\nu^2}} \right\} \\ &\qquad \times (\cos\nu(y-\eta) - \cos\nu(y+\eta))\, f(\xi,\eta)d\xi d\eta. \end{aligned}$$

To find the inverse Laplace transform that is contained in the above integral, we make use of relation (7.6), along with the translation theorem yielding

$$u(x,y,t) = \int_0^b \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-(\frac{(x-\xi)^2}{4\varkappa t}+\beta t)}}{2b\sqrt{\varkappa \pi t}} \sum_{n=1}^{\infty} \mathrm{e}^{-\nu^2 \varkappa t}$$
$$\times (\cos \nu(y-\eta) - \cos \nu(y+\eta))\, f(\xi,\eta) d\xi d\eta$$

which reveals the following formula for the Green's function

$$G(x,y,t;\xi,\eta) = \frac{\mathrm{e}^{-(\frac{(x-\xi)^2}{4\varkappa t}+\beta t)}}{2b\sqrt{\varkappa \pi t}}$$
$$\times \sum_{n=1}^{\infty} \mathrm{e}^{-\nu^2 \varkappa t} (\cos \nu(y-\eta) - \cos \nu(y+\eta))$$

for the homogeneous initial-boundary-value problem corresponding to (7.48)–(7.50). We can also write the above in terms of the Jacobi Theta function ϑ_3 as

$$G(x,y,t;\xi,\eta) = \frac{\mathrm{e}^{-(\frac{(x-\xi)^2}{4\varkappa t}+\beta t)}}{4b\sqrt{\varkappa \pi t}} \left[\vartheta_3\left(\frac{y-\eta}{2b}, \frac{\varkappa t}{b^2}\right) - \vartheta_3\left(\frac{y+\eta}{2b}, \frac{\varkappa t}{b^2}\right) \right]. \quad (7.56)$$

Example 7.6. Consider the initial-boundary-value problem

$$\frac{\partial u}{\partial t} = \varkappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \beta u, \quad (x,y) \in \Omega, \quad t > 0, \quad (7.57)$$

$$u(x,y,0) = f(x,y), \quad (7.58)$$

$$u(x,0,t) = \frac{\partial}{\partial y} u(x,b,t) = 0, \quad \lim_{x \to \pm\infty} |u(x,y,t)| < \infty \quad (7.59)$$

on the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$, and, following our familiar approach, construct the Green's function for the corresponding homogeneous problem.

The problem in (7.57)–(7.59) gives rise to the following boundary-value problem

$$\nabla^2 U(x,y,p) - \frac{p+\beta}{\varkappa} U(x,y,p) = -\frac{1}{\varkappa} f(x,y)$$
$$U(x,0,p) = \frac{\partial}{\partial y} U(x,b,p) = 0, \quad \lim_{x \to \pm\infty} |U(x,y,p)| < \infty$$

for the Laplace transform $U(x,y,p)$ of $u(x,y,t)$, in the static Klein–Gordon equation.

It is evident that the set

$$\sin\frac{n\pi y}{2b}, \quad n = 1, 3, 5, \ldots,$$

represents the eigenfunctions to the above problem, which is why, in the following the eigenfunction expansion method, we expand $U(x, y, p)$ and $f(x, y)$ in the series

$$U(x, y, p) = \sum_{n=1,3,5,\ldots}^{\infty} U_n(x, p)\sin\nu y,\; f(x, y) = \sum_{n=1,3,5,\ldots}^{\infty} f_n(x)\sin\nu y,$$

where $\nu = n\pi/(2b)$. This yields the boundary-value problem in (7.53) and (7.54) for the coefficients $U_n(x, p)$ of the first of the above expansions.

Proceeding with our derivation procedure (the remaining phase of which is exactly the same as in Example 7.5), we finally obtain the Green's function for the homogeneous initial-boundary-value problem corresponding to (7.56)–(7.58) in the form

$$G(x, y, t; \xi, \eta) = \frac{\mathrm{e}^{-(\frac{(x-\xi)^2}{4\varkappa t}+\beta t)}}{2b\sqrt{\varkappa\pi t}} \times \sum_{n=1,3,5,\ldots} \mathrm{e}^{-\frac{n^2\pi^2}{4b^2}\varkappa t}\left(\cos\frac{n\pi(y-\eta)}{2b} - \cos\frac{n\pi(y+\eta)}{2b}\right).$$

The above formula can be expressed in terms of the Jacobi Theta function ϑ_3. However, some preparatory work is required: performing the following transformation

$$\begin{aligned} G(x, y, t; \xi, \eta) = {} & \frac{\mathrm{e}^{-(\frac{(x-\xi)^2}{4\varkappa t}+\beta t)}}{2b\sqrt{\varkappa\pi t}} \\ & \times \left\{ \sum_{n=1,3,5,\ldots} \mathrm{e}^{-\frac{n^2\pi^2}{4b^2}\varkappa t}\left(\cos\frac{n\pi(y-\eta)}{2b} - \cos\frac{n\pi(y+\eta)}{2b}\right)\right. \\ & + \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{(2n)^2\pi^2}{4b^2}\varkappa t}\left(\cos\frac{2n\pi(y-\eta)}{2b} - \cos\frac{2n\pi(y+\eta)}{2b}\right) \\ & \left. - \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{(2n)^2\pi^2}{4b^2}\varkappa t}\left(\cos\frac{2n\pi(y-\eta)}{2b} - \cos\frac{2n\pi(y+\eta)}{2b}\right)\right\} \end{aligned} \tag{7.60}$$

we realize that the sum of the first and the second series in the above represents a single series

$$\sum_{n=1}^{\infty} \mathrm{e}^{-\frac{n^2\pi^2}{4b^2}\varkappa t}\left(\cos\frac{n\pi(y-\eta)}{2b} - \cos\frac{n\pi(y+\eta)}{2b}\right)$$

whilst the third series simplifies to

$$\sum_{n=1}^{\infty} \mathrm{e}^{-\frac{n^2\pi^2}{b^2}\varkappa t} \left(\cos \frac{n\pi(y-\eta)}{b} - \cos \frac{n\pi(y+\eta)}{b} \right)$$

converting (7.60) into

$$\begin{aligned} G(x,y,t;\xi,\eta) = {} & \frac{\mathrm{e}^{-(\frac{(x-\xi)^2}{4\varkappa t}+\beta t)}}{2b\sqrt{\varkappa \pi t}} \\ & \times \left\{ \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{n^2\pi^2}{4b^2}\varkappa t} \left(\cos \frac{n\pi(y-\eta)}{2b} - \cos \frac{n\pi(y+\eta)}{2b} \right) \right. \\ & \quad \left. - \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{n^2\pi^2}{b^2}\varkappa t} \left(\cos \frac{n\pi(y-\eta)}{b} - \cos \frac{n\pi(y+\eta)}{b} \right) \right\} \end{aligned}$$

which ultimately reads, in terms of the Jacobi Theta function, as

$$\begin{aligned} G(x,y,t;\xi,\eta) = {} & \frac{\mathrm{e}^{-(\frac{(x-\xi)^2}{4\varkappa t}+\beta t)}}{4b\sqrt{\varkappa \pi t}} \left[\vartheta_3\left(\frac{y-\eta}{4b}, \frac{\varkappa t}{4b^2} \right) - \vartheta_3\left(\frac{y+\eta}{4b}, \frac{\varkappa t}{4b^2} \right) \right. \\ & \left. - \vartheta_3\left(\frac{y-\eta}{2b}, \frac{\varkappa t}{b^2} \right) + \vartheta_3\left(\frac{y+\eta}{2b}, \frac{\varkappa t}{b^2} \right) \right]. \end{aligned} \tag{7.61}$$

Example 7.7. In order to construct another Green's function for the diffusion equation, consider the mixed initial-boundary-value problem

$$\frac{\partial u}{\partial t} = \varkappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \beta u, \quad (x,y) \in \Omega, \quad t > 0, \tag{7.62}$$

$$u(x,y,0) = f(x,y), \tag{7.63}$$

$$u(x,0,t) = u(x,b,t) = 0, \tag{7.64}$$

$$\frac{\partial u(0,y,t)}{\partial x} - \gamma u(0,y,t) = 0, \quad \gamma \geq 0, \quad \lim_{x\to\infty} |u(x,y,t)| < \infty \tag{7.65}$$

on the semi-infinite strip $\Omega = \{0 < x < \infty, 0 < y < b\}$.

In accordance with the procedure based on the Laplace transform, the problem in (7.62)–(7.65) gives rise to the following boundary-value problem

$$\nabla^2 U(x,y,p) - \frac{p+\beta}{\varkappa} U(x,y,p) = -\frac{1}{\varkappa} f(x,y), \tag{7.66}$$

$$U(x,0,p) = U(x,b,p) = 0, \tag{7.67}$$

$$\frac{\partial U(0,y,p)}{\partial x} - \gamma U(0,y,p) = 0, \quad \lim_{x\to\infty} |U(x,y,p)| < \infty \tag{7.68}$$

for the transform $U(x,y,p)$ of $u(x,y,t)$.

To solve (7.66)–(7.68), following the method of eigenfunction expansion, we expand $U(x, y, t)$ and the right-hand side function $f(x, y)$ of (7.66) in a Fourier sine-series, with respect to y

$$U(x, y, p) = \sum_{n=1}^{\infty} U_n(x, p) \sin \nu y, \quad f(x, y) = \sum_{n=1}^{\infty} f_n(x) \sin \nu y, \tag{7.69}$$

where $\nu = n\pi/b$.

This yields the boundary-value problem

$$\frac{d^2 U_n(x, p)}{dx^2} - \omega^2 U_n(x, p) = -\frac{1}{\varkappa} f_n(x), \quad \omega^2 = \frac{\varkappa \nu^2 + p + \beta}{\varkappa}, \tag{7.70}$$

$$\frac{dU_n(0, p)}{dx} - \gamma U_n(0, p) = 0, \quad \lim_{x\to\infty} |U_n(x, p)| < \infty \tag{7.71}$$

for the coefficient $U_n(x, p)$ of the first series in (7.69).

The general solution of the equation in (7.70) has been found earlier in this section. Using our current notation, it reads

$$U_n(x, p) = \int_0^x \frac{1}{2\varkappa\omega} [e^{\omega(\xi - x)} - e^{\omega(x-\xi)}] f_n(\xi) d\xi + D_1 e^{\omega x} + D_2 e^{-\omega x}. \tag{7.72}$$

In light of the boundary conditions in (7.71), the constants of integration in the above are found as

$$D_1 = \frac{1}{2\varkappa\omega} \int_0^\infty e^{-\omega\xi} f_n(\xi) d\xi$$

and

$$D_2 = \frac{\omega - \gamma}{2\varkappa\omega(\omega + \gamma)} \int_0^\infty e^{-\omega\xi} f_n(\xi) d\xi.$$

Upon substituting them in (7.72), we obtain

$$U_n(x, p) = \int_0^x \frac{1}{2\varkappa\omega} [e^{\omega(\xi - x)} - e^{\omega(x-\xi)}] f_n(\xi) d\xi + \int_0^\infty \frac{1}{2\varkappa\omega} \left[e^{\omega(x-\xi)} + \frac{\omega - \gamma}{\omega + \gamma} e^{-\omega(x+\xi)} \right] f_n(\xi) d\xi. \tag{7.73}$$

To adjust the above formula to the needs of our upcoming development, we transform the constant factor of the second exponential function in the second of the above integrals as

$$\frac{\omega - \gamma}{\omega + \gamma} = 1 - \frac{2\gamma}{\omega + \gamma}.$$

This simplifies (7.73) to the compact form

$$U_n(x,p) = \int_0^\infty \frac{1}{2\varkappa\omega}[\mathrm{e}^{-\omega|x-\xi|} + \mathrm{e}^{-\omega(x+\xi)} - \frac{2\gamma}{\omega+\gamma}\mathrm{e}^{-\omega(x+\xi)}] f_n(\xi)d\xi$$

and allows us to write down the solution to the problem in (7.66)–(7.68) as

$$U(x,y,p) = \int_0^b \int_0^\infty \sum_{n=1}^\infty \frac{1}{b\varkappa\omega}\left[\mathrm{e}^{-\omega|x-\xi|} + \mathrm{e}^{-\omega(x+\xi)} - \frac{2\gamma}{\omega+\gamma}\mathrm{e}^{-\omega(x+\xi)}\right] \sin\nu y \sin\nu\eta f(\xi,\eta)d\xi d\eta.$$

Hence, the solution to (7.62)–(7.65) must be the inverse Laplace transform of $U(x,y,p)$, appearing as the volume integral

$$u(x,y,t) = \int_0^b \int_0^\infty \frac{1}{b\varkappa}\sum_{n=1}^\infty \mathbf{L}^{-1}\left\{\frac{1}{\omega}\left[\mathrm{e}^{-\omega|x-\xi|} + \mathrm{e}^{-\omega(x+\xi)} - \frac{2\gamma}{\omega+\gamma}\mathrm{e}^{-\omega(x+\xi)}\right]\right\} \sin\nu y \sin\nu\eta f(\xi,\eta)d\xi d\eta.$$

The kernel-function of which

$$G(x,y,t;\xi,\eta) = \frac{1}{b\varkappa}\sum_{n=1}^\infty \mathbf{L}^{-1}\left\{\frac{1}{\omega}\left[\mathrm{e}^{-\omega|x-\xi|} + \mathrm{e}^{-\omega(x+\xi)} - \frac{2\gamma}{\omega+\gamma}\mathrm{e}^{-\omega(x+\xi)}\right]\right\} \sin\nu y \sin\nu\eta \tag{7.74}$$

represents the Green's function for the homogeneous problem corresponding to (7.62)–(7.65).

To find the inverse transform

$$\mathbf{L}^{-1}\left\{\frac{1}{\omega}\left[\mathrm{e}^{-\omega|x-\xi|} + \mathrm{e}^{-\omega(x+\xi)} - \frac{2\gamma}{\omega+\gamma}\mathrm{e}^{-\omega(x+\xi)}\right]\right\}$$

we recall the abbreviated notations introduced earlier for the parameters ω and ν, which reduce the above to

$$\mathbf{L}^{-1}\left\{\frac{\sqrt{\varkappa}\mathrm{e}^{-\frac{|x-\xi|}{\sqrt{\varkappa}}\sqrt{p+\beta+\varkappa\nu^2}}}{\sqrt{p+\beta+\varkappa\nu^2}} + \frac{\sqrt{\varkappa}\mathrm{e}^{-\frac{x+\xi}{\sqrt{\varkappa}}\sqrt{p+\beta+\varkappa\nu^2}}}{\sqrt{p+\beta+\varkappa\nu^2}} - \frac{2\gamma\varkappa\mathrm{e}^{-\frac{x+\xi}{\sqrt{\varkappa}}\sqrt{p+\beta+\varkappa\nu^2}}}{\sqrt{p+\beta+\varkappa\nu^2}\left(\gamma\sqrt{\varkappa}+\sqrt{p+\beta+\varkappa\nu^2}\right)}\right\}.$$

The inverse Laplace transform of the first two additive components in the above equation can be obtained with the aid of the relation in (7.6) and the translation theorem, whereas for the last component we employ the relation in (7.8) and the translation theorem. Recalling now the series expansion (7.52) of the Jacobi Theta function ϑ_3, the sought-after Green's function, the series expansion of which was just exhibited in (7.74), ultimately converts to

$$G(x, y, t; \xi, \eta) = \frac{e^{-\beta t}}{4b\sqrt{\varkappa \pi t}} \left[e^{-\frac{(x-\xi)^2}{4\varkappa t}} + e^{-\frac{(x+\xi)^2}{4\varkappa t}} \right.$$
$$\left. -2\gamma\sqrt{\varkappa\pi t}\, e^{\gamma(x+\xi)} e^{\gamma^2 \varkappa t} \operatorname{erfc}\left(\gamma\sqrt{\varkappa t} + \frac{x+\xi}{2\sqrt{\varkappa t}}\right)\right]$$
$$\times \left[\vartheta_3\left(\frac{y-\eta}{2b}, \frac{\varkappa t}{b^2}\right) - \vartheta_3\left(\frac{y+\eta}{2b}, \frac{\varkappa t}{b^2}\right)\right] \tag{7.75}$$

Returning to the problem in (7.62)–(7.65), now note that, if we assume the parameter γ to be zero, the mixed boundary condition in (7.65) on the boundary segment $x = 0$ of Ω reduces to the Neumann type. Clearly, the Green's function for such a setting

$$G(x, y, t; \xi, \eta) = \frac{e^{-\beta t}}{4b\sqrt{\varkappa \pi t}} [e^{-\frac{(x-\xi)^2}{4\varkappa t}} + e^{-\frac{(x+\xi)^2}{4\varkappa t}}]$$
$$\times \left[\vartheta_3\left(\frac{y-\eta}{2b}, \frac{\varkappa t}{b^2}\right) - \vartheta_3\left(\frac{y+\eta}{2b}, \frac{\varkappa t}{b^2}\right)\right] \tag{7.76}$$

follows from that in (7.75), if γ is set equal to zero. In one of the chapter exercises, the reader is challenged to derive the above Green's function directly.

Example 7.8. Consider the mixed initial-boundary-value problem

$$\frac{\partial u}{\partial t} = \varkappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \beta u, \quad (x, y) \in \Omega, \quad t > 0, \tag{7.77}$$

$$u(x, y, 0) = f(x, y), \tag{7.78}$$

$$\frac{\partial u(x, 0, t)}{\partial y} = \frac{\partial u(x, b, t)}{\partial y} = 0, \tag{7.79}$$

$$u(0, y, t) = 0, \quad \lim_{x\to\infty} |u(x, y, t)| < \infty \tag{7.80}$$

on the semi-infinite strip $\Omega = \{0 < x < \infty, 0 < y < b\}$.

The problem in (7.77)–(7.80) gives rise to the following boundary-value problem

$$\nabla^2 U(x,y,p) - \frac{p+\beta}{\varkappa} U(x,y,p) = -\frac{1}{\varkappa} f(x,y), \tag{7.81}$$

$$\frac{\partial U(x,0,p)}{\partial y} = \frac{\partial U(x,b,p)}{\partial y} = 0, \tag{7.82}$$

$$U(0,y,p) = 0, \quad \lim_{x\to\infty} |U(x,y,p)| < \infty \tag{7.83}$$

for the Laplace transform $U(x,y,p)$ of $u(x,y,t)$.

In contrast to the problem in the previous example (equations (7.66)–(7.68)), the set

$$\cos\frac{n\pi y}{b}, \quad n = 0,1,2,\ldots.$$

provides eigenfunctions to the problem in (7.81)–(7.83). Hence, proceeding with the method of eigenfunction expansion, we expand $U(x,y,t)$ and $f(x,y)$ in (7.81) into a Fourier cosine-series with respect to y

$$U(x,y,p) = \sum_{n=0}^{\infty} U_n(x,p)\cos\nu y, \quad f(x,y) = \sum_{n=0}^{\infty} f_n(x)\cos\nu y \tag{7.84}$$

with $\nu = n\pi/b$.

This yields the boundary-value problem

$$\frac{d^2U_n(x,p)}{dx^2} - \omega^2 U_n(x,p) = -\frac{1}{\varkappa} f_n(x), \quad \omega^2 = \frac{\varkappa\nu^2 + p + \beta}{\varkappa},$$

$$U_n(0,p) = 0, \quad \lim_{x\to\infty} |U_n(x,p)| < \infty$$

for $U_n(x,p)$ in the first series of (7.84).

Following the method of variation of parameters, we find the solution of the above problem in the form

$$U_n(x,p) = \int_0^\infty \frac{1}{2\varkappa\omega}[\mathrm{e}^{-\omega|x-\xi|} - \mathrm{e}^{-\omega(x+\xi)}] f_n(\xi)d\xi. \tag{7.85}$$

Notice that the Fourier–Euler formula

$$f_n(\xi) = \frac{\varepsilon_n}{b}\int_0^b f(\xi,\eta)\cos\nu\eta d\eta, \quad \varepsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n > 0, \end{cases}$$

reduces (7.85) to the integral form

$$U_n(x,p) = \int_0^b \int_0^\infty \frac{\varepsilon_n}{2b\varkappa\omega}[\mathrm{e}^{-\omega|x-\xi|} - \mathrm{e}^{-\omega(x+\xi)}]\cos\nu\eta f(\xi,\eta)d\xi d\eta,$$

which allows us to write down the solution of equations (7.81)–(7.83) as

$$U(x, y, p) = \int_0^b \int_0^\infty \frac{1}{2b\varkappa} \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\omega} [\mathrm{e}^{-\omega|x-\xi|} - \mathrm{e}^{-\omega(x+\xi)}] \times \cos \nu y \cos \nu\eta f(\xi, \eta) d\xi d\eta.$$

Taking the inverse Laplace transform of the above, the solution to the problem in (7.77)–(7.80) appears as the volume integral

$$u(x, y, t) = \int_0^b \int_0^\infty \frac{1}{4b\varkappa} \sum_{n=0}^{\infty} \mathbf{L}^{-1} \left\{ \frac{\varepsilon_n}{\omega} [\mathrm{e}^{-\omega|x-\xi|} - \mathrm{e}^{-\omega(x+\xi)}] \right\} \times (\cos \nu(y-\eta) + \cos \nu(y+\eta) f(\xi, \eta) d\xi d\eta,$$

the kernel-function of which

$$G(x, y, t; \xi, \eta) = \frac{1}{4b\varkappa} \sum_{n=0}^{\infty} \mathbf{L}^{-1} \left\{ \frac{\varepsilon_n}{\omega} [\mathrm{e}^{-\omega|x-\xi|} - \mathrm{e}^{-\omega(x+\xi)}] \right\} \times (\cos \nu(y-\eta) + \cos \nu(y+\eta)), \tag{7.86}$$

therefore represents the Green's function for the homogeneous problem corresponding to (7.77)–(7.80).

It is important to note that the parameter ω has a slightly different form for the zero term ($n = 0$) of the series in (7.86) compared with the rest of its terms. Taking this into account, we ultimately arrive at the following representation

$$G(x, y, t; \xi, \eta) = \frac{\mathrm{e}^{-\beta t}}{4b\sqrt{\varkappa\pi t}} \left[\mathrm{e}^{-\frac{(x-\xi)^2}{4\varkappa t}} - \mathrm{e}^{-\frac{(x+\xi)^2}{4\varkappa t}} \right] \times \left[\vartheta_3 \left(\frac{y-\eta}{2b}, \frac{\varkappa t}{b^2} \right) + \vartheta_3 \left(\frac{y+\eta}{2b}, \frac{\varkappa t}{b^2} \right) \right] \tag{7.87}$$

for the sought-after Green's function, expressed in terms of the Jacobi Theta function ϑ_3.

Example 7.9. Consider the Dirichlet problem

$$\frac{\partial u}{\partial t} = \varkappa \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \beta u, \quad (x, y) \in \Omega, \quad t > 0, \tag{7.88}$$

$$u(x, y, 0) = f(x, y), \tag{7.89}$$

$$u(x, 0, t) = u(x, b, t) = u(0, y, t) = u(a, y, t) = 0 \tag{7.90}$$

on the rectangle $\Omega = \{0 < x < a, 0 < y < b\}$.

The problem described above gives rise to the following boundary-value problem

$$\nabla^2 U(x,y,p) - \frac{p+\beta}{\varkappa} U(x,y,p) = -\frac{1}{\varkappa} f(x,y), \tag{7.91}$$

$$U(x,0,p) = U(x,b,p) = U(0,y,p) = U(a,y,p) = 0 \tag{7.92}$$

for the static Klein–Gordon equation, for the Laplace transform $U(x,y,p)$ of $u(x,y,t)$.

Two scenarios for exploiting the method of eigenfunction expansion are possible in order to treat the problem appearing in (7.91) and (7.92). In accordance with the first of them, we expand $U(x,y,t)$ and $f(x,y)$ of (7.91) in the Fourier double-series

$$U(x,y,p) = \sum_{m,n=1}^{\infty} U_{mn} \sin\mu x \sin\nu y,\ f(x,y) = \sum_{m,n=1}^{\infty} f_{mn} \sin\mu x \sin\nu y \tag{7.93}$$

with $\mu = m\pi/a$ and $\nu = n\pi/b$. Substituting these into (7.91), we arrive at

$$(\mu^2+\nu^2)U_{mn} + \frac{p+\beta}{\varkappa} U_{mn} = \frac{1}{\varkappa} f_{mn}.$$

Solving this equation for U_{mn}, whilst the Fourier coefficients f_{mn} are expressed in terms of $f(x,y)$ as

$$f_{mn} = \frac{4}{ab}\int_0^b \int_0^a f(\xi,\eta)\sin\mu\xi \sin\nu\eta d\xi d\eta$$

we get

$$U_{mn} = \frac{4}{ab}\int_0^b \int_0^a f(\xi,\eta)\frac{\sin\mu\xi \sin\nu\eta}{\varkappa(\mu^2+\nu^2)+p+\beta} d\xi d\eta.$$

Now substituting the above in (7.93), we obtain $U(x,y,p)$ in the form

$$U(x,y,p) = \frac{4}{ab}\int_0^b \int_0^a \sum_{m,n=1}^{\infty} \frac{\sin\mu x \sin\nu y \sin\mu\xi \sin\nu\eta}{\varkappa(\mu^2+\nu^2)+p+\beta} f(\xi,\eta) d\xi d\eta \tag{7.94}$$

the inverse Laplace transform of which, representing the solution $u(x,y,t)$ to equations (7.88)–(7.90), can be readily obtained with the aid of the trivial transform

$$\mathbf{L}^{-1}\left\{\frac{1}{p-\sigma}\right\} = \mathrm{e}^{\sigma t}$$

converting (7.94) into

$$u(x,y,t) = \frac{4\mathrm{e}^{-\beta t}}{ab}\int_0^b \int_0^a \sum_{m,n=1}^{\infty} \mathrm{e}^{-\varkappa(\mu^2+\nu^2)}$$
$$\times \sin\mu x \sin\nu y \sin\mu\xi \sin\nu\eta f(\xi,\eta) d\xi d\eta.$$

This formula reveals the sought-after Green's function

$$G(x,y,t;\xi,\eta)=\frac{4\mathrm{e}^{-\beta t}}{ab}\sum_{m,n=1}^{\infty}\mathrm{e}^{-\varkappa(\mu^2+\nu^2)t}\sin\mu x\sin\nu y\sin\mu\xi\sin\nu\eta. \qquad (7.95)$$

Proceeding with the second eigenfunction expansion scenario for the boundary-value problem appearing in (7.91) and (7.92), we expand its solution $U(x,y,t)$ as well as the right-hand side function $f(x,y)$ of (7.91) in the Fourier sine-series

$$U(x,y,p)=\sum_{n=1}^{\infty}U_n(x,p)\sin\nu y,\quad f(x,y)=\sum_{n=1}^{\infty}f_n(x)\sin\nu y, \qquad (7.96)$$

where $\nu=n\pi/b$.

This yields the boundary-value problem

$$\frac{d^2U_n(x,p)}{dx^2}-\omega^2U_n(x,p)=-\frac{1}{\varkappa}f_n(x),\quad \omega^2=\frac{\varkappa\nu^2+p+\beta}{\varkappa},$$
$$U_n(0,p)=0,\quad U_n(a,p)=0$$

for the coefficients $U_n(x,p)$ of the first series in (7.96).

Following the method of variation of parameters, we find the solution of the above problem in the form

$$U_n(x,p)=\frac{1}{2\varkappa\omega}\int_0^a\frac{1}{\mathrm{e}^{\omega a}-\mathrm{e}^{-\omega a}}$$
$$\times\left(\mathrm{e}^{-\omega(|x-\xi|-a)}+\mathrm{e}^{\omega(|x-\xi|-a)}-\mathrm{e}^{\omega(x+\xi-a)}-\mathrm{e}^{\omega(a-x-\xi)}\right)f_n(\xi)d\xi.$$

To ease the necessary algebra that follows, we multiply the numerator and the denominator of the kernel in the above integrand by $\mathrm{e}^{-\omega a}$, which yields

$$U_n(x,p)=\frac{1}{2\varkappa\omega}\int_0^a\frac{1}{1-\mathrm{e}^{-2\omega a}}$$
$$\times\left(\mathrm{e}^{-\omega|x-\xi|}+\mathrm{e}^{\omega(|x-\xi|-2a)}-\mathrm{e}^{\omega(x+\xi-2a)}-\mathrm{e}^{-\omega(x+\xi)}\right)f_n(\xi)d\xi. \qquad (7.97)$$

In light of the evident relation

$$\sum_{k=0}^{\infty}\mathrm{e}^{-2\omega ka}=\frac{1}{1-\mathrm{e}^{-2\omega a}}$$

equation (7.97) transforms into

$$U_n(x,p) = \frac{1}{2\varkappa\omega}\int_0^a (\mathrm{e}^{-\omega|x-\xi|} + \mathrm{e}^{\omega(|x-\xi|-2a)} - \mathrm{e}^{\omega(x+\xi-2a)} - \mathrm{e}^{-\omega(x+\xi)}) \sum_{k=0}^{\infty} \mathrm{e}^{-2\omega ka} f_n(\xi)d\xi$$

$$= \frac{1}{2\varkappa\omega}\int_0^a \sum_{k=0}^{\infty} (\mathrm{e}^{-\omega(|x-\xi|+2ka)} + \mathrm{e}^{\omega(|x-\xi|-2a(k+1))} - \mathrm{e}^{\omega(x+\xi-2a(k+1))} - \mathrm{e}^{-\omega(x+\xi+2ka)}) f_n(\xi)d\xi.$$

Further following our procedure, we find the solution to the boundary-value problem in (7.91) and (7.92) as

$$U(x,y,p) = \int_0^b \int_0^a \frac{1}{b\varkappa}\sum_{n=1}^{\infty}\left[\frac{1}{\omega}\sum_{k=0}^{\infty}(\mathrm{e}^{-\omega(|x-\xi|+2ka)} + \mathrm{e}^{\omega(|x-\xi|-2a(k+1))} - \mathrm{e}^{\omega(x+\xi-2a(k+1))} - \mathrm{e}^{-\omega(x+\xi+2ka)})\right] \sin\nu y \sin\nu\eta f(\xi,\eta)d\xi d\eta.$$

Now, taking the inverse Laplace transform of the above, the solution to the (7.88)–(7.90) appears as the volume integral

$$u(x,y,t) = \int_0^b \int_0^a \frac{\mathrm{e}^{-\beta t}}{b\sqrt{\varkappa\pi t}} \sum_{k=0}^{\infty}(\mathrm{e}^{-\frac{(|x-\xi|+2ka)^2}{4\varkappa t}} + \mathrm{e}^{-\frac{(|x-\xi|-2a(k+1))^2}{4\varkappa t}} - \mathrm{e}^{-\frac{(x+\xi-2a(k+1))^2}{4\varkappa t}} - \mathrm{e}^{-\frac{(x+\xi+2ka)^2}{4\varkappa t}}) \times \sum_{n=1}^{\infty} \mathrm{e}^{-\varkappa\nu^2 t} \sin\nu y \sin\nu\eta f(\xi,\eta)d\xi d\eta$$

the kernel function of which

$$G(x,y,t;\xi,\eta) = \frac{\mathrm{e}^{-\beta t}}{b\sqrt{\varkappa\pi t}} \sum_{k=0}^{\infty}(\mathrm{e}^{-\frac{(|x-\xi|+2ka)^2}{4\varkappa t}} + \mathrm{e}^{-\frac{(|x-\xi|-2a(k+1))^2}{4\varkappa t}} - \mathrm{e}^{-\frac{(x+\xi-2a(k+1))^2}{4\varkappa t}} - \mathrm{e}^{-\frac{(x+\xi+2ka)^2}{4\varkappa t}}) \sum_{n=1}^{\infty} \mathrm{e}^{-\varkappa\nu^2 t} \sin\nu y \sin\nu\eta$$

represents the Green's function for the homogeneous problem corresponding to (7.88)–(7.90).

The second series in the above formula for the Green's function can customarily be converted to Jacobi Theta function ϑ_3 form. Doing so, we finally arrive at

$$\begin{aligned} G(x,y,t;\xi,\eta) = \frac{\mathrm{e}^{-\beta t}}{b\sqrt{\varkappa\pi t}} \sum_{k=0}^{\infty} (&\mathrm{e}^{-\frac{(|x-\xi|+2ka)^2}{4\varkappa t}} + \mathrm{e}^{-\frac{(|x-\xi|-2a(k+1))^2}{4\varkappa t}} \\ &- \mathrm{e}^{-\frac{(x+\xi-2a(k+1))^2}{4\varkappa t}} - \mathrm{e}^{-\frac{(x+\xi+2ka)^2}{4\varkappa t}}) \\ &\times \left[\vartheta_3\left(\frac{y-\eta}{2b}, \frac{\varkappa t}{b^2}\right) - \vartheta_3\left(\frac{y+\eta}{2b}, \frac{\varkappa t}{b^2}\right) \right] \end{aligned} \tag{7.98}$$

Note that of the two alternative expressions of the Green's function on the rectangle derived in the current example (see (7.95) and (7.98)), the first one is computationally preferable for *large* values of t, whereas the second works better for *small* values of t.

In the last example in this section, we will highlight technical details in our procedure for the construction of the Green's function for a simple problem stated in polar coordinates.

Example 7.10. Consider the initial boundary-value problem

$$\frac{\partial u}{\partial t} = \varkappa\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}\right), \quad (r,\varphi) \in \Omega, \quad t > 0, \tag{7.99}$$

$$u(r,\varphi,0) = f(r,\varphi) \tag{7.100}$$

$$u(r,0,t) = u(r,\pi,t) = 0 \tag{7.101}$$

on the upper half-plane $\Omega = \{0 < r < \infty, 0 < \varphi < \pi\}$.

For the above to be a well-posed problem, its solution is assumed bounded for r going to zero and infinity.

Upon applying the Laplace transform

$$\mathbf{L}\{u(r,\varphi,t)\} = U(r,\varphi,p)$$

the problem in (7.99)–(7.101) transforms into a boundary-value problem for the static Klein–Gordon equation written in polar coordinates, that is,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} - \frac{p}{\varkappa}\right) U(r,\varphi,p) = -\frac{1}{\varkappa} f(r,\varphi), (r,\varphi) \in \Omega. \tag{7.102}$$

The above equation is subject to the boundary conditions

$$U(r,0,p) = U(r,\pi,p) = 0 \tag{7.103}$$

$$\lim_{r\to 0} |U(r,\varphi,p)| < \infty, \quad \lim_{r\to\infty} |U(r,\varphi,p)| < \infty. \tag{7.104}$$

Using the expansions

$$U(r,\varphi,p)=\sum_{n=1}^{\infty}U_n(r,p)\sin n\varphi,\ f(r,\varphi)=\sum_{n=1}^{\infty}f_n(r)\sin n\varphi, \tag{7.105}$$

we obtain the following boundary-value problem

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\left(\frac{p}{\varkappa}+\frac{n^2}{r^2}\right)\right)U_n(r,p)=-\frac{1}{\varkappa}f_n(r), \tag{7.106}$$

$$\lim_{r\to 0}|U_n(r,p)|<\infty, \quad \lim_{r\to\infty}|U_n(r,p)|<\infty \tag{7.107}$$

in $U_n(r,p)$.

Notice that in Chapter 3 (see Example 3.21), we considered a problem of the type in (7.106) and (7.107), and the Green's function $G_n(r,p;\varrho)$ for the corresponding homogeneous problem was obtained. Using our current notation, it appears in the form

$$G_n(r,p;\varrho)=\begin{cases} I_n(\lambda r)K_n(\lambda\varrho), & r\le\varrho,\\ I_n(\lambda\varrho)K_n(\lambda r), & r\ge\varrho,\end{cases} \tag{7.108}$$

with $\lambda=\sqrt{p/\varkappa}$, whilst $I_n(\lambda r)$ and $K_n(\lambda r)$ represent the nth-order modified Bessel functions [73, 74] of the first and the second kind, respectively, which allows us to write down the solution $U(r,\varphi,p)$ of the problem in (7.102)–(7.104) as

$$\begin{aligned} U(r,\varphi,p)=\int_0^{\pi}\int_0^{\infty}\frac{2}{\varkappa\pi}\sum_{n=1}^{\infty}I_n(\lambda r)K_n(\lambda\varrho)\\ \times\sin n\varphi\sin n\psi f(\varrho,\psi)\varrho d\varrho d\psi, \quad r\le\varrho. \end{aligned} \tag{7.109}$$

Note that for $r\ge\varrho$, the product $I_n(\lambda r)K_n(\lambda\varrho)$ in (7.109) must, in accordance with (7.108), be replaced with $I_n(\lambda\varrho)K_n(\lambda r)$. Taking the inverse Laplace transform of $U(r,\varphi,p)$, we obtain the solution $u(r,\varphi,t)$ to equations (7.99)–(7.101) as

$$u(r,\varphi,t)=\int_0^{\pi}\int_0^{\infty}G(x,y,t;\xi,\eta)f(\varrho,\psi)\varrho d\varrho d\psi$$

with

$$\begin{aligned} G(x,y,t;\xi,\eta)&=\mathbf{L}^{-1}\left\{\frac{2}{\varkappa\pi}\sum_{n=1}^{\infty}I_n\left(\sqrt{\frac{p}{\varkappa}}r\right)K_n\left(\sqrt{\frac{p}{\varkappa}}\varrho\right)\sin n\varphi\sin n\psi\right\}\\ &=\mathbf{L}^{-1}\left\{\frac{1}{\varkappa\pi}\sum_{n=1}^{\infty}I_n\left(\sqrt{\frac{p}{\varkappa}}r\right)K_n\left(\sqrt{\frac{p}{\varkappa}}\varrho\right)[\cos n(\varphi-\psi)-\cos n(\varphi+\psi)]\right\} \end{aligned} \tag{7.110}$$

the Green's function to the homogeneous problem corresponding to (7.99)–(7.101).

Recall the so-called *summation theorem* for the modified Bessel functions (see, for example, [37, 74]):

$$K_0(\omega|(r\cos\varphi - \varrho\cos\psi) + i(r\sin\varphi - \varrho\sin\psi)|)$$
$$= I_0(\omega r)K_0(\omega\varrho) + 2\sum_{n=1}^{\infty} I_n(\omega r)K_n(\omega\varrho)\cos n(\varphi - \psi), \quad \omega > 0.$$

In view of the above, the series in (7.110) can be summed, transforming the expression for $G(x, y, t; \xi, \eta)$ into

$$G(x, y, t; \xi, \eta) = \frac{1}{2\varkappa\pi}\mathbf{L}^{-1}\left\{K_0\left(\sqrt{\frac{p}{\varkappa}}|z - \zeta|\right) - K_0\left(\sqrt{\frac{p}{\varkappa}}|z - \overline{\zeta}|\right)\right\},$$

where z and ζ are introduced, for compactness, as $z = r(\cos\varphi + i\sin\varphi)$ and $\zeta = \varrho(\cos\psi + i\sin\psi)$.

To perform this transformation, we recall the relation from (7.7) yielding

$$G(x, y, t; \xi, \eta) = \frac{1}{4\varkappa\pi t}[\mathrm{e}^{-\frac{|z-\zeta|^2}{4\varkappa t}} - \mathrm{e}^{-\frac{|z-\overline{\zeta}|^2}{4\varkappa t}}] \tag{7.111}$$

which is the well-known expression for the Green's function for the Dirichlet problem for the diffusion equation on the upper half-plane $y > 0$ [13, 57, 66]. In standard textbooks, this Green's function is usually derived with help of the method of images, starting from the fundamental solution

$$\frac{1}{4\pi\varkappa t}\exp\left(-\frac{|z - \zeta|^2}{4\varkappa t}\right)$$

of the diffusion equation in two dimensions.

7.3 Matrices of Green's Type

In this section, we will discuss the construction procedure for matrices of Green's type for diffusion (heat conduction) phenomena taking place in media whose conductive properties vary discontinuously as functions of the spatial coordinates, within the region to be considered.

Example 7.11. For the first illustrative example, we obtain the matrix of Green's type for the assembly of thin, semi-infinite rods, as depicted in Figure 7.1. Each rod is composed of a homogeneous conducting material the heat conductivity and thermal diffusivity of which are h_i and $\varkappa_i$, respectively, with an initial temperature $f_i(x)$. We assume ideal contact at the contact point $x = 0$, that is, both the temperature and the heat flux are assumed continuous for $x = 0$.

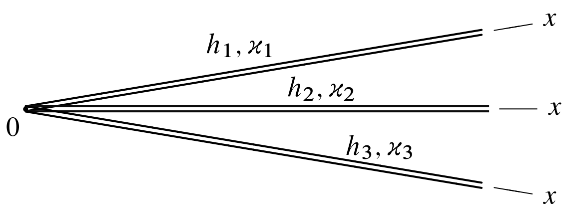

Figure 7.1. Assembly of semi-infinite rods.

We also assume that the heat flow within each rod moves only in the x direction, i.e. the lateral surfaces of the rods are insulated. The resulting problem is associated with the set ($i = 1, 2, 3$) of three heat equations

$$\frac{\partial u_i(x,t)}{\partial t} = \varkappa_i \frac{\partial^2 u_i(x,t)}{\partial x^2}, \quad x \in (0, \infty), \quad t > 0, \tag{7.112}$$

subject to initial conditions

$$u_i(x, 0) = f_i(x), \quad i = 1, 2, 3, \tag{7.113}$$

and put into the format of the system format by the conditions of ideal contact

$$u_1(0,t) = u_2(0,t) = u_3(0,t). \tag{7.114}$$

and

$$h_1 \frac{\partial u_1(0,t)}{\partial x} + h_2 \frac{\partial u_2(0,t)}{\partial x} + h_3 \frac{\partial u_3(0,t)}{\partial x} = 0. \tag{7.115}$$

In addition, we assume that the temperature $u_i(x,t)$ is finite at infinity for all times $t > 0$

$$\lim_{x \to \infty} |u_i(x,t)| < \infty, \quad i = 1, 2, 3. \tag{7.116}$$

To proceed with our construction procedure for the matrix of Green's type for the homogeneous problem corresponding to (7.112)–(7.116), we take the Laplace transform

$$\mathbf{L}\{u_i(x,t)\} = U_i(x.p), \quad i = 1, 2, 3,$$

of $u_i(x,t)$ with respect to the time variable t.

By virtue of the properties of linearity and differentiation of the original of the Laplace transform (see Section 7.1), the system (7.112)–(7.116) transforms into the following multi point posed boundary-value problem in $U_i(x, p)$

$$\frac{d^2U_i(x,p)}{dx^2} - \frac{p}{\varkappa_i}U_i(x,p) = -\frac{1}{\varkappa_i}f_i(x), \quad i = 1,2,3, \tag{7.117}$$

$$U_1(0,p) = U_2(0,p) = U_3(0,p), \tag{7.118}$$

$$h_1\frac{dU_1(0,p)}{dx} + h_2\frac{dU_2(0,p)}{dx} + h_3\frac{dU_3(0,p)}{dx} = 0, \tag{7.119}$$

$$\lim_{x\to\infty}|U_i(x,p)| < \infty \quad i = 1,2,3. \tag{7.120}$$

Using the method of variation of parameters, the general solution of equation (7.117) is found (see Example 2.1 in Section 2) to be

$$U_i(x,p) = \int_0^x \frac{1}{2\sqrt{\varkappa_i p}}[\mathrm{e}^{\frac{\xi-x}{\sqrt{\varkappa_i}}\sqrt{p}} - \mathrm{e}^{\frac{x-\xi}{\sqrt{\varkappa_i}}\sqrt{p}}]f_i(\xi)d\xi$$
$$+ M_i\mathrm{e}^{\sqrt{\frac{p}{\varkappa_i}}x} + N_i\mathrm{e}^{-\sqrt{\frac{p}{\varkappa_i}}x}, \quad i = 1,2,3, \tag{7.121}$$

where the constants of integration M_i and N_i can be obtained from the boundary and contact conditions stated in (7.117)–(7.120). That is, the boundedness conditions in (7.120) yield

$$M_i = \frac{1}{2\sqrt{\varkappa_i p}}\int_0^\infty \mathrm{e}^{-\sqrt{\frac{p}{\varkappa_i}}\xi}f_i(\xi)d\xi, \quad i = 1,2,3.$$

Based on these expressions for M_i, the values of N_i can be found from the contact conditions. That is, the continuity conditions formulated at $x = 0$ in (7.118) lead to

$$N_1 - N_2 = \frac{1}{2\sqrt{p}}\int_0^\infty \left[\frac{\mathrm{e}^{-\sqrt{\frac{p}{\varkappa_2}}\xi}}{\sqrt{\varkappa_2}}f_2(\xi) - \frac{\mathrm{e}^{-\sqrt{\frac{p}{\varkappa_1}}\xi}}{\sqrt{\varkappa_1}}f_1(\xi)\right]d\xi \tag{7.122}$$

and

$$N_2 - N_3 = \frac{1}{2\sqrt{p}}\int_0^\infty \left[\frac{\mathrm{e}^{-\sqrt{\frac{p}{\varkappa_3}}\xi}}{\sqrt{\varkappa_3}}f_3(\xi) - \frac{\mathrm{e}^{-\sqrt{\frac{p}{\varkappa_2}}\xi}}{\sqrt{\varkappa_2}}f_2(\xi)\right]d\xi. \tag{7.123}$$

The continuity condition for the heat flux at $x = 0$ in equation (7.119) provides us with

$$\frac{h_1}{\sqrt{\varkappa_1}}N_1 + \frac{h_2}{\sqrt{\varkappa_2}}N_2 + \frac{h_3}{\sqrt{\varkappa_3}}N_3$$
$$= \frac{1}{2\sqrt{p}}\int_0^\infty \left[\frac{h_1}{\varkappa_1}\mathrm{e}^{-\sqrt{\frac{p}{\varkappa_1}}\xi}f_1(\xi) + \frac{h_2}{\varkappa_2}\mathrm{e}^{-\sqrt{\frac{p}{\varkappa_2}}\xi}f_2(\xi) + \frac{h_3}{\varkappa_3}\mathrm{e}^{-\sqrt{\frac{p}{\varkappa_3}}\xi}f_3(\xi)\right]d\xi. \tag{7.124}$$

The relations (7.122)–(7.124) form a well-posed system of linear algebraic equations for N_i. After solving it, we obtain

$$N_1 = \frac{1}{2H\sqrt{p}}\int_0^\infty\left[\left(\frac{h_1}{\sqrt{\varkappa_1}} - \frac{h_2}{\sqrt{\varkappa_2}} - \frac{h_3}{\sqrt{\varkappa_3}}\right)\frac{e^{-\sqrt{\frac{p}{\varkappa_1}}\xi}}{\sqrt{\varkappa_1}} f_1(\xi)\right.$$
$$\left. + \frac{2h_2}{\varkappa_2}e^{-\sqrt{\frac{p}{\varkappa_2}}\xi} f_2(\xi) + \frac{2h_3}{\varkappa_3}e^{-\sqrt{\frac{p}{\varkappa_3}}\xi} f_3(\xi)\right]d\xi$$

$$N_2 = \frac{1}{2H\sqrt{p}}\int_0^\infty\left[\left(\frac{h_2}{\sqrt{\varkappa_2}} - \frac{h_1}{\sqrt{\varkappa_1}} - \frac{h_3}{\sqrt{\varkappa_3}}\right)\frac{e^{-\sqrt{\frac{p}{\varkappa_2}}\xi}}{\sqrt{\varkappa_2}} f_2(\xi)\right.$$
$$\left. + \frac{2h_1}{\varkappa_1}e^{-\sqrt{\frac{p}{\varkappa_1}}\xi} f_1(\xi) + \frac{2h_3}{\varkappa_3}e^{-\sqrt{\frac{p}{\varkappa_3}}\xi} f_3(\xi)\right]d\xi,$$

and

$$N_3 = \frac{1}{2H\sqrt{p}}\int_0^\infty\left[\left(\frac{h_3}{\sqrt{\varkappa_3}} - \frac{h_2}{\sqrt{\varkappa_2}} - \frac{h_1}{\sqrt{\varkappa_1}}\right)\frac{e^{-\sqrt{\frac{p}{\varkappa_3}}\xi}}{\sqrt{\varkappa_3}} f_3(\xi)\right.$$
$$\left. + \frac{2h_1}{\varkappa_1}e^{-\sqrt{\frac{p}{\varkappa_1}}\xi} f_1(\xi) + \frac{2h_2}{\varkappa_2}e^{-\sqrt{\frac{p}{\varkappa_2}}\xi} f_2(\xi)\right]d\xi,$$

where

$$H = \frac{h_1}{\sqrt{\varkappa_1}} + \frac{h_2}{\sqrt{\varkappa_2}} + \frac{h_3}{\sqrt{\varkappa_3}}.$$

For the compactness of following, we introduce the column vector-functions

$$\mathbf{U}(x,p) = (U_1(x,p), U_2(x,p), U_3(x,p))^T$$

and

$$\mathbf{F}(x) = (f_1(x), f_2(x), f_3(x))^T\,.$$

Upon substituting the M_i and N_i, that we just found, into (7.121) and doing some algebra, we obtain the vector $\mathbf{U}(x,p)$ in terms of the vector $\mathbf{F}(x)$:

$$\mathbf{U}(x,p) = \int_0^\infty \mathbf{G}(x,p;\xi)\mathbf{F}(\xi)d\xi.$$

Clearly, the kernel-matrix $\mathbf{G}(x,p;\xi) = (G_{ij}(x,p;\xi))_{3\times 3}$ of this integral represents the matrix of Green's type for the homogeneous boundary-value problem corresponding to (7.112)–(7.116). In the following, we display the elements $G_{ij}(x,p;\xi)$

of $\mathbf{G}(x, p; \xi)$ in a row-by-row manner. The elements of its first row are found as

$$G_{11}(x, p; \xi) = \frac{1}{2H\sqrt{\varkappa_1 p}} \left[H\mathrm{e}^{-|x-\xi|\sqrt{\frac{p}{\varkappa_1}}} + \left(\frac{h_1}{\sqrt{\varkappa_1}} - \frac{h_2}{\sqrt{\varkappa_2}} - \frac{h_3}{\sqrt{\varkappa_3}} \right) \mathrm{e}^{-(x+\xi)\sqrt{\frac{p}{\varkappa_1}}} \right],$$

$$G_{12}(x, p; \xi) = \frac{h_2}{H\varkappa_2\sqrt{p}} \mathrm{e}^{-(\frac{x}{\sqrt{\varkappa_1}} + \frac{\xi}{\sqrt{\varkappa_2}})\sqrt{p}},$$

and

$$G_{13}(x, p; \xi) = \frac{h_3}{H\varkappa_3\sqrt{p}} \mathrm{e}^{-(\frac{x}{\sqrt{\varkappa_1}} + \frac{\xi}{\sqrt{\varkappa_3}})\sqrt{p}}.$$

For the elements $G_{2j}(x, p; \xi)$ of the second row of $\mathbf{G}(x, p; \xi)$ we have

$$G_{21}(x, p; \xi) = \frac{h_1}{H\varkappa_1\sqrt{p}} \mathrm{e}^{-(\frac{x}{\sqrt{\varkappa_2}} + \frac{\xi}{\sqrt{\varkappa_1}})\sqrt{p}},$$

$$G_{22}(x, p; \xi) = \frac{1}{2H\sqrt{\varkappa_2 p}} \left[H\mathrm{e}^{-|x-\xi|\sqrt{\frac{p}{\varkappa_2}}} + \left(\frac{h_2}{\sqrt{\varkappa_2}} - \frac{h_1}{\sqrt{\varkappa_1}} - \frac{h_3}{\sqrt{\varkappa_3}} \right) \mathrm{e}^{-(x+\xi)\sqrt{\frac{p}{\varkappa_2}}} \right],$$

and

$$G_{23}(x, p; \xi) = \frac{h_3}{H\varkappa_3\sqrt{p}} \mathrm{e}^{-(\frac{x}{\sqrt{\varkappa_2}} + \frac{\xi}{\sqrt{\varkappa_3}})\sqrt{p}}.$$

Finally, the elements $G_{3j}(x, p; \xi)$ of the third row are found as

$$G_{31}(x, p; \xi) = \frac{h_1}{H\varkappa_1\sqrt{p}} \mathrm{e}^{-(\frac{x}{\sqrt{\varkappa_3}} + \frac{\xi}{\sqrt{\varkappa_1}})\sqrt{p}},$$

$$G_{32}(x, p; \xi) = \frac{h_2}{H\varkappa_2\sqrt{p}} \mathrm{e}^{-(\frac{x}{\sqrt{\varkappa_3}} + \frac{\xi}{\sqrt{\varkappa_2}})\sqrt{p}},$$

and

$$G_{33}(x, p; \xi) = \frac{1}{2H\sqrt{\varkappa_3 p}} \left[H\mathrm{e}^{-|x-\xi|\sqrt{\frac{p}{\varkappa_3}}} + \left(\frac{h_3}{\sqrt{\varkappa_3}} - \frac{h_1}{\sqrt{\varkappa_1}} - \frac{h_2}{\sqrt{\varkappa_2}} \right) \mathrm{e}^{-(x+\xi)\sqrt{\frac{p}{\varkappa_3}}} \right].$$

Hence, the solution vector of the original initial-boundary-value problem in (7.112)–(7.116) can be found as the inverse Laplace transform of the vector $\mathbf{U}(x, p; \xi)$, in the form

$$\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))^T.$$

That is

$$\begin{aligned}\mathbf{u}(x,t) &= \mathbf{L}^{-1}\{\mathbf{U}(x,p)\}\\ &= \int_0^\infty \mathbf{L}^{-1}\{\mathbf{G}(x,p;\xi)\}\mathbf{F}(\xi)d\xi = \int_0^\infty \mathbf{g}(x,t;\xi)\mathbf{F}(\xi)d\xi.\end{aligned}$$

Hence, the inverse Laplace transforms of $\mathbf{G}(x, p; \xi)$ represent the sought-after matrix of Green's type $\mathbf{g}(x,t;\xi)$ that we are looking for. The elements $g_{ij}(x,t;\xi)$ of $\mathbf{g}(x,t;\xi)$ are obtained from the corresponding elements $G_{ij}(x,p;\xi)$ of $\mathbf{G}(x,p;\xi)$, with the help of the relation in (7.6). The elements of the first row of $\mathbf{g}(x,t;\xi)$ are found as

$$\begin{aligned}g_{11}(x,t;\xi) &= \frac{1}{2H\sqrt{\pi\varkappa_1 t}}\Bigg[H\mathrm{e}^{-\frac{(x-\xi)^2}{4\varkappa_1 t}}\\ &\quad + \left(\frac{h_1}{\sqrt{\varkappa_1}} - \frac{h_2}{\sqrt{\varkappa_2}} - \frac{h_3}{\sqrt{\varkappa_3}}\right)\mathrm{e}^{-\frac{(x+\xi)^2}{4\varkappa_1 t}}\Bigg],\\ g_{12}(x,t;\xi) &= \frac{h_2}{H\varkappa_2\sqrt{\pi t}}\mathrm{e}^{-\frac{1}{4t}\left(\frac{x}{\sqrt{\varkappa_1}}+\frac{\xi}{\sqrt{\varkappa_2}}\right)^2},\end{aligned}$$

and

$$g_{13}(x,t;\xi) = \frac{h_3}{H\varkappa_3\sqrt{\pi t}}\mathrm{e}^{-\frac{1}{4t}\left(\frac{x}{\sqrt{\varkappa_1}}+\frac{\xi}{\sqrt{\varkappa_3}}\right)^2}$$

whilst for the elements of the second row of $g(x,t;\xi)$, we obtain

$$\begin{aligned}g_{21}(x,t;\xi) &= \frac{h_1}{H\varkappa_1\sqrt{\pi t}}\mathrm{e}^{-\frac{1}{4t}\left(\frac{x}{\sqrt{\varkappa_2}}+\frac{\xi}{\sqrt{\varkappa_1}}\right)^2},\\ g_{22}(x,t;\xi) &= \frac{1}{2H\sqrt{\pi\varkappa_2 t}}\Bigg[H\mathrm{e}^{-\frac{(x-\xi)^2}{4\varkappa_2 t}}\\ &\quad + \left(\frac{h_2}{\sqrt{\varkappa_2}} - \frac{h_1}{\sqrt{\varkappa_1}} - \frac{h_3}{\sqrt{\varkappa_3}}\right)\mathrm{e}^{-\frac{(x+\xi)^2}{4\varkappa_2 t}}\Bigg],\end{aligned}$$

and

$$g_{23}(x,t;\xi) = \frac{h_3}{H\varkappa_3\sqrt{\pi t}}\mathrm{e}^{-\frac{1}{4t}\left(\frac{x}{\sqrt{\varkappa_2}}+\frac{\xi}{\sqrt{\varkappa_3}}\right)^2}.$$

Finally, the elements of the third row of $g(x,t;\xi)$ are found as

$$g_{31}(x,t;\xi) = \frac{h_1}{H\varkappa_1\sqrt{\pi t}} e^{-\frac{1}{4t}\left(\frac{x}{\sqrt{\varkappa_3}}+\frac{\xi}{\sqrt{\varkappa_1}}\right)^2},$$
$$g_{32}(x,t;\xi) = \frac{h_2}{H\varkappa_2\sqrt{\pi t}} e^{-\frac{1}{4t}\left(\frac{x}{\sqrt{\varkappa_3}}+\frac{\xi}{\sqrt{\varkappa_2}}\right)^2},$$

and

$$g_{33}(x,t;\xi) = \frac{1}{2H\sqrt{\pi\varkappa_3 t}}\left[H e^{-\frac{(x-\xi)^2}{4\varkappa_3 t}} + \left(\frac{h_3}{\sqrt{\varkappa_3}} - \frac{h_1}{\sqrt{\varkappa_1}} - \frac{h_2}{\sqrt{\varkappa_2}}\right) e^{-\frac{(x+\xi)^2}{4\varkappa_3 t}}\right].$$

In the following, we examine a two-dimensional initial boundary-value problem for a set of the diffusion equations.

Example 7.12. Construct the matrix of Green's type for the initial-boundary-value problem

$$\frac{\partial u_i(x,y,t)}{\partial t} = \varkappa_i \nabla^2 u_i(x,y,t), \quad (x,y) \in \Omega_i, \quad t>0, \quad i=1,2, \tag{7.125}$$

$$u_i(x,y,0) = f_i(x,y), \quad u_i(x,0,t) = u_i(x,b,t) = 0, \tag{7.126}$$

$$\lim_{x\to-\infty} |u_1(x,y,t)| < \infty, \quad \lim_{x\to\infty} |u_2(x,y,t)| < \infty, \tag{7.127}$$

$$u_1(0,y,t) = u_2(0,y,t), \quad \lambda_1 \frac{\partial u_1(0,y,t)}{\partial x} = \lambda_2 \frac{\partial u_2(0,y,t)}{\partial x} \tag{7.128}$$

stated on the infinite strip $\Omega = \{-\infty < x < \infty, 0 < y < b\}$ composed of two semi-infinite strips $\Omega_1 = \{-\infty < x < 0, 0 < y < b\}$ and $\Omega = \{0 < x < \infty, 0 < y < b\}$.

The above problem might be thought of, in physical terms, as a problem that models the process of heat conduction within a thin plate, with its middle plane occupying the region Ω, one segment of which (Ω_1) is made of a material whose conductivity and thermal diffusivity are λ_1 and $\varkappa_1$, respectively, whilst the thermal constants of the other segment (Ω_2) are λ_2 and $\varkappa_2$. The initial temperature of the plate is given by the functions $f_1(x,y)$ and $f_2(x,y)$. The lateral surfaces of the plate are insulated, the contour lines $y=0$ and $y=b$ are kept at zero temperature at all times, and ideal contact is assumed at the intersection line of the segments $x=0$.

Applying the Laplace transform

$$\mathbf{L}\{u_i(x,y,t)\} = U_i(x,y,p), \quad i=1,2,$$

produces the boundary-value problem

$$\nabla^2 U_i(x,y,p) - \frac{p}{\varkappa_i} U_i(x,y,p) = -\frac{1}{\varkappa_i} f_i(x,y), \quad (x,y) \in \Omega_i,\ i = 1,2, \quad (7.129)$$

$$U_i(x,0,p) = U_i(x,b,p) = 0, \quad (7.130)$$

$$\lim_{x\to-\infty} |U_1(x,y,p)| < \infty, \quad \lim_{x\to\infty} |U_2(x,y,p)| < \infty, \quad (7.131)$$

$$U_1(0,y,p) = U_2(0,y,p), \quad \lambda_1 \frac{\partial U_1(0,y,p)}{\partial x} = \lambda_2 \frac{\partial U_2(0,y,p)}{\partial x} \quad (7.132)$$

for the transforms $U_i(x,y,p)$ of $u_i(x,y,t)$.

Upon expanding $U_i(x,y,p)$ and $f_i(x,y)$ in the Fourier series

$$U_i(x,y,p) = \sum_{n=1}^{\infty} U_{i,n}(x,p) \sin \nu y, \quad f_i(x,y) = \sum_{n=1}^{\infty} f_{i,n}(x) \sin \nu y, \quad i = 1,2, \quad (7.133)$$

with $\nu = n\pi/b$, we arrive at the boundary-value problem

$$\frac{d^2 U_{i,n}(x,p)}{dx^2} - \left(\nu^2 + \frac{p}{\varkappa_i}\right) U_{i,n}(x,p) = -\frac{1}{\varkappa_i} f_{i,n}(x), \quad i = 1,2, \quad (7.134)$$

$$\lim_{x\to-\infty} |U_{1,n}(x,p)| < \infty, \quad \lim_{x\to\infty} |U_{2,n}(x,p)| < \infty, \quad (7.135)$$

$$U_{1,n}(0,p) = U_{2,n}(0,p), \quad \lambda_1 \frac{dU_{1,n}(0,p)}{dx} = \lambda_2 \frac{dU_{2,n}(0,p)}{dx} \quad (7.136)$$

for the Fourier coefficients $U_{i,n}(x,p)$ for the first of the expansions in (7.133).

After the problem in (7.134)–(7.136) is treated by our customary method of variation of parameters, the function $U_{1,n}(x,p)$ is obtained as

$$U_{1,n}(x,p) = \int_{-\infty}^{x} \frac{e^{\omega_1(\xi-x)} - e^{\omega_1(x-\xi)}}{2\varkappa_1\omega_1} f_{1,n}(\xi) d\xi$$
$$+ \int_{-\infty}^{0} \frac{(\lambda\omega_2+\omega_1)e^{\omega_1(x-\xi)} - (\lambda\omega_2-\omega_1)e^{\omega_1(x+\xi)}}{2\varkappa_1\omega_1(\lambda\omega_2+\omega_1)} f_{1,n}(\xi) d\xi$$
$$+ \int_{0}^{\infty} \frac{\lambda e^{\omega_1 x - \omega_2\xi}}{\varkappa_2(\lambda\omega_2+\omega_1)} f_{2,n}(\xi) d\xi, \quad \lambda = \frac{\lambda_2}{\lambda_1}, \quad \omega_i = \sqrt{\nu^2 + \frac{p}{\varkappa_i}}.$$

When the first two integrals are combined, the above reduces to

$$U_{1,n}(x,p) = \int_{-\infty}^{0} g_{11}^n(x,\xi) f_{1,n}(\xi) d\xi + \int_{0}^{\infty} g_{12}^n(x,\xi) f_{2,n}(\xi) d\xi,$$

where

$$g_{11}^n(x,\xi) = \frac{(\lambda\omega_2+\omega_1)e^{-\omega_1|x-\xi|} - (\lambda\omega_2-\omega_1)e^{\omega_1(x+\xi)}}{2\varkappa_1\omega_1(\lambda\omega_2+\omega_1)}$$

and

$$g_{12}^{n}(x,\xi)=\frac{\lambda e^{\omega_1 x-\omega_2\xi}}{\varkappa_2(\lambda\omega_2+\omega_1)}.$$

For $U_{2,n}(x,p)$, we similarly have

$$U_{2,n}(x,p)=\int_{-\infty}^{0} g_{21}^{n}(x,\xi)f_{1,n}(\xi)d\xi+\int_{0}^{\infty} g_{22}^{n}(x,\xi)f_{2,n}(\xi)d\xi,$$

where

$$g_{21}^{n}(x,\xi)=\frac{e^{\omega_1\xi-\omega_2 x}}{\varkappa_1(\lambda\omega_2+\omega_1)}$$

and

$$g_{22}^{n}(x,\xi)=\frac{(\lambda\omega_2+\omega_1)e^{-\omega_2|x-\xi|}+(\lambda\omega_2-\omega_1)e^{-\omega_2(x+\xi)}}{2\varkappa_2\omega_2(\lambda\omega_2+\omega_1)}.$$

To obtain the matrix of Green's type for the homogeneous problem corresponding to
(7.125)–(7.128), we encourage the reader to follow our customary derivation procedure to completion. We will make the rest of our presentation more compact by turning to a particular case of the problem under consideration. In doing so, let the thermal diffusivity of the materials of Ω_1 and Ω_2 be identical. That is, $\varkappa_1=\varkappa_2=\varkappa$ which implies

$$\omega_1=\omega_2=\sqrt{\nu^2+\frac{p}{\varkappa}}.$$

This reduces the expressions for $g_{i,j}^{n}(x,\xi)$ just displayed to

$$g_{11}^{n}(x,\xi)=\frac{(\lambda+1)e^{-\frac{|x-\xi|}{\sqrt{\varkappa}}\sqrt{p+\nu^2\varkappa}}-(\lambda-1)e^{\frac{x+\xi}{\sqrt{\varkappa}}\sqrt{p+\nu^2\varkappa}}}{2\sqrt{\varkappa}(\lambda+1)\sqrt{p+\nu^2\varkappa}}, \tag{7.137}$$

$$g_{12}^{n}(x,\xi)=\frac{\lambda e^{\frac{x-\xi}{\sqrt{\varkappa}}\sqrt{p+\nu^2\varkappa}}}{\sqrt{\varkappa}\sqrt{p+\nu^2\varkappa}},\quad g_{21}^{n}(x,\xi)=\frac{e^{\frac{\xi-x}{\sqrt{\varkappa}}\sqrt{p+\nu^2\varkappa}}}{\sqrt{\varkappa}\sqrt{p+\nu^2\varkappa}},$$

and

$$g_{22}^{n}(x,\xi)=\frac{(\lambda+1)e^{-\frac{|x-\xi|}{\sqrt{\varkappa}}\sqrt{p+\nu^2\varkappa}}+(\lambda-1)e^{-\frac{x+\xi}{\sqrt{\varkappa}}\sqrt{p+\nu^2\varkappa}}}{2\sqrt{\varkappa}(\lambda+1)\sqrt{p+\nu^2\varkappa}}.$$

Following our routine, the solution to (7.129)–(7.132) is obtained as

$$U_i(x,y,p) = \int_0^b \int_{-\infty}^0 \frac{2}{b} \sum_{n=1}^{\infty} g_{i1}^n(x,\xi) \sin \nu y \sin \nu \eta f_1(\xi,\eta) d\xi d\eta$$
$$+ \int_0^b \int_0^{\infty} \frac{2}{b} \sum_{n=1}^{\infty} g_{i2}^n(x,\xi) \sin \nu y \sin \nu \eta f_2(\xi,\eta) d\xi d\eta, \quad i = 1,2.$$

Taking the inverse Laplace transform of $U_i(x,y,p)$, we finally obtain the solution to (7.125)–(7.128) in the form

$$u_i(x,y,t) = \int_0^b \int_{-\infty}^0 g_{i1}(x,y,t;\xi,\eta) f_1(\xi,\eta) d\xi d\eta$$
$$+ \int_0^b \int_0^{\infty} g_{i2}(x,y,t;\xi,\eta) f_2(\xi,\eta) d\xi d\eta, \quad i = 1,2, \tag{7.138}$$

with the kernel-functions $g_{ij}(x,y,t;\xi,\eta)$ representing matrix elements of the sought-after matrix of Green's type

$$\mathbf{G}(x,y,t;\xi,\eta) = \left(g_{ij}(x,y,t;\xi,\eta)\right)_{i,j=1,2},$$

expressed in terms of $g_{ij}^n(x,\xi)$ as

$$g_{ij}(x,y,t;\xi,\eta) = \frac{2}{b} \sum_{n=1}^{\infty} \mathbf{L}^{-1}\{g_{ij}^n(x,\xi)\} \sin \nu y \sin \nu \eta.$$

Taking, for example, the element $g_{11}(x,y,t;\xi,\eta)$ and describing the procedure of its inverse Laplace transform in detail, we write this element as

$$g_{11}(x,y,t;\xi,\eta) = \frac{1}{b} \sum_{n=1}^{\infty} \mathbf{L}^{-1}\left\{g_{11}^n(x,\xi)\right\} (\cos \nu(y-\eta) - \cos \nu(y+\eta)). \tag{7.139}$$

After recalling $g_{11}^n(x,\xi)$ as displayed in (7.137), we obtain its inverse transform with the aid of the relation in (7.6) and the translation theorem, transforming equation (7.139) to

$$g_{11}(x,y,t;\xi,\eta) = \frac{1}{2b\sqrt{\varkappa \pi t}} \left(\mathrm{e}^{-\frac{(x-\xi)^2}{4\varkappa t}} - \frac{\lambda - 1}{\lambda + 1} \mathrm{e}^{-\frac{(x+\xi)^2}{4\varkappa t}} \right)$$
$$\times \sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 \frac{\varkappa t}{b^2} n^2} \left[\cos 2n\pi \frac{(y-\eta)}{2b} - \cos 2n\pi \frac{(y+\eta)}{2b} \right].$$

This reduces to the following formula, containing the Jacobi Theta function

$$g_{11}(x, y, t; \xi, \eta) = \frac{1}{4b\sqrt{\varkappa\pi t}} \left(e^{-\frac{(x-\xi)^2}{4\varkappa t}} - \frac{\lambda - 1}{\lambda + 1} e^{-\frac{(x+\xi)^2}{4\varkappa t}} \right) \times \left[\vartheta_3 \left(\frac{y-\eta}{2b}, \frac{\varkappa t}{b^2} \right) - \vartheta_3 \left(\frac{y+\eta}{2b}, \frac{\varkappa t}{b^2} \right) \right]. \tag{7.140}$$

The other matrix elements of $\mathbf{G}(x, y, t; \xi, \eta)$ are obtained as

$$g_{12}(x, y, t; \xi, \eta) = \frac{\lambda}{2(\lambda + 1)b\sqrt{\varkappa\pi t}} e^{-\frac{(x-\xi)^2}{4\varkappa t}} \times \left[\vartheta_3 \left(\frac{y-\eta}{2b}, \frac{\varkappa t}{b^2} \right) - \vartheta_3 \left(\frac{y+\eta}{2b}, \frac{\varkappa t}{b^2} \right) \right], \tag{7.141}$$

$$g_{21}(x, y, t; \xi, \eta) = \frac{1}{2(\lambda + 1)b\sqrt{\varkappa\pi t}} e^{-\frac{(x-\xi)^2}{4\varkappa t}} \times \left[\vartheta_3 \left(\frac{y-\eta}{2b}, \frac{\varkappa t}{b^2} \right) - \vartheta_3 \left(\frac{y+\eta}{2b}, \frac{\varkappa t}{b^2} \right) \right] \tag{7.142}$$

and

$$g_{22}(x, y, t; \xi, \eta) = \frac{1}{4b\sqrt{\varkappa\pi t}} \left(e^{-\frac{(x-\xi)^2}{4\varkappa t}} + \frac{\lambda - 1}{\lambda + 1} e^{-\frac{(x+\xi)^2}{4\varkappa t}} \right) \times \left[\vartheta_3 \left(\frac{y-\eta}{2b}, \frac{\varkappa t}{b^2} \right) - \vartheta_3 \left(\frac{y+\eta}{2b}, \frac{\varkappa t}{b^2} \right) \right]. \tag{7.143}$$

In this section, we have considered several illustrative examples demonstrating the possibilities to obtain matrices of Green's type for the diffusion equation in media, whose conductive properties vary discontinuously as a function of the spatial variables within the region of interest. Many other problems of this kind can be treated similarly.

7.4 Chapter Exercises

1. Construct the Green's function for the diffusion equation

$$\frac{\partial u}{\partial t} = \varkappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0,$$

that we obtained earlier in Section 7.2.1 (see equation (7.35)), subject to the initial condition $u(x, 0) = 0$, and the boundary conditions

$$\frac{\partial u(0, t)}{\partial x} = 0, \quad \lim_{x \to \infty} |u(x, t)| < \infty.$$

2. Construct the Green's function for the diffusion equation

$$\frac{\partial u}{\partial t} = \varkappa \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < a, \quad t > 0,$$

subject to the initial condition $u(x, 0) = 0$, and the boundary conditions:

(a) $\frac{\partial u(0,t)}{\partial x} = 0, u(a, t) = 0$;

(b) $\frac{\partial u(0,t)}{\partial x} - \gamma u(0, t) = 0, u(a, t) = 0$.

3. Construct the Green's function for the diffusion equation

$$\frac{\partial u}{\partial t} = \varkappa \nabla^2 u - \beta u, \quad -\infty < x < \infty, \quad 0 < y < b, \quad t > 0,$$

subject to the initial condition $u(x, y, 0) = 0$, and the boundary conditions

$$\frac{\partial u(x, 0, t)}{\partial y} = \frac{\partial u(x, b, t)}{\partial y} = 0, \quad \lim_{x \to \pm\infty} |u(x, y, t)| = 0.$$

4. Construct the Green's function for the diffusion equation

$$\frac{\partial u}{\partial t} = \varkappa \nabla^2 u - \beta u, \quad 0 < x < \infty, \quad 0 < y < b, \quad t > 0,$$

subject to the initial condition $u(x, y, 0) = 0$, and the boundary conditions:

(a) $\frac{\partial u(0,y,t)}{\partial x} = 0, u(x, 0, t) = u(x, b, t) = 0$;

(b) $u(0, y, t) = 0, u(x, 0, t) = \frac{\partial u(x,b,t)}{\partial y} = 0$;

(c) $\frac{\partial u(0,y,t)}{\partial x} - \gamma u(0, y, t) = 0, u(x, 0, t) = \frac{\partial u(x,b,t)}{\partial y} = 0$;

(d) $\frac{\partial u(0,y,t)}{\partial x} = 0, u(x, 0, t) = \frac{\partial u(x,b,t)}{\partial y} = 0$;

(e) $\frac{\partial u(0,y,t)}{\partial x} = 0, \ \frac{\partial u(x,0,t)}{\partial y} = \frac{\partial u(x,b,t)}{\partial y} = 0$.

5. Construct the Green's function for the diffusion equation

$$\frac{\partial u}{\partial t} = \varkappa \nabla^2 u - \beta u, \quad 0 < x < a, \quad 0 < y < b, \quad t > 0,$$

subject to the initial condition $u(x, y, 0) = 0$, and the boundary conditions

$$\frac{\partial u(0, y, t)}{\partial x} = 0, \quad \gamma u(a, y, t) = 0$$

and

$$u(x, 0, t) = u(x, b, t) = 0.$$

Chapter 8
Black–Scholes Equation

The partial differential equations that we have dealt with so far in this manual, were well-represented in standard textbooks, used in the field. This is primarily so, due to a wide range of applications of those equations in various areas of engineering and natural sciences. That is, the diffusion equation, which has been reviewed methodically in the previous chapter, finds numerous applications in continuum mechanics [13, 27, 29] (in both fluid and solid sections of this science). As to the two-dimensional static Klein–Gordon equation, that we considered in Chapter 3, its application might be associated, for example, with a simulation of the phenomenon of steady-state heat conduction in thin-walled structures, whose lateral surfaces are uninsulated. The higher-order partial differential equations, that we analyzed in Chapter 4, serve as mathematical models for a score of problems in different areas of structural mechanics [7, 19, 24, 26, 32, 38, 42, 47, 56, 59, 60, 64, 71, 72]. It is, of course, needless to list the vast number of various applications of the classical Laplace equation in engineering and science.

Our focus in the present chapter is on an equation that has not yet been integrated into the standard courses on applied partial differential equations. Hence, the equation to be looked at herein stays apart from all other equations listed above. The history of its application is relatively short, compared with the others: indeed, the *Black–Scholes* equation, which will be explored in this chapter, was introduced as a model in financial mathematics only in recent decades [8, 52, 55, 75], which is why it cannot be considered a customary equation in the area of applied partial differential equations. However, the equation is becoming increasingly popular in its field where it is widely used for qualitative as well as quantitative analysis of stock option pricing problems. This attracts numerous researchers working in this rapidly developing area of financial engineering, that is slowly but surely becoming an integral part of applied mathematics.

Our approach to the Black–Scholes equation and its treatment are very similar to those of the diffusion equation that we described in detail in Chapter 7: the use of the Laplace transform will be combined with the method based on eigenfunction expansion, in order to treat a number of terminal-boundary-value problems for the governing equation. Some necessary preparatory work is presented in Section 8.1, where we will show that a special solution of the Black–Scholes equation, which is, quite customarily in the field, referred to as its *Green's function*, represents, in fact, its *fundamental solution*. In Section 8.2 the emphasis will be on the construction of those Green's functions that were never before exposed in books. In Section 8.3 our focus will be on several methodological issues related to our approach to the construction of Green's functions for parabolic type equations. Section 8.4 changes our view angle, as we focus on numerical results, illustrating the computational potential of the procedures, based on Green's functions.

8.1 The Fundamental Solution

The intention in this section is to establish a terminological basis for our work on Green's functions for the Black–Scholes equation

$$\frac{\partial V(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0 \tag{8.1}$$

which represents a linear, backward in time, parabolic type partial differential equation with variable coefficients. In shorthand form, it will in this book, frequently be referred to as BSE.

To explain the process modeled by (8.1), we note that the function $V = V(S,t)$ represents, in financial terms, the *price of the derivative product*, with the independent variables S and t being the *share price of the underlying asset* and *time*, respectively. The constant parameters σ and r are the *volatility of the underlying asset* and the *risk-free interest rate*, respectively.

Green's function-based methods, which are advocated throughout this book as a tool for solving boundary-value and initial-boundary-value problems for partial differential equations, may also be productive in dealing with the BSE. It is worth noting, however, that only a limited number of computer-friendly formulas for Green's functions of this equation are accessible to users. This circumstance considerably restrains the applicability range for the Green's function methods in the field.

The objective in this chapter is to develop a Green's function-based analytical approach to a class of terminal-boundary-value problems posed the BSE. Note that the term *terminal* applies in this case due to the fact that the equation is backward in time. A number of Green's functions will be constructed later, but before going any further with their construction, we will focus on a terminological issue that plays an important role in the rest of our presentation.

Consider the inhomogeneous BSE

$$\frac{\partial V(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = F(S,t) \tag{8.2}$$

on the region $\Omega = (S_1 < S < S_2) \times (T > t > -\infty)$ of the S,t plane. Let the above equation be subjected to the terminal condition

$$V(S,T) = \varphi(S) \tag{8.3}$$

with $\varphi(S)$ representing, in financial terms, the *pay-off function* of a given derivative problem at the *expiration time* T. Let also the boundary conditions

$$V(S_1,t) = B_1(t) \quad \text{and} \quad V(S_2,t) = B_2(t) \tag{8.4}$$

be imposed on the boundary lines $S = S_1$ and $S = S_2$ of Ω.

It is evident that, upon introducing a new unknown function $v(S,t)$ which can, for example, be expressed in terms of $V(S,t)$ as

$$v(S,t) = V(S,t) - \frac{S - S_1}{S_2 - S_1}[B_2(t) - B_1(t)] + B_1(t) \tag{8.5}$$

the terminal boundary-value problem in (8.2)–(8.4) reduces to

$$\frac{\partial v(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v(S,t)}{\partial S^2} + rS\frac{\partial v(S,t)}{\partial S} - rv(S,t) = H(S,t), \tag{8.6}$$

$$v(S,T) = f(S) \tag{8.7}$$

with the homogeneous boundary conditions

$$v(S_1,t) = 0 \quad \text{and} \quad v(S_2,t) = 0 \tag{8.8}$$

imposed on the boundary lines $S = S_1$ and $S = S_2$ of Ω.

In view of relation (8.5), the right-hand side functions $H(S,t)$ and $f(S)$ in (8.6) and (8.7) are expressed in terms of the right-hand sides $F(S,t)$ and $\varphi(S)$ from (8.2) and (8.3) as

$$H(S,t) = F(S,t) + \frac{S - S_1}{S_2 - S_1}[r\,(B_2(t) - B_1(t)) + B_1'(t) - B_2'(t)]$$
$$-B_1'(t) + rB_1(t) + rS\frac{B_1(t) - B_2(t)}{S_2 - S_1}$$

and

$$f(S) = \varphi(S) - \frac{S - S_1}{S_2 - S_1}[B_2(T) - B_1(T)] - B_1(T).$$

Let $G(S,t;\eta)$ represent the Green's function for the homogeneous ($H(S,t) \equiv 0$ and $f(S) \equiv 0$) terminal-boundary-value problem corresponding in (8.6)–(8.8). If so, then, as follows from the qualitative theory of partial differential equations [3, 18, 22, 25, 39, 53, 54, 57, 61, 66, 67, 77], the solution to (8.6)–(8.8) itself can be expressed as the sum

$$v(S,t) = \int_{S_1}^{S_2} G(S,t;\eta)f(\eta)d\eta + \int_t^T \int_{S_1}^{S_2} G(S,t-\tau;\eta)H(\eta,\tau)d\eta d\tau \tag{8.9}$$

of two integral representations.

This implies that, in order to obtain a computer-friendly analytical solution to the terminal-boundary-value problem in (8.2)–(8.4), we have to find a compact form of the Green's function $G(S,t;\eta)$ of the homogeneous problem corresponding to (8.6)–(8.8).

Before going any further with the presentation of a construction procedure for Green's functions for various terminal-boundary-value problems for the BSE, we turn to the following function

$$\Phi(S,t;\eta) = \frac{e^{-r(T-t)}}{\sigma\eta\sqrt{2\pi(T-t)}} \exp\left(-\frac{[\ln\frac{S}{\eta} + (r - \frac{\sigma^2}{2})(T-t)]^2}{2\sigma^2(T-t)}\right) \tag{8.10}$$

which is well known in financial mathematics, and is traditionally referred to, in the literature related to the field, as the Green's function of the Black–Scholes equation [55, 65, 75].

In order to be more accurate in our terminology, we now have to present the reader some specifics of $\Phi(S,t;\eta)$: we will specify a terminal-boundary-value problem for the BSE, for which $\Phi(S,t;\eta)$ represents the actual Green's function. In doing so, we will consider the problem

$$v(S,T) = f(S), \tag{8.11}$$

$$\lim_{S\to 0} |v(S,t)| < \infty \quad \text{and} \quad \lim_{S\to\infty} |v(S,t)| < \infty \tag{8.12}$$

for the homogeneous Black–Scholes equation

$$\frac{\partial v(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v(S,t)}{\partial S^2} + rS\frac{\partial v(S,t)}{\partial S} - rv(S,t) = 0 \tag{8.13}$$

on the quarter-plane $\Omega = (0 < S < \infty) \times (T > t > -\infty)$ of the S,t-space, and will show that this is the actual problem.

Since the Black–Scholes equation is used to model problems, occuring in financial mathematics, and if we interpret the independent variables S and t appropriately, the quarter-plane Ω represents the whole S,t-space. This makes it quite reasonable to call $\Phi(S,t;\eta)$ either the *fundamental solution* of the BSE or its *whole space Green's function*. Note also that, following conventional Green's function terminology, the variable $\eta \in (0,\infty)$ in (8.10) can be referred to as the *source point* variable.

A special comment is required with regard to the symbolism used in setting up the *boundary* conditions in (8.12). Both end-points of the domain of the independent variable S represent the so-called *singular points* [25, 53, 66] of the Black–Scholes equation. Hence, the corresponding *boundary* conditions cannot formally assign definite values to the solution of the governing differential equation. Instead, the conditions in (8.12) imply that the sought-after solution has to be bounded for S going to both zero and infinity.

For several decades, the function in (8.10) was the only available Green's function for the BSE in financial mathematics. In this chapter, we describe a nontrivial approach [49], which will enable us to construct Green's functions for the Black–Scholes equation not only for the boundary conditions in (8.12), but also for a variety of others. This approach follows from the technique for the diffusion equation. Similar to the latter, it is based on a combination of the classical integral Laplace transform formalism and the method of variation of parameters.

Since the BSE may be considered as *of the Cauchy–Euler type* [20, 53, 66] for the variable S, it can be reduced, in financial mathematics (see, for example [55, 65]), to an equation with constant coefficients. That is, after introducing new independent variables x and τ

$$x = \ln S \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T - t) \tag{8.14}$$

the equation in (8.13) reads

$$\frac{\partial u(x,\tau)}{\partial \tau} = \frac{\partial^2 u(x,\tau)}{\partial x^2} + (c-1)\frac{\partial u(x,\tau)}{\partial x} - cu(x,\tau) \tag{8.15}$$

representing a parabolic diffusion-type single-parameter partial differential equation forward in time.

The parameter c in (8.15) is defined in terms of the parameters r and σ of the Black–Scholes equation as $c = 2r/\sigma^2$.

The change of variables in (8.14) converts the conditions in (8.11) and (8.12) into

$$u(x,0) = f(\mathrm{e}^x), \tag{8.16}$$

$$\lim_{x\to-\infty} |u(x,\tau)| < \infty \quad \text{and} \quad \lim_{x\to\infty} |u(x,\tau)| < \infty \tag{8.17}$$

imposed on equation (8.15) on the half-plane $(-\infty < x < \infty) \times (0 < \tau < \infty)$.

Applying the Laplace integral transform

$$U(x;s) = L\{u(x,\tau)\} = \int_0^\infty \mathrm{e}^{-s\tau} u(x,\tau)d\tau$$

to the problem in (8.15)–(8.17), we arrive at the following boundary-value problem for the transform $U(x;s)$:

$$\frac{d^2U(x;s)}{dx^2} + (c-1)\frac{dU(x;s)}{dx} - (s+c)U(x;s) = -f(\mathrm{e}^x), \tag{8.18}$$

$$\lim_{x\to-\infty} |U(x;s)| < \infty, \quad \lim_{x\to\infty} |U(x;s)| < \infty. \tag{8.19}$$

Note that (8.18) is a linear inhomogeneous ordinary differential equation with constant coefficients, since s now is only a parameter and $U(x;s)$ is treated as a single-variable function of x.

To find a fundamental set of solutions for the homogeneous equation corresponding to (8.18), we consider its characteristic equation

$$k^2 + (c-1)k - (s+c) = 0$$

the roots of which are found as

$$k_1 = \alpha + \omega \quad \text{and} \quad k_2 = \alpha - \omega$$

with $\omega = \sqrt{s+\beta}$, whilst the parameters α and β are defined in terms of c as

$$\alpha = \frac{1-c}{2} \quad \text{and} \quad \beta = \left(\frac{1+c}{2}\right)^2. \tag{8.20}$$

This yields two linearly independent particular solutions of the homogeneous equation corresponding to (8.18) as

$$U_1(x;s) = \mathrm{e}^{(\alpha+\omega)x} \quad \text{and} \quad U_2(x;s) = \mathrm{e}^{(\alpha-\omega)x}$$

with their linear combination

$$U(x;s) = A(x;s)\mathrm{e}^{(\alpha+\omega)x} + B(x;s)\mathrm{e}^{(\alpha-\omega)x} \tag{8.21}$$

representing, in accordance with the method of variation of parameters, the general solution to (8.18). Following this procedure, as described in detail earlier in our book, we arrive at the well-posed system

$$\begin{pmatrix} \mathrm{e}^{(\alpha+\omega)x} & \mathrm{e}^{(\alpha-\omega)x} \\ (\alpha+\omega)\mathrm{e}^{(\alpha+\omega)x} & (\alpha-\omega)\mathrm{e}^{(\alpha-\omega)x} \end{pmatrix} \times \begin{pmatrix} A'(x;s) \\ B'(x;s) \end{pmatrix} = \begin{pmatrix} 0 \\ -f(\mathrm{e}^x) \end{pmatrix}$$

of linear algebraic equations for the derivatives of the coefficients $A(x;s)$ and $B(x;s)$ with respect to x from the linear combination in (8.21). We obtain the solution of the above system as

$$A'(x;s) = -\frac{\mathrm{e}^{-\alpha x}\mathrm{e}^{-x\omega}}{2\omega} f(\mathrm{e}^x) \quad \text{and} \quad B'(x;s) = \frac{\mathrm{e}^{-\alpha x}\mathrm{e}^{x\omega}}{2\omega} f(\mathrm{e}^x).$$

After integration, the functions $A(x;s)$ and $B(x;s)$ themselves are found in the form

$$A(x;s) = -\frac{1}{2\omega}\int_{-\infty}^{x} \mathrm{e}^{-\alpha\xi}\mathrm{e}^{-\xi\omega} f(\mathrm{e}^{\xi})d\xi + M(s)$$

and

$$B(x;s) = \frac{1}{2\omega}\int_{-\infty}^{x} \mathrm{e}^{-\alpha\xi}\mathrm{e}^{\xi\omega} f(\mathrm{e}^{\xi})d\xi + N(s).$$

Substitution of these into (8.21) yields the general solution of (8.18) in the form

$$U(x;s) = \frac{1}{2\omega}\int_{-\infty}^{x} \mathrm{e}^{\alpha(x-\xi)}\left(\mathrm{e}^{\omega(\xi-x)} - \mathrm{e}^{\omega(x-\xi)}\right) f(\mathrm{e}^{\xi})d\xi$$
$$+ M(s)\mathrm{e}^{(\alpha+\omega)x} + N(s)\mathrm{e}^{(\alpha-\omega)x} \qquad (8.22)$$

The *constants of integration* $M(s)$ and $N(s)$ can be obtained through imposing the boundary conditions in (8.19). Omitting the details, we get

$$N(s) = 0, \quad M(s) = \frac{1}{2\omega}\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha\xi}\mathrm{e}^{-\xi\omega} f(\mathrm{e}^{\xi})d\xi.$$

Upon substituting these into (8.22), we obtain the solution to the boundary-value problem in (8.18) and (8.19) as

$$U(x;s) = \int_{-\infty}^{x} \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\omega}(\mathrm{e}^{\omega(\xi-x)} - \mathrm{e}^{\omega(x-\xi)}) f(\mathrm{e}^{\xi})d\xi$$
$$+ \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\omega}\mathrm{e}^{\omega(x-\xi)} f(\mathrm{e}^{\xi})d\xi$$

which can be rewritten in a compact single-integral form as

$$U(x;s) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\omega}\mathrm{e}^{-\omega|x-\xi|} f(\mathrm{e}^{\xi})d\xi. \qquad (8.23)$$

The solution $u(x,\tau)$ to the initial boundary-value problem in (8.15)–(8.17) can be obtained from $U(x;s)$ with the aid of the inverse Laplace transform

$$u(x,\tau) = L^{-1}\{U(x;s)\}.$$

To obtain it, we keep in mind that the parameter ω has earlier been introduced in terms of the parameter s of the Laplace transform as $\sqrt{s+\beta}$, yielding

$$u(x,\tau) = \int_{-\infty}^{\infty} \mathrm{e}^{\alpha(x-\xi)} L^{-1}\left\{\frac{\mathrm{e}^{-|x-\xi|\sqrt{s+\beta}}}{2\sqrt{s+\beta}}\right\} f(\mathrm{e}^{\xi})d\xi.$$

Referring back to Chapter 7, we recall the transform of (7.6) and the Translation Theorem, which converts the above expression for $u(x,\tau)$ into

$$u(x,\tau) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\alpha(x-\xi)}\mathrm{e}^{-\beta\tau}}{2\sqrt{\pi\tau}}\mathrm{e}^{-\frac{(x-\xi)^2}{4\tau}} f(\mathrm{e}^{\xi})d\xi. \qquad (8.24)$$

To obtain the solution $v(S,t)$ to the system in (8.11)–(8.13), we perform backward substitutions in accordance with the relations in (8.14). This implies that x, τ and ξ must be replaced with S, t and η , respectively as

$$x = \ln S, \quad \tau = \frac{\sigma^2}{2}(T-t), \quad \text{and} \quad \xi = \ln\eta.$$

Clearly, the differential of the variable of integration ξ in (8.24) converts to the form

$$d\xi = \frac{1}{\eta} d\eta$$

whilst the interval of integration $(-\infty, \infty)$ in (8.24) transforms, according to the relation $\xi = \ln \eta$, to the interval $[0, \infty)$ with respect to η. With all this in mind, we arrive at the solution of the terminal-boundary-value problem in (8.11)–(8.13) as

$$v(S,t) = \int_0^\infty \frac{1}{\sigma\eta\sqrt{2\pi(T-t)}} \times \exp\left(\alpha \ln \frac{S}{\eta} - \beta\frac{\sigma^2}{2}(T-t) - \frac{(\ln \frac{S}{\eta})^2}{2\sigma^2(T-t)}\right) f(\eta) d\eta \tag{8.25}$$

revealing, in light of relation (8.9), the Green's function to the problem in (8.11)–(8.13) as

$$G(S,t;\eta) = \frac{1}{\sigma\eta\sqrt{2\pi(T-t)}} \exp\left(\alpha \ln \frac{S}{\eta} - \beta\frac{\sigma^2}{2}(T-t) - \frac{(\ln \frac{S}{\eta})^2}{2\sigma^2(T-t)}\right). \tag{8.26}$$

At this point, it is not evident that the above expression for $G(S,t;\eta)$ and the one for $\Phi(S,t;\eta)$ displayed in (8.10) are identical. To verify their identity, we express α and β in (8.26) in terms of the original parameters σ and r of the BSE as

$$\alpha = \frac{\frac{\sigma^2}{2} - r}{\sigma^2} \quad \text{and} \quad \beta = \left(\frac{r + \frac{\sigma^2}{2}}{\sigma^2}\right)^2$$

and rewrite (8.26) accordingly as

$$G(S,t;\eta) = \frac{1}{\sigma\eta\sqrt{2\pi(T-t)}} \times \exp\left(\frac{\frac{\sigma^2}{2} - r}{\sigma^2} \ln \frac{S}{\eta} - \frac{(r + \frac{\sigma^2}{2})^2}{2\sigma^2}(T-t) - \frac{(\ln \frac{S}{\eta})^2}{2\sigma^2(T-t)}\right) \tag{8.27}$$

Now multiplying the above expression by the product of the two factors

$$e^{-r(T-t)} e^{r(T-t)}$$

this product being equal to unity, we leave the first of these factors (the negative exponent) in its current form, combine the second factor with the existing exponential term in (8.27), and subsequently rewrite the latter as

$$G(S,t;\eta) = \frac{e^{-r(T-t)}}{\sigma\eta\sqrt{2\pi(T-t)}} \times \exp\left(-\frac{r - \frac{\sigma^2}{2}}{\sigma^2} \ln \frac{S}{\eta} + r(T-t) - \frac{(r + \frac{\sigma^2}{2})^2}{2\sigma^2}(T-t) - \frac{(\ln \frac{S}{\eta})^2}{2\sigma^2(T-t)}\right).$$

Combining the second and third additive terms in the argument of the extended exponential function above, we reduce it to

$$-\frac{r-\frac{\sigma^2}{2}}{\sigma^2}\ln\frac{S}{\eta}-\frac{(r-\frac{\sigma^2}{2})^2}{2\sigma^2}(T-t)-\frac{(\ln\frac{S}{\eta})^2}{2\sigma^2(T-t)}$$

which immediately transforms to

$$-\frac{(\ln\frac{S}{\eta})^2+2(r-\frac{\sigma^2}{2})(T-t)\ln\frac{S}{\eta}+(r-\frac{\sigma^2}{2})^2(T-t)^2}{2\sigma^2(T-t)}.$$

It is evident that the numerator in the above fraction represents a complete square, reducing the above to

$$-\frac{[\ln\frac{S}{\eta}+(r-\frac{\sigma^2}{2})(T-t)]^2}{2\sigma^2(T-t)}.$$

Hence, the expression for $\Phi(S,t;\eta)$ presented in (8.10) is indeed identical to $G(S,t;\eta)$ in (8.26). This implies that $\Phi(S,t;\eta)$ does in fact represent the Green's function for the terminal-boundary-value problem posed in (8.11)–(8.13). In other words, our approach has proved to be effective, and in the next section we bring a thorough justification of its successful applicability to other terminal-boundary-value problems for the BSE.

8.2 Other Green's Functions

As we already mentioned before, our approach to the construction of Green's functions for the BSE follows from the procedure for the diffusion equation, proposed in Chapter 7. It is based on a combination of two classical methods in applied mathematics, which are: the method of Laplace integral transform and the method of variation of parameters traditionally used to find the general solution of linear high-order ordinary differential equations. In the following series of examples we will describe the procedure in detail.

Example 8.1. Consider the terminal-boundary-value problem

$$\frac{\partial v(S,t)}{\partial t}+\frac{\sigma^2S^2}{2}\frac{\partial^2 v(S,t)}{\partial S^2}+rS\frac{\partial v(S,t)}{\partial S}-rv(S,t)=0, \tag{8.28}$$

$$v(S,T)=f(S), \tag{8.29}$$

$$v(S_1,t)=0 \quad \text{and} \quad v(S_2,t)=0, \tag{8.30}$$

posed in the region $\Omega=(S_1<S<S_2)\times(T>t>-\infty)$.

Recall from [55, 66] that the solution of the above problem can be written in terms of the Green's function $G(S, t; \eta)$ of the corresponding homogeneous ($f(S) \equiv 0$) problem as

$$v(S, t) = \int_{S_1}^{S_2} G(S, t; \eta) f(\eta) d\eta. \tag{8.31}$$

The above relation determines the strategy, that we will follow for the actual derivation of a compact formula for the Green's function $G(S, t; \eta)$.

As in the previous section, we emphasize the single-parameter forward in time parabolic equation

$$\frac{\partial u(x, \tau)}{\partial \tau} = \frac{\partial^2 u(x, \tau)}{\partial x^2} + (c - 1)\frac{\partial u(x, \tau)}{\partial x} - cu(x, \tau) \tag{8.32}$$

arising from (8.28) after introducing new independent variables x and τ

$$x = \ln S \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T - t) \tag{8.33}$$

with c in (8.32) defined in terms of r and σ as $c = 2r/\sigma^2$.

Note that equation (8.32) has, in contrast to the Black–Scholes equation, constant coefficients. This simplifies the situation significantly.

After introducing the variables x and τ in accordance with the relations of (8.33), the terminal-boundary-value problem in (8.28)–(8.30) transforms in the following initial-boundary-value problem

$$u(x, 0) = f(\mathrm{e}^x), \tag{8.34}$$

$$u(a, \tau) = 0 \quad \text{and} \quad u(b, \tau) = 0 \tag{8.35}$$

for equation (8.32) on the semi-infinite strip-shaped region $(a < x < b) \times (0 < \tau < \infty)$ in the x, τ-plane, with

$$a = \ln S_1 \quad \text{and} \quad b = \ln S_2.$$

Applying the Laplace transform

$$U(x; s) = L\{u(x, \tau)\} = \int_0^\infty \mathrm{e}^{-s\tau} u(x, \tau) d\tau$$

to the system in (8.32), (8.34) and (8.35), we obtain the following boundary-value problem

$$\frac{d^2 U(x; s)}{dx^2} + (c - 1)\frac{dU(x; s)}{dx} - (s + c)U(x; s) = -f(\mathrm{e}^x), \tag{8.36}$$

$$U(a; s) = 0, \quad U(b; s) = 0 \tag{8.37}$$

for the Laplace transform $U(x; s)$ of $u(x, \tau)$.

We recall from Section 8.1 (observe the relation in (8.22)), that the general solution to the equation in (8.36) can be written, in accordance with the method of variation of parameters, as

$$U(x,s)=\int_a^x \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\omega}(\mathrm{e}^{\omega(\xi-x)}-\mathrm{e}^{\omega(x-\xi)})f(\mathrm{e}^{\xi})d\xi \\ +M(s)\mathrm{e}^{(\alpha+\omega)x}+N(s)\mathrm{e}^{(\alpha-\omega)x}, \tag{8.38}$$

where ω is defined as $\omega=\sqrt{s+\beta}$, whilst α and β are expressed as

$$\alpha=\frac{1-c}{2} \quad \text{and} \quad \beta=\left(\frac{1+c}{2}\right)^2.$$

After imposing the boundary conditions in (8.37), we arrive at the system of linear algebraic equations

$$\begin{pmatrix} \mathrm{e}^{(\alpha+\omega)a} & \mathrm{e}^{(\alpha-\omega)a} \\ \mathrm{e}^{(\alpha+\omega)b} & \mathrm{e}^{(\alpha-\omega)b} \end{pmatrix}\times\begin{pmatrix} M(s) \\ N(s) \end{pmatrix}=\begin{pmatrix} 0 \\ \Psi(s) \end{pmatrix} \tag{8.39}$$

in $M(s)$ and $N(s)$, where

$$\Psi(s)=-\int_a^b \frac{1}{2\omega}[\mathrm{e}^{(\alpha-\omega)(b-\xi)}-\mathrm{e}^{(\alpha+\omega)(b-\xi)}]f(\mathrm{e}^{\xi})d\xi.$$

Solving the system in (8.39), we obtain

$$M(s)=\int_a^b \frac{\mathrm{e}^{(\alpha-\omega)a}\mathrm{e}^{\alpha(b-\xi)}[\mathrm{e}^{\omega(\xi-b)}-\mathrm{e}^{\omega(b-\xi)}]}{2\omega[\mathrm{e}^{\omega(a-b)}-\mathrm{e}^{\omega(b-a)}]}f(\mathrm{e}^{\xi})d\xi$$

and

$$N(s)=-\int_a^b \frac{\mathrm{e}^{(\alpha+\omega)a}\mathrm{e}^{\alpha(b-\xi)}[\mathrm{e}^{\omega(\xi-b)}-\mathrm{e}^{\omega(b-\xi)}]}{2\omega[\mathrm{e}^{\omega(a-b)}-\mathrm{e}^{\omega(b-a)}]}f(\mathrm{e}^{\xi})d\xi.$$

Upon substituting these in (8.38), the latter reads as

$$U(x,s)=\int_a^x \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\omega}(\mathrm{e}^{\omega(\xi-x)}-\mathrm{e}^{\omega(x-\xi)})f(\mathrm{e}^{\xi})d\xi \\ +\int_a^b \frac{\mathrm{e}^{\alpha(x-\xi)}[\mathrm{e}^{\omega(x-a)}-\mathrm{e}^{\omega(a-x)}][\mathrm{e}^{\omega(\xi-b)}-\mathrm{e}^{\omega(b-\xi)}]}{2\omega[\mathrm{e}^{\omega(a-b)}-\mathrm{e}^{\omega(b-a)}]}f(\mathrm{e}^{\xi})d\xi$$

which can be expressed, in single-integral form, making use of absolute value function notation, as

$$U(x;s)=\int_a^b \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\omega\left[\mathrm{e}^{\omega(a-b)}-\mathrm{e}^{\omega(b-a)}\right]} \\ \times\{\mathrm{e}^{\omega[(x+\xi)-(a+b)]}+\mathrm{e}^{\omega[(a+b)-(x+\xi)]} \\ -\mathrm{e}^{\omega[(a-b)+|x-\xi|]}-\mathrm{e}^{\omega[(b-a)-|x-\xi|]}\}f(\mathrm{e}^{\xi})d\xi.$$

Transforming the factor $e^{\omega(a-b)} - e^{\omega(b-a)}$ in the denominator to

$$e^{\omega(a-b)\omega} - e^{\omega(b-a)} = -e^{\omega(b-a)}[1 - e^{2\omega(a-b)}]$$

we rewrite $U(x;s)$ as

$$\begin{aligned} U(x;s) = &- \int_a^b \frac{e^{\alpha(x-\xi)}}{2\omega e^{\omega(b-a)}\left[1 - e^{2\omega(a-b)}\right]} \\ &\times \{e^{\omega[(x+\xi)-(a+b)]} + e^{\omega[(a+b)-(x+\xi)]} \\ &- e^{\omega[(a-b)+|x-\xi|]} - e^{\omega[(b-a)-|x-\xi|]}\} f(e^{\xi}) d\xi. \end{aligned} \tag{8.40}$$

Note that an immediate inverse Laplace transform of $U(x;s)$ is problematic, if the latter is kept in its current form. Hence, we first adjust it by representing the factor $1/[1-e^{2\omega(a-b)}]$ in the integrand of (8.40) as the sum of the geometric series

$$\frac{1}{1 - e^{2\omega(a-b)}} = \sum_{n=0}^{\infty} e^{2n\omega(a-b)}$$

the common ratio $e^{2\omega(a-b)}$ of which appears to be a negative exponent ($a < b$) and is, therefore, less than unity, transforming (8.40) to

$$\begin{aligned} U(x;s) = &\int_a^b \frac{e^{\alpha(x-\xi)}}{2\omega} \sum_{n=0}^{\infty} \{e^{\omega[|x-\xi|+2(n+1)(a-b)]} \\ &+ e^{\omega[2n(a-b)-|x-\xi|]} - e^{\omega[(x+\xi)+2n(a-b)-2b]} \\ &- e^{\omega[2a+2n(a-b)-(x+\xi)]}\} f(e^{\xi}) d\xi. \end{aligned}$$

The inverse Laplace transform of the above can be performed term-by-term. Recall that ω, in terms of the parameter s of the Laplace transform, reads as $\sqrt{s+\beta}$. This yields the solution $u(x,\tau)$ of the initial-boundary-value problem in (8.32), (8.34) and (8.35) as

$$\begin{aligned} u(x,\tau) &= L^{-1}\{U(x,s)\} \\ &= \int_a^b \frac{e^{\alpha(x-\xi)} e^{-\beta\tau}}{2\sqrt{\pi\tau}} \sum_{n=0}^{\infty} \left\{ \exp\left(-\frac{[|x-\xi| + 2(n+1)(a-b)]^2}{4\tau}\right) \right. \\ &+ \exp\left(-\frac{[|x-\xi| - 2n(a-b)]^2}{4\tau}\right) - \exp\left(-\frac{[2b - (x+\xi) - 2n(a-b)]^2}{4\tau}\right) \\ &\left. - \exp\left(-\frac{[(x+\xi) - 2a - 2n(a-b)]^2}{4\tau}\right)\right\} f(e^{\xi}) d\xi \end{aligned}$$

which can be written in a more compact form by rearranging the summation in the above series. In doing so, we combine the first two exponents with other two, whilst summing from negative to positive infinity. Recalling the transform of (7.6) and the Translation Theorem for the Laplace transform in Chapter 7, we obtain

$$u(x,\tau)=\int_a^b \frac{e^{\alpha(x-\xi)}e^{-\beta\tau}}{2\sqrt{\pi\tau}}\sum_{m=-\infty}^{\infty}\left\{\exp\left(-\frac{[|x-\xi|+2m(a-b)]^2}{4\tau}\right)\right.$$
$$\left.-\exp\left(-\frac{[2b-(x+\xi)-2m(a-b)]^2}{4\tau}\right)\right\} f(e^{\xi})d\xi.$$

We obtain the solution $v(S,t)$ to the system in (8.28)–(8.30) from the above through backward substitution of x, τ and ξ with S, t and η, respectively, which we can do in accordance with the relations in (8.33). When α and β are expressed in terms of the original parameters r and σ of the Black–Scholes equation, we obtain $v(S,t)$ in the form

$$v(S,t)=\int_{S_1}^{S_2}\frac{\exp(-\frac{r-\frac{\sigma^2}{2}}{\sigma^2}\ln\frac{S}{\eta}-\frac{(r+\frac{\sigma^2}{2})^2}{2\sigma^2}(T-t))}{\sigma\eta\sqrt{2\pi(T-t)}}$$
$$\times\sum_{m=-\infty}^{\infty}\left\{\exp\left(-\frac{(\ln\frac{S}{\eta}+2m\ln\frac{S_1}{S_2})^2}{2\sigma^2(T-t)}\right)\right.$$
$$\left.-\exp\left(-\frac{(\ln\frac{S_2^2}{S\eta}-2m\ln\frac{S_1}{S_2})^2}{2\sigma^2(T-t)}\right)\right\}f(\eta)d\eta$$

transforming, after combining the sums of the additive components in the series factor into a single logarithmic function, to

$$v(S,t)=\int_{S_1}^{S_2}\frac{\exp(-\frac{r-\frac{\sigma^2}{2}}{\sigma^2}\ln\frac{S}{\eta}-\frac{(r+\frac{\sigma^2}{2})^2}{2\sigma^2}(T-t))}{\sigma\eta\sqrt{2\pi(T-t)}}$$
$$\times\sum_{m=-\infty}^{\infty}\left\{\exp\left(-\frac{(\ln\frac{SS_1^{2m}}{\eta S_2^{2m}})^2}{2\sigma^2(T-t)}\right)-\exp\left(-\frac{(\ln\frac{S_2^{2(m+1)}}{S\eta S_1^{2m}})^2}{2\sigma^2(T-t)}\right)\right\}f(\eta)d\eta. \tag{8.41}$$

Since the solution to the problem in (8.28)–(8.30) is obtained in the integral form of (8.31), the kernel of equation (8.41)

$$G(S,t;\eta) = \frac{\exp(-\frac{r-\frac{\sigma^2}{2}}{\sigma^2}\ln\frac{S}{\eta} - \frac{(r+\frac{\sigma^2}{2})^2}{2\sigma^2}(T-t))}{\sigma\eta\sqrt{2\pi(T-t)}}$$
$$\times \sum_{m=-\infty}^{\infty}\left\{\exp\left(-\frac{(\ln\frac{SS_1^{2m}}{\eta S_2^{2m}})^2}{2\sigma^2(T-t)}\right) - \exp\left(-\frac{(\ln\frac{S_2^{2(m+1)}}{S\eta S_1^{2m}})^2}{2\sigma^2(T-t)}\right)\right\} \quad (8.42)$$

represents the Green's function to the homogeneous setting corresponding to that in (8.28)–(8.30).

It is evident that the above series converges at a high rate, unless the variable t is in the immediate proximity of the expiration time T, which implies that, when computing values of $G(S,t;\eta)$, we can attain any accuracy level as required for practical application by truncating its series appropriately to the Mth partial sum as

$$G(S,t;\eta) \approx \frac{\exp(-\frac{r-\frac{\sigma^2}{2}}{\sigma^2}\ln\frac{S}{\eta} - \frac{(r+\frac{\sigma^2}{2})^2}{2\sigma^2}(T-t))}{\sigma\eta\sqrt{2\pi(T-t)}}$$
$$\times \sum_{m=-M}^{M}\left\{\exp\left(-\frac{(\ln\frac{SS_1^{2m}}{\eta S_2^{2m}})^2}{2\sigma^2(T-t)}\right) - \exp\left(-\frac{(\ln\frac{S_2^{2(m+1)}}{S\eta S_1^{2m}})^2}{2\sigma^2(T-t)}\right)\right\}. \quad (8.43)$$

We performed a multi-parameter numerical experiment in order to come up with practical recommendations for the choice of the truncation parameter M in (8.43). We calculated approximate values of $G(S,t;\eta)$ for a wide range of the parameters r, σ, S_1 and S_2. The experiment suggests unconditionally, that $M \geq 5$ is a sufficient condition for obtaining values of $G(S,t;\eta)$ from (8.43), accurate to the sixth decimal place, even for values of t in a very proximity of T.

Later, in Section 8.4, we will present graphical evidence of the computer-friendly nature of the form in (8.43).

Example 8.2. With the experience gained from working on Example 8.1, consider another terminal-boundary-value problem for the BSE, namely

$$\frac{\partial v(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v(S,t)}{\partial S^2} + rS\frac{\partial v(S,t)}{\partial S} - rv(S,t) = 0, \quad (8.44)$$

$$v(S,T) = f(S), \quad (8.45)$$

$$|v(0,t)| < \infty \quad \text{and} \quad \frac{\partial v(D,t)}{\partial S} + \gamma v(D,t) = 0, \quad \gamma \geq 0, \quad (8.46)$$

defined on the S, t space for the region $\Omega = (0 < S < D) \times (T > t > -\infty)$, where D is a positive constant.

Note that, in (8.46), we impose different types of boundary conditions on the two bounding segments $S = 0$ and $S = D$ of Ω. In mathematical physics the second of these conditions is referred to as either the *mixed* type or the *Robin* type [13, 17, 29]. To our best knowledge, mixed boundary conditions have not yet been considered in association with the Black–Scholes equation. It is in fact unclear if such problem settings might be premature with regard to financial engineering. But from standpoint of mathematics, they look feasible and might find realistic applications within the field in years to come.

In order to construct Green's function to the homogeneous ($f(S) \equiv 0$) problem corresponding to (8.44)–(8.46), we follow our procedure from Example 8.1. Upon introducing new independent variables x and τ, as suggested in (8.33), we convert the system (8.44)–(8.46) to the following initial-boundary-value problem

$$u(x, 0) = f(\mathrm{e}^x), \tag{8.47}$$

$$|u(-\infty, \tau)| < \infty \quad \text{and} \quad \frac{\partial u(b, \tau)}{\partial x} + \varrho u(b, \tau) = 0 \tag{8.48}$$

for equation (8.32) on the quarter-plane $(-\infty < x < b) \times (0 < \tau < \infty)$. The parameters b and ϱ in (8.48) are defined in terms of D and γ from the original problem as

$$b = \ln D \quad \text{and} \quad \varrho = D\gamma.$$

Applying the Laplace transform to the problem in (8.32), (8.47) and (8.48), we arrive at the boundary-value problem

$$\frac{d^2U(x; s)}{dx^2} + (c - 1)\frac{dU(x; s)}{dx} - (s + c)U(x; s) = -f(\mathrm{e}^x), \tag{8.49}$$

$$|U(-\infty; s)| < \infty, \quad \frac{dU(b; s)}{dx} + \varrho U(b; s) = 0 \tag{8.50}$$

for the transform $U(x; s)$ of $u(x, \tau)$.

The method of variation of parameters gives the general solution to the equation in (8.49) of the form

$$U(x; s) = \frac{1}{2\omega}\int_{-\infty}^{x} \mathrm{e}^{\alpha(x-\xi)}[\mathrm{e}^{\omega(\xi-x)} - \mathrm{e}^{\omega(x-\xi)}]f(\mathrm{e}^{\xi})d\xi$$
$$+ M(s)\mathrm{e}^{(\alpha+\omega)x} + N(s)\mathrm{e}^{(\alpha-\omega)x} \tag{8.51}$$

with α and ω as introduced earlier, in Example 8.1.

To determine $M(s)$ and $N(s)$, we take advantage of the boundary conditions in (8.50). When x goes to negative infinity, the integral component in (8.51) vanishes, whereas the $M(s)$-containing component approaches zero. Hence, for the first condition in (8.50) to hold, $N(s)$ must be zero

$$N(s) = 0 \tag{8.52}$$

because the exponential factor of the $N(s)$-containing component in (8.51) is unbounded for x going to negative infinity.

Hence, the derivative of $U(x;s)$ in (8.51) reads as

$$\begin{aligned}\frac{dU(x;s)}{dx} &= M(s)(\alpha+\omega)\mathrm{e}^{(\alpha+\omega)x} \\ &\quad + \frac{1}{2\omega}\int_{-\infty}^{x} \mathrm{e}^{\alpha(x-\xi)}[(\alpha-\omega)\mathrm{e}^{\omega(\xi-x)} - (\alpha+\omega)\mathrm{e}^{\omega(x-\xi)}]f(\mathrm{e}^{\xi})d\xi\end{aligned}$$

and the second condition in (8.50) yields the following equation

$$\begin{aligned}&\frac{1}{2\omega}\int_{-\infty}^{b} \mathrm{e}^{\alpha(b-\xi)}[(\alpha-\omega)\mathrm{e}^{\omega(\xi-b)} - (\alpha+\omega)\mathrm{e}^{\omega(b-\xi)}]f(\mathrm{e}^{\xi})d\xi \\ &\quad + \frac{\varrho}{2\omega}\int_{-\infty}^{b} \mathrm{e}^{\alpha(b-\xi)}[\mathrm{e}^{\omega(\xi-b)} - \mathrm{e}^{\omega(b-\xi)}]f(\mathrm{e}^{\xi})d\xi \\ &\quad + M(s)(\alpha+\omega)\mathrm{e}^{(\alpha+\omega)b} + \varrho M(s)\mathrm{e}^{(\alpha+\omega)b} = 0\end{aligned}$$

from which $M(s)$ is found as

$$M(s) = -\frac{1}{2\omega}\int_{-\infty}^{b} \mathrm{e}^{-\alpha\xi-b\omega}\left[\frac{\vartheta-\omega}{\vartheta+\omega}\mathrm{e}^{\omega(\xi-b)} - \mathrm{e}^{\omega(b-\xi)}\right]f(\mathrm{e}^{\xi})d\xi,$$

where $\vartheta = \varrho + \alpha$.

Now substituting the above expression for $M(s)$ into (8.51) and taking into account (8.52), we obtain the solution to the boundary-value problem in (8.49) and (8.50) as

$$\begin{aligned}U(x;s) &= \frac{1}{2\omega}\int_{-\infty}^{x} \mathrm{e}^{\alpha(x-\xi)}[\mathrm{e}^{\omega(\xi-x)} - \mathrm{e}^{\omega(x-\xi)}]f(\mathrm{e}^{\xi})d\xi \\ &\quad - \frac{1}{2\omega}\int_{-\infty}^{b} \mathrm{e}^{\alpha(x-\xi)}\mathrm{e}^{\omega(x-b)}[\frac{\vartheta-\omega}{\vartheta+\omega}\mathrm{e}^{\omega(\xi-b)} - \mathrm{e}^{\omega(b-\xi)}]f(\mathrm{e}^{\xi})d\xi\end{aligned}$$

To obtain the inverse Laplace transform $u(x,\tau)$ of the above, which represents the solution to the initial boundary-value problem in (8.32), (8.47) and (8.48), we simplify the above expression for $U(x;s)$. After some tedious but straightforward algebra, we obtain its more compact form

$$U(x;s) = \int_{-\infty}^{b} \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\omega}\left[\mathrm{e}^{-\omega|x-\xi|} - \frac{\vartheta-\omega}{\vartheta+\omega}\mathrm{e}^{\omega(x+\xi-2b)}\right]f(\mathrm{e}^{\xi})d\xi$$

which is still inconvenient for immediate inverse Laplace transform. To make things easier, we rewrite $U(x;s)$ in the equivalent form

$$U(x;s) = \int_{-\infty}^{b} \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\omega}\left[\mathrm{e}^{-\omega|x-\xi|} - \left(\frac{2\vartheta}{\vartheta+\omega} - 1\right)\mathrm{e}^{\omega(x+\xi-2b)}\right] f(\mathrm{e}^{\xi})d\xi.$$

Upon recalling the expression for ω in terms of the parameter s of the Laplace transform, the above transforms to

$$U(x;s) = \int_{-\infty}^{b} \frac{\mathrm{e}^{\alpha(x-\xi)}}{2}\left\{\frac{\mathrm{e}^{-\sqrt{s+\beta}|x-\xi|}}{\sqrt{s+\beta}} - \left[\frac{2\vartheta}{(\vartheta+\sqrt{s+\beta})\sqrt{s+\beta}} - \frac{1}{\sqrt{s+\beta}}\right]\mathrm{e}^{\sqrt{s+\beta}(x+\xi-2b)}\right\} f(\mathrm{e}^{\xi})d\xi.$$

The inverse transform of the above can be obtained with the aid of the relations in Chapter 7 (see (7.8)). The solution $u(x,\tau)$ of the initial-boundary-value problem in (8.32), (8.47) and (8.48) is found in the form

$$u(x,\tau) = \int_{-\infty}^{b} \mathrm{e}^{\alpha(x-\xi)-\beta\tau} \times \left\{\frac{1}{2\sqrt{\pi\tau}}\left[\exp\left(-\frac{(x-\xi)^2}{4\tau}\right) + \exp\left(-\frac{(x+\xi-2b)^2}{4\tau}\right)\right] - \vartheta\mathrm{e}^{\vartheta^2\tau-\vartheta(x+\xi-2b)}\operatorname{erfc}\left(\vartheta\sqrt{\tau} - \frac{x+\xi-2b}{2\sqrt{\tau}}\right)\right\} f(\mathrm{e}^{\xi})d\xi \tag{8.53}$$

Recall that the conventional abbreviation $\operatorname{erfc}(\cdot)$ represents the complementary Gauss error function [1, 3, 27, 37, 66], which is defined as

$$\operatorname{erfc}(\varphi) = \frac{2}{\sqrt{\pi}}\int_{\varphi}^{\infty} \mathrm{e}^{-x^2}dx.$$

The solution $v(S,t)$ of the terminal-boundary-value problem in (8.44)–(8.46) can be obtained from (8.53) by backward substitution of the original parameters, implying

$$v(S,t) = \int_{0}^{D} \frac{1}{\eta}\exp\left(\alpha\ln\frac{S}{\eta} - \beta\frac{\sigma^2}{2}(T-t)\right) \times \left\{\frac{1}{\sigma\sqrt{2\pi(T-t)}}\left[\exp\left(-\frac{(\ln\frac{S}{\eta})^2}{2\sigma^2(T-t)}\right) + \exp\left(-\frac{(\ln\frac{S\eta}{D^2})^2}{2\sigma^2(T-t)}\right)\right] - \vartheta\mathrm{e}^{\frac{\vartheta^2\sigma^2}{2}(T-t)-\vartheta\ln\frac{S\eta}{D^2}}\operatorname{erfc}\left(\frac{\vartheta}{2}\sqrt{2\sigma^2(T-t)} - \frac{\ln\frac{S\eta}{D^2}}{\sqrt{2\sigma^2(T-t)}}\right)\right\} f(\eta)d\eta$$

from which we arrive at a conclusion that the kernel $G(S,t;\eta)$ of the above integral represents the Green's function to the homogeneous problem corresponding to that in (8.44)–(8.46). After some trivial algebra, it can be written in the form

$$\begin{aligned} G(S,t;\eta) = {} & \frac{1}{\eta}\left(\frac{S}{\eta}\right)^{\alpha} \exp\left(-\beta\frac{\sigma^2}{2}(T-t)\right) \\ & \times \left\{ \frac{1}{\sigma\sqrt{2\pi(T-t)}} \left[\exp\left(-\frac{(\ln\frac{S}{\eta})^2}{2\sigma^2(T-t)}\right) + \exp\left(-\frac{(\ln\frac{S\eta}{D^2})^2}{2\sigma^2(T-t)}\right)\right] \right. \\ & \left. - \vartheta\left(\frac{S\eta}{D^2}\right)^{-\vartheta} \mathrm{e}^{\frac{\vartheta^2\sigma^2}{2}(T-t)} \operatorname{erfc}\left(\frac{\vartheta}{2}\sqrt{2\sigma^2(T-t)} - \frac{\ln\frac{S\eta}{D^2}}{\sqrt{2\sigma^2(T-t)}}\right)\right\}. \end{aligned} \tag{8.54}$$

Recall that α, β and ϑ in (8.54) read, in terms of the original parameters σ and r of the Black–Scholes equation and the D and γ in (8.46), as

$$\alpha = \frac{\frac{\sigma^2}{2} - r}{\sigma^2}, \quad \beta = \left(\frac{r + \frac{\sigma^2}{2}}{\sigma^2}\right)^2 \quad \text{and} \quad \vartheta = D\gamma + \alpha.$$

Example 8.3. Note that the problem specification in (8.44)–(8.46) allows two particular cases that might be of interest in option pricing valuation. One of these cases is for γ equal to zero, transforming the boundary conditions in (8.46) to

$$|v(0,t)| < \infty \quad \text{and} \quad \frac{\partial v(D,t)}{\partial S} = 0. \tag{8.55}$$

Later, in the Chapter Exercises, we challenge the reader to implement our technique and derive the Green's function for the problem in (8.44), (8.45), and (8.55) from scratch. In the following, we will obtain it, by taking advantage of the formula for the Green's function that appeared in (8.54). Making subsequent changes necessitated by setting $\gamma = 0$, which implies $\vartheta = \alpha$, we arrive at

$$\begin{aligned} G(S,t;\eta) = {} & \frac{1}{\eta}\left(\frac{S}{\eta}\right)^{\alpha} \exp\left(-\beta\frac{\sigma^2}{2}(T-t)\right) \\ & \times \left\{ \frac{1}{\sigma\sqrt{2\pi(T-t)}} \left[\exp\left(-\frac{\left(\ln\frac{S}{\eta}\right)^2}{2\sigma^2(T-t)}\right) + \exp\left(-\frac{(\ln\frac{S\eta}{D^2})^2}{2\sigma^2(T-t)}\right)\right] \right. \\ & \left. - \alpha\left(\frac{S\eta}{D^2}\right)^{-\alpha} \mathrm{e}^{\frac{\alpha^2\sigma^2}{2}(T-t)} \operatorname{erfc}\left(\frac{\alpha}{2}\sqrt{2\sigma^2(T-t)} - \frac{\ln\frac{S\eta}{D^2}}{\sqrt{2\sigma^2(T-t)}}\right)\right\} \end{aligned} \tag{8.56}$$

which is the sought-after Green's function.

Example 8.4. The second particular case for the problem in (8.44)–(8.46) is when γ goes to infinity. This transforms the boundary conditions in (8.46) to

$$|v(0,t)| < \infty \quad \text{and} \quad v(D,t) = 0. \tag{8.57}$$

Later, in the Chapter Exercises, we challenge the reader to implement our technique and derive the Green's function for the setting in (8.44), (8.45), and (8.57) from scratch. In the following, we will get it from (8.54) directly . Note that this is not as straightforward as in Example 8.3: taking the limit in (8.54) for γ going to infinity is not a trivial exercise. That is why, in order to get it done, we revisit the expression

$$U(x;s) = \int_{-\infty}^{b} \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\omega} \left[\mathrm{e}^{-\omega|x-\xi|} - \frac{\vartheta - \omega}{\vartheta + \omega} \mathrm{e}^{\omega(x+\xi-2b)}\right] f(\mathrm{e}^{\xi}) d\xi \tag{8.58}$$

that we obtained earlier for the Laplace transform $U(x;s)$ of $u(x,\tau)$ in Example 8.2, for the problem in (8.44)–(8.46). Observing that

$$\lim_{\gamma\to\infty} \frac{\vartheta - \omega}{\vartheta + \omega} = \lim_{\gamma\to\infty} \frac{(D\gamma + \alpha) - \omega}{(D\gamma + \alpha) + \omega} = 1$$

we write (8.58) for the problem in (8.44), (8.45), and (8.57) as

$$\begin{aligned} U(x;s) &= \int_{-\infty}^{b} \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\omega} [\mathrm{e}^{-\omega|x-\xi|} - \mathrm{e}^{\omega(x+\xi-2b)}] f(\mathrm{e}^{\xi}) d\xi \\ &= \int_{-\infty}^{b} \frac{\mathrm{e}^{\alpha(x-\xi)}}{2} \left[\frac{\mathrm{e}^{-\sqrt{s+\beta}|x-\xi|}}{\sqrt{s+\beta}} - \frac{\mathrm{e}^{\sqrt{s+\beta}(x+\xi-2b)}}{\sqrt{s+\beta}}\right] f(\mathrm{e}^{\xi}) d\xi. \end{aligned}$$

The inverse Laplace transform of the above is no longer a problem and we find

$$\begin{aligned} u(x,\tau) = \int_{-\infty}^{b} & \frac{\mathrm{e}^{\alpha(x-\xi)-\beta\tau}}{2\sqrt{\pi\tau}} \\ & \times \left[\exp\left(-\frac{(x-\xi)^2}{4\tau}\right) - \exp\left(-\frac{(x+\xi-2b)^2}{4\tau}\right)\right] f(\mathrm{e}^{\xi}) d\xi \end{aligned}$$

allowing to find the solution of the problem in (8.44), (8.45) and (8.57) as

$$\begin{aligned} v(S,t) = \int_{0}^{D} & \frac{\exp(\alpha \ln \frac{S}{\eta} - \beta \frac{\sigma^2}{2}(T-t))}{\eta\sigma\sqrt{2\pi(T-t)}} \\ & \times \left[\exp\left(-\frac{(\ln \frac{S}{\eta})^2}{2\sigma^2(T-t)}\right) - \exp\left(-\frac{(\ln \frac{S\eta}{D^2})^2}{2\sigma^2(T-t)}\right)\right] f(\eta) d\eta \end{aligned}$$

which simplifies to

$$v(S,t) = \int_0^D \frac{1}{\eta}\left(\frac{S}{\eta}\right)^{\alpha} \frac{\exp(-\beta\frac{\sigma^2}{2}(T-t))}{\sigma\sqrt{2\pi(T-t)}} \times \left[\exp\left(-\frac{(\ln\frac{S}{\eta})^2}{2\sigma^2(T-t)}\right) - \exp\left(-\frac{(\ln\frac{S\eta}{D^2})^2}{2\sigma^2(T-t)}\right)\right] f(\eta)d\eta$$

from which follows that

$$G(S,t;\eta) = \frac{S^{\alpha}}{\eta^{\alpha+1}} \cdot \frac{\exp(-\beta\frac{\sigma^2}{2}(T-t))}{\sigma\sqrt{2\pi(T-t)}} \times \left[\exp\left(-\frac{(\ln\frac{S}{\eta})^2}{2\sigma^2(T-t)}\right) - \exp\left(-\frac{(\ln\frac{S\eta}{D^2})^2}{2\sigma^2(T-t)}\right)\right] \tag{8.59}$$

represents the Green's function to the homogeneous problem corresponding to that in (8.44), (8.45), and (8.57).

Example 8.5. Consider another terminal boundary-value problem

$$\frac{\partial v(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v(S,t)}{\partial S^2} + rS\frac{\partial v(S,t)}{\partial S} - rv(S,t) = 0, \tag{8.60}$$

$$v(S,T) = f(S), \tag{8.61}$$

$$\frac{\partial v(S_1,t)}{\partial S} = 0 \quad \text{and} \quad \frac{\partial v(S_2,t)}{\partial S} = 0 \tag{8.62}$$

on the region $\Omega = (S_1 < S < S_2) \times (T > t > -\infty)$, and construct Green's function to the corresponding homogeneous ($f(S) \equiv 0$) problem.

Omitting the customary details of the derivation procedure, which have been thoroughly explained earlier, we present a brief sketch, whilst focusing on issues that are specific to the problem under consideration.

After a change of variables as specified in (8.33), the problem in (8.60)–(8.62) reduces to the initial-boundary-value problem

$$\frac{\partial u(x,\tau)}{\partial \tau} = \frac{\partial^2 u(x,\tau)}{\partial x^2} + (c-1)\frac{\partial u(x,\tau)}{\partial x} - cu(x,\tau), \tag{8.63}$$

$$u(x,0) = f(\mathrm{e}^x), \tag{8.64}$$

$$\frac{\partial u(a,\tau)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u(b,\tau)}{\partial x} = 0, \tag{8.65}$$

where c is introduced as $2r/\sigma^2$, whilst

$$a = \ln S_1 \quad \text{and} \quad b = \ln S_2.$$

Applying the Laplace transform

$$U(x;s) = L\{u(x,\tau)\} = \int_0^\infty e^{-s\tau} u(x,\tau)d\tau$$

to (8.63)–(8.65), we arrive at the boundary-value problem

$$\frac{d^2U(x;s)}{dx^2} + (c-1)\frac{dU(x;s)}{dx} - (s+c)U(x;s) = -f(e^x), \tag{8.66}$$

$$\frac{\partial U(a;s)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial U(b;s)}{\partial x} = 0 \tag{8.67}$$

for $U(x;s)$. Following the procedure of variation of parameters, we find the solution of the system (8.66)–(8.67) as

$$\begin{aligned} U(x;s) = &\int_a^x \frac{e^{\alpha(x-\xi)}}{2\omega}[e^{\omega(\xi-x)} - e^{\omega(x-\xi)}]f(e^\xi)d\xi \\ &+ \int_a^b \frac{e^{\alpha(x-\xi)}[(\alpha-\omega)e^{\omega(\xi-b)} - (\alpha+\omega)e^{\omega(b-\xi)}]}{2\omega(\alpha^2-\omega^2)[e^{\omega(a-b)} - e^{\omega(b-a)}]} \\ &\qquad \times [(\alpha-\omega)e^{\omega(x-a)} - (\alpha+\omega)e^{\omega(a-x)}]f(e^\xi)d\xi \end{aligned}$$

with ω defined as $\omega = \sqrt{s+\beta}$, whilst α and β are expressed in terms of the parameter c that we introduced recently as

$$\alpha = \frac{1-c}{2} \quad \text{and} \quad \beta = \left(\frac{1+c}{2}\right)^2.$$

The above expression for $U(x;s)$ reduces to single-integral form:

$$\begin{aligned} U(x;s) = &\int_a^b \frac{e^{\alpha(x-\xi)}}{2\omega(\alpha^2-\omega^2)[e^{\omega(a-b)} - e^{\omega(b-a)}]} \\ &\times \{(\alpha-\omega)^2 e^{\omega[(x+\xi)-(a+b)]} + (\alpha+\omega)^2 e^{\omega[(a+b)-(x+\xi)]} \\ &\quad - (\alpha^2-\omega^2)[e^{\omega[|x-\xi|+(a-b)]} - e^{\omega[(b-a)-|x-\xi|]}]\}f(e^\xi)d\xi \end{aligned} \tag{8.68}$$

To ease performing the inverse transform of $U(x;s)$, we rewrite (as was suggested earlier in Example 8.1) one of the factors in the integrand in (8.68) as

$$\frac{1}{e^{\omega(a-b)} - e^{\omega(b-a)}} = -\frac{e^{\omega(a-b)}}{1 - e^{2\omega(a-b)}} = -e^{\omega(a-b)}\sum_{n=0}^\infty e^{2n\omega(a-b)}.$$

This converts (8.68) to

$$\begin{aligned} U(x;s) = &\int_a^b \frac{e^{\alpha(x-\xi)}}{2\omega}\sum_{n=0}^\infty \Big\{e^{\omega[|x-\xi|+2(n+1)(a-b)]} + e^{\omega[2n(a-b)-|x-\xi|]} \\ &- \frac{\alpha-\omega}{\alpha+\omega}[e^{\omega[(x+\xi)+2n(a-b)-2b]} + e^{\omega[2n(a-b)-(x+\xi)+2a]}]\Big\} f(e^\xi)d\xi, \end{aligned}$$

which can be transformed as

$$U(x;s) = \int_a^b \frac{e^{\alpha(x-\xi)}}{2\omega} \sum_{n=0}^{\infty} \left\{ e^{\omega[|x-\xi|+2(n+1)(a-b)]} + e^{\omega[2n(a-b)-|x-\xi|]} \right.$$
$$\left. - \left(1 - \frac{2\omega}{\alpha+\omega}\right) [e^{\omega[(x+\xi)+2n(a-b)-2b]} + e^{\omega[2n(a-b)-(x+\xi)+2a]}] \right\} f(e^{\xi}) d\xi$$
$$= \int_a^b e^{\alpha(x-\xi)} \left\{ \frac{1}{2\omega} \sum_{n=0}^{\infty} [e^{\omega[|x-\xi|+2(n+1)(a-b)]} + e^{\omega[2n(a-b)-|x-\xi|]} \right.$$
$$- e^{\omega[(x+\xi)+2n(a-b)-2b]} - e^{\omega[2n(a-b)-(x+\xi)+2a]}]$$
$$\left. + \frac{1}{\alpha+\omega} \sum_{n=0}^{\infty} [e^{\omega[(x+\xi)+2n(a-b)-2b]} + e^{\omega[2n(a-b)-(x+\xi)+2a]}] \right\} f(e^{\xi}) d\xi. \tag{8.69}$$

It is evident that the inverse transform of the first series component

$$\frac{1}{2\omega} \sum_{n=0}^{\infty} [e^{\omega[|x-\xi|+2(n+1)(a-b)]} + e^{\omega[2n(a-b)-|x-\xi|]}$$
$$- e^{\omega[(x+\xi)+2n(a-b)-2b]} - e^{\omega[2n(a-b)-(x+\xi)+2a]}] \tag{8.70}$$

in (8.69) can be obtained term-by-term with the aid of the relation from (7.6) and the Translation Theorem. Indeed, recalling the expression for ω in terms of s in the Laplace transform, we turn the inverse transform of (8.70) into

$$\frac{e^{-\beta\tau}}{2\sqrt{\pi\tau}} \sum_{n=0}^{\infty} [e^{-\frac{[|x-\xi|+2(n+1)(a-b)]^2}{4\tau}} + e^{-\frac{[2n(a-b)-|x-\xi|]^2}{4\tau}}$$
$$- e^{-\frac{[(x+\xi)+2n(a-b)-2b]^2}{4\tau}} - e^{-\frac{[2n(a-b)-(x+\xi)+2a]^2}{4\tau}}]. \tag{8.71}$$

It appears that the inverse transform of the second series component in (8.69)

$$\frac{1}{\alpha+\omega} \sum_{n=0}^{\infty} [e^{\omega[(x+\xi)+2n(a-b)-2b]} + e^{\omega[2n(a-b)-(x+\xi)+2a]}] \tag{8.72}$$

cannot be directly obtained by means of the standard relations available from Chapter 7 without some preparatory work. In doing so, we recall the relations

$$\mathbf{L}^{-1}\left\{\frac{1}{\sqrt{s}} e^{-k\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi\tau}} e^{-\frac{k^2}{4\tau}} \tag{8.73}$$

and

$$\mathbf{L}^{-1}\left\{\frac{e^{-k\sqrt{s}}}{\sqrt{s}(\sqrt{s}+\alpha)}\right\}=e^{\alpha(\alpha\tau+k)}\mathrm{erfc}\left(\alpha\sqrt{\tau}+\frac{k}{2\sqrt{\tau}}\right) \tag{8.74}$$

listed in Chapter 7 as (7.6) and (7.8). In light of the decomposition

$$\frac{1}{\sqrt{s}(\sqrt{s}+\alpha)}=\frac{1}{\alpha}\left(\frac{1}{\sqrt{s}}-\frac{1}{\sqrt{s}+\alpha}\right)$$

the relation in (8.74) transforms to

$$\mathbf{L}^{-1}\left\{\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}\right\}-\mathbf{L}^{-1}\left\{\frac{1}{\sqrt{s}+\alpha}e^{-k\sqrt{s}}\right\}=\alpha e^{\alpha(\alpha\tau+k)}\mathrm{erfc}\left(\alpha\sqrt{\tau}+\frac{k}{2\sqrt{\tau}}\right)$$

which gives birth to the new relation

$$\mathbf{L}^{-1}\left\{\frac{1}{\sqrt{s}+\alpha}e^{-k\sqrt{s}}\right\}=\frac{1}{\sqrt{\pi\tau}}e^{-\frac{k^2}{4\tau}}-\alpha e^{\alpha(\alpha\tau+k)}\mathrm{erfc}\left(\alpha\sqrt{\tau}+\frac{k}{2\sqrt{\tau}}\right). \tag{8.75}$$

By using the above, the inverse Laplace transform of (8.72) is found in the form

$$\begin{aligned}
&e^{-\beta\tau}\Bigg\{\frac{1}{\sqrt{\pi\tau}}\sum_{n=0}^{\infty}[e^{-\frac{[(x+\xi)+2n(a-b)-2b]^2}{4\tau}}+e^{-\frac{[2n(a-b)-(x+\xi)+2a]^2}{4\tau}}]\\
&\quad-\alpha\sum_{n=0}^{\infty}\Bigg[e^{\alpha(\alpha\tau+(x+\xi)+2n(a-b)-2b)}\mathrm{erfc}\left(\alpha\sqrt{\tau}+\frac{(x+\xi)+2n(a-b)-2b}{2\sqrt{\tau}}\right)\\
&\quad+e^{\alpha(\alpha\tau+2n(a-b)-(x+\xi)+2a)}\mathrm{erfc}\left(\alpha\sqrt{\tau}+\frac{2n(a-b)-(x+\xi)+2a}{2\sqrt{\tau}}\right)\Bigg]\Bigg\}
\end{aligned} \tag{8.76}$$

With (8.71) and (8.76) in mind, we finally arrive at the inverse Laplace transform of (8.69), which appears as

$$\begin{aligned}
&u(x,\tau)=L^{-1}\{U(x;s)\}=\int_a^b e^{\alpha(x-\xi)}e^{-\beta\tau}\\
&\times\Bigg\{\frac{1}{2\sqrt{\pi\tau}}\sum_{n=0}^{\infty}[e^{-\frac{[|x-\xi|+2(n+1)(a-b)]^2}{4\tau}}+e^{-\frac{[2n(a-b)-|x-\xi|]^2}{4\tau}}\\
&\quad+e^{-\frac{[(x+\xi)+2n(a-b)-2b]^2}{4\tau}}+e^{-\frac{[2n(a-b)-(x+\xi)+2a]^2}{4\tau}}]\\
&\quad-\alpha\sum_{n=0}^{\infty}\Bigg[e^{\alpha(\alpha\tau+(x+\xi)+2n(a-b)-2b)}\mathrm{erfc}\left(\alpha\sqrt{\tau}+\frac{(x+\xi)+2n(a-b)-2b}{2\sqrt{\tau}}\right)\\
&\quad+e^{\alpha(\alpha\tau+2n(a-b)-(x+\xi)+2a)}\mathrm{erfc}\left(\alpha\sqrt{\tau}+\frac{2n(a-b)-(x+\xi)+2a}{2\sqrt{\tau}}\right)\Bigg]\Bigg\}f(e^{\xi})d\xi.
\end{aligned}$$

The first series component of the above reduces to the more compact form

$$u(x,\tau) = \int_a^b e^{\alpha(x-\xi)} e^{-\beta\tau}$$
$$\times \Bigg\{ \frac{1}{2\sqrt{\pi\tau}} \sum_{m=-\infty}^{\infty} [e^{-\frac{[|x-\xi|+2m(a-b)]^2}{4\tau}} + e^{-\frac{[2b-2m(a-b)-(x+\xi)]^2}{4\tau}}]$$
$$-\alpha \sum_{n=0}^{\infty} \Bigg[e^{\alpha(\alpha\tau+(x+\xi)+2n(a-b)-2b)} \operatorname{erfc}\left(\alpha\sqrt{\tau} + \frac{(x+\xi)+2n(a-b)-2b}{2\sqrt{\tau}}\right)$$
$$+ e^{\alpha(\alpha\tau+2n(a-b)-(x+\xi)+2a)} \operatorname{erfc}\left(\alpha\sqrt{\tau} + \frac{2n(a-b)-(x+\xi)+2a}{2\sqrt{\tau}}\right)\Bigg]\Bigg\} f(e^{\xi}) d\xi.$$

After replacing the variables x, ξ, and τ with S, η, and t, in accordance with (8.33)

$$x = \ln S, \quad \xi = \ln \eta, \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T-t),$$

we arrive at the solution $v(S,t)$ of the problem in (8.60)–(8.62)

$$v(S,t) = \int_{S_1}^{S_2} \left(\frac{S}{\eta}\right)^{\alpha} \exp\left(-\frac{\beta\sigma^2}{2}(T-t)\right) \Bigg\{ \frac{1}{\sigma\sqrt{2\pi(T-t)}}$$
$$\times \sum_{m=-\infty}^{\infty} \left[\exp\left(-\frac{(\ln \frac{SS_1^{2m}}{\eta S_2^{2m}})^2}{2\sigma^2(T-t)} \right) + \exp\left(-\frac{(\ln \frac{S_2^{2(m+1)}}{S\eta S_1^{2m}})^2}{2\sigma^2(T-t)} \right) \right]$$
$$-\alpha \exp\left(\frac{\alpha^2\sigma^2}{2}(T-t)\right) \sum_{n=0}^{\infty} \Bigg[\left(\frac{S\eta S_1^{2n}}{S_2^{2(n+1)}}\right)^{\alpha} \operatorname{erfc}\left(\alpha\sigma\sqrt{\frac{T-t}{2}} + \frac{\ln(\frac{S\eta S_1^{2n}}{S_2^{2(n+1)}})}{\sigma\sqrt{2(T-t)}}\right)$$
$$+ \left(\frac{S_1^{2(n+1)}}{S\eta S_2^{2n}}\right)^{\alpha} \operatorname{erfc}\left(\alpha\sigma\sqrt{\frac{T-t}{2}} + \frac{\ln\left(\frac{S_1^{2(n+1)}}{S\eta S_2^{2n}}\right)}{\sigma\sqrt{2(T-t)}}\right)\Bigg]\Bigg\} \frac{1}{\eta} f(\eta) d\eta.$$

Thus, the sought-after Green's function reads as

$$G(S,t;\eta) = \frac{S^{\alpha}}{\eta^{\alpha+1}} \exp\left(-\frac{\beta\sigma^2}{2}(T-t)\right) \Bigg\{ \frac{1}{\sigma\sqrt{2\pi(T-t)}}$$
$$\times \sum_{m=-\infty}^{\infty} \left[\exp\left(-\frac{\left(\ln \frac{SS_1^{2m}}{\eta S_2^{2m}}\right)^2}{2\sigma^2(T-t)} \right) + \exp\left(-\frac{\left(\ln \frac{S_2^{2(m+1)}}{S\eta S_1^{2m}}\right)^2}{2\sigma^2(T-t)} \right) \right]$$

$$-\alpha\exp\left(\frac{\alpha^2\sigma^2}{2}(T-t)\right)\sum_{n=0}^{\infty}\left[\left(\frac{S\eta S_1^{2n}}{S_2^{2(n+1)}}\right)^{\alpha}\operatorname{erfc}\left(\alpha\sigma\sqrt{\frac{T-t}{2}}+\frac{\ln(\frac{S\eta S_1^{2n}}{S_2^{2(n+1)}})}{\sigma\sqrt{2(T-t)}}\right)\right.$$

$$\left.\left.+\left(\frac{S_1^{2(n+1)}}{S\eta S_2^{2n}}\right)^{\alpha}\operatorname{erfc}\left(\alpha\sigma\sqrt{\frac{T-t}{2}}+\frac{\ln(\frac{S_1^{2(n+1)}}{S\eta S_2^{2n}})}{\sigma\sqrt{2(T-t)}}\right)\right]\right\}. \tag{8.77}$$

Since an estimation of the series convergence in the above formula does not look trivial, we might use numerical data to help decide whether this form is computer-friendly or not. To undertake this endeavor, we suggest, in Chapter Exercises, that the reader conduct a multi-parameter numerical experiment.

Example 8.6. Construct the Green's function of the homogeneous ($f(S)\equiv 0$) problem corresponding to

$$\frac{\partial v(S,t)}{\partial t}+\frac{\sigma^2S^2}{2}\frac{\partial^2 v(S,t)}{\partial S^2}+rS\frac{\partial v(S,t)}{\partial S}-rv(S,t)=0, \tag{8.78}$$

$$v(S,T)=f(S), \tag{8.79}$$

$$\frac{\partial v(D,t)}{\partial S}-\gamma v(D,t)=0 \quad\text{and}\quad \lim_{S\to\infty}|v(S,t)|<\infty \tag{8.80}$$

on the region $\Omega=(D<S<\infty)\times(T>t>-\infty)$.

Since all stages of the derivation procedure, which we will in this case apply, have been explained earlier, our presentation is limited to a schematic description. After the customary change of variables introduced in (8.33), the problem in (8.78)–(8.80) reduces to the initial-boundary-value problem

$$\frac{\partial u(x,\tau)}{\partial\tau}=\frac{\partial^2 u(x,\tau)}{\partial x^2}+(c-1)\frac{\partial u(x,\tau)}{\partial x}-cu(x,\tau), \tag{8.81}$$

$$u(x,0)=f(\mathrm{e}^x), \tag{8.82}$$

$$\frac{\partial u(a,\tau)}{\partial x}-\vartheta u(a,\tau)=0 \quad\text{and}\quad \lim_{x\to\infty}|u(x,\tau)|<\infty \tag{8.83}$$

in $u(x,\tau)$, where $c=2r/\sigma^2$, $a=\ln D$, and $\vartheta=\gamma D$.

The system in (8.81)–(8.83) gives rise, in turn, to the boundary-value problem

$$\frac{d^2U(x;s)}{dx^2}+(c-1)\frac{dU(x;s)}{dx}-(s+c)U(x;s)=-f(\mathrm{e}^x), \tag{8.84}$$

$$\frac{dU(a;s)}{dx}-\vartheta U(a;s)=0 \quad\text{and}\quad \lim_{x\to\infty}|U(x;s)|<\infty \tag{8.85}$$

for the Laplace transform $U(x;s)$ of $u(x,\tau)$. Using the method of variation of parameters, the solution of (8.84)–(8.85) is then found in the form

$$U(x;s) = \int_a^\infty \left\{ \frac{1}{2\omega}[\mathrm{e}^{\alpha(x-\xi)}\mathrm{e}^{-\omega|x-\xi|} - \mathrm{e}^{-\alpha(x+\xi)}\mathrm{e}^{-\omega(x+\xi-2a)}] \right. \\ \left. + \frac{\mathrm{e}^{-\alpha(x+\xi)}\mathrm{e}^{-\omega(x+\xi-2a)}}{\omega + (\vartheta - \alpha)} \right\} f(\mathrm{e}^\xi)d\xi, \tag{8.86}$$

where, as we recall, ω is defined through the parameter s of the Laplace transform as $\omega = \sqrt{s+\beta}$, whilst α and β are expressed in terms of the parameter c as

$$\alpha = \frac{1-c}{2} \quad \text{and} \quad \beta = \left(\frac{1+c}{2}\right)^2.$$

Clearly, the inverse Laplace transform in (8.86) represents the solution of (8.81)–(8.83). It can be found directly by using the relations in (8.73), (8.75) together with the Translation Theorem, yielding

$$u(x,\tau) = \int_a^\infty \mathrm{e}^{-\beta\tau} \left\{ \frac{\mathrm{e}^{\alpha(x-\xi)}}{2\sqrt{\pi\tau}}\mathrm{e}^{-\frac{(x-\xi)^2}{4\tau}} + \frac{\mathrm{e}^{-\alpha(x+\xi)}}{2\sqrt{\pi\tau}}\mathrm{e}^{-\frac{(x+\xi-2a)^2}{4\tau}} \right. \\ - (\vartheta-\alpha)\mathrm{e}^{(\vartheta-\alpha)((\vartheta-\alpha)\tau+(x+\xi-2a))} \\ \left. \times \operatorname{erfc}\left((\vartheta-\alpha)\sqrt{\tau} + \frac{x+\xi-2a}{2\sqrt{\tau}}\right)\right\} f(\mathrm{e}^\xi)d\xi.$$

After replacing x, ξ, and τ with S, η, and t, respectively, in accordance with (8.33)

$$x = \ln S, \quad \xi = \ln\eta, \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T-t)$$

the solution $v(S,t)$ of the problem in (8.78)–(8.80) appears as

$$v(S,t) = \int_D^\infty \frac{1}{\eta}\exp\left(-\frac{\beta\sigma^2}{2}(T-t)\right)\left\{\frac{1}{\sigma\sqrt{2\pi(T-t)}}\right. \\ \times \left[\left(\frac{S}{\eta}\right)^\alpha \exp\left(-\frac{(\ln(\frac{S}{\eta}))^2}{2\sigma^2(T-t)}\right) + \frac{1}{(S\eta)^\alpha}\exp\left(-\frac{(\ln(\frac{S\eta}{D^2}))^2}{2\sigma^2(T-t)}\right)\right] \\ - \frac{(\vartheta-\alpha)S\eta}{D^2}\mathrm{e}^{\frac{(\vartheta-\alpha)^2\sigma^2}{2}(T-t)} \\ \left. \times \operatorname{erfc}\left((\vartheta-\alpha)\sigma\sqrt{\frac{T-t}{2}} + \frac{\ln(\frac{S\eta}{D^2})}{\sigma\sqrt{2(T-t)}}\right)\right\} f(\eta)d\eta.$$

Hence, the Green's function for the homogeneous terminal boundary-value problem corresponding to that in (8.78)–(8.80) reads

$$\begin{aligned} G(S,t;\eta) = \frac{1}{\eta}\exp\Big(-\frac{\beta\sigma^2}{2}(T-t)\Big)\Bigg\{&\frac{1}{\sigma\sqrt{2\pi(T-t)}} \\ &\times\Bigg[\Big(\frac{S}{\eta}\Big)^{\alpha}\exp\Big(-\frac{(\ln(\frac{S}{\eta}))^2}{2\sigma^2(T-t)}\Big)+\frac{1}{(S\eta)^{\alpha}}\exp\Big(-\frac{(\ln(\frac{S\eta}{D^2}))^2}{2\sigma^2(T-t)}\Big)\Bigg] \\ &-\frac{(\vartheta-\alpha)S\eta}{D^2}\mathrm{e}^{\frac{(\vartheta-\alpha)^2\sigma^2}{2}(T-t)} \\ &\times\operatorname{erfc}\Bigg((\vartheta-\alpha)\sigma\sqrt{\frac{T-t}{2}}+\frac{\ln(\frac{S\eta}{D^2})}{\sigma\sqrt{2(T-t)}}\Bigg)\Bigg\}. \end{aligned} \tag{8.87}$$

In the Chapter Exercises, we challenge the reader to take a close look at two particular cases following from this problem. These are the cases when γ in (8.80) is either zero, reducing the first boundary condition to the Neumann type, or goes to infinity, in which case the condition reduces to the Dirichlet type.

8.3 A Methodologically Valuable Example

In this section, we will focus our attention on a particular case of an equation, encountered before in Section 8.2: if the parameter c in (8.15) becomes unity ($c = 1$), the equation reduces to

$$\frac{\partial u(x,\tau)}{\partial \tau} = \frac{\partial^2 u(x,\tau)}{\partial x^2} - u(x,\tau). \tag{8.88}$$

Consider the initial-boundary-value problem

$$u(x,0) = f(x), \tag{8.89}$$

$$u(0,\tau) = 0 \quad \text{and} \quad \lim_{x\to\infty}|u(x,\tau)| < \infty \tag{8.90}$$

posed for (8.88) on the quarter-plane $\Omega = (0 < x < \infty) \times (0 < \tau < \infty)$ of the (x,τ)-space.

The problem in (8.88)–(8.90) is probably not of direct interest to financial mathematics. It is, however, of methodological value with regard to the approach to the construction of Green's functions for the Black–Scholes equation, that we have implemented in this chapter.

If we apply the Laplace transform

$$U(x;s) = L\{u(x,\tau)\} = \int_0^\infty \mathrm{e}^{-s\tau}u(x,\tau)d\tau$$

to the system in (8.88)–(8.90), we arrive at the boundary-value problem

$$\frac{d^2U(x;s)}{dx^2} - (s+1)U(x;s) = -f(x), \tag{8.91}$$

$$U(0;s) = 0, \quad \lim_{x\to\infty} |U(x;s)| < \infty \tag{8.92}$$

in $U(x;s)$.

Proceeding in accordance with the method of variation of parameters, and expressing the general solution of (8.91) as

$$U(x;s) = A(x;s)\mathrm{e}^{x\sqrt{s+1}} + B(x;s)\mathrm{e}^{-x\sqrt{s+1}} \tag{8.93}$$

we arrive at

$$A'(x;s) = \frac{\mathrm{e}^{-x\sqrt{s+1}}}{2\sqrt{s+1}} f(x) \quad \text{and} \quad B'(x;s) = \frac{\mathrm{e}^{x\sqrt{s+1}}}{2\sqrt{s+1}} f(x).$$

Upon integrating the above equations, the functions $A(x;s)$ and $B(x;s)$ themselves are found in the form

$$A(x;s) = -\frac{1}{2\sqrt{s+1}} \int_0^x \mathrm{e}^{-\xi\sqrt{s+1}} f(\xi)d\xi + M(s)$$

and

$$B(x;s) = \frac{1}{2\sqrt{s+1}} \int_0^x \mathrm{e}^{\xi\sqrt{s+1}} f(\xi)d\xi + N(s).$$

Substitution of these into (8.93) yields

$$U(x;s) = \frac{1}{2\sqrt{s+1}} \int_0^x (\mathrm{e}^{(\xi-x)\sqrt{s+1}} - \mathrm{e}^{(x-\xi)\sqrt{s+1}}) f(\xi)d\xi + M(s)\mathrm{e}^{x\sqrt{s+1}} + N(s)\mathrm{e}^{-x\sqrt{s+1}}.$$

The ‘constants of integration’ $M(s)$ and $N(s)$ can be obtained by imposing the boundary conditions in (8.92). Omitting the details, we have

$$U(x;s) = \frac{1}{2\sqrt{s+1}} \int_0^x (\mathrm{e}^{(\xi-x)\sqrt{s+1}} - \mathrm{e}^{(x-\xi)\sqrt{s+1}}) f(\xi)d\xi + \frac{1}{2\sqrt{s+1}} \int_0^\infty (\mathrm{e}^{(x-\xi)\sqrt{s+1}} - \mathrm{e}^{-(x+\xi)\sqrt{s+1}}) f(\xi)d\xi$$

which, in compact single-integral form, reads as

$$U(x;s) = \frac{1}{2\sqrt{s+1}} \int_0^\infty (\mathrm{e}^{-|x-\xi|\sqrt{s+1}} - \mathrm{e}^{-(x+\xi)\sqrt{s+1}}) f(\xi)d\xi$$

representing the solution of the boundary-value problem in (8.91)–(8.92), which reveals the Green's function of the corresponding homogeneous problem as

$$G(x,s;\xi) = \frac{1}{2\sqrt{s+1}}(e^{-|x-\xi|\sqrt{s+1}} - e^{-(x+\xi)\sqrt{s+1}}).$$

Hence, Green's function $g(x,\tau;\xi)$ for the homogeneous initial-boundary-value problem corresponding to (8.88)–(8.90) can be obtained from $G(x,s;\xi)$ by performing the inverse Laplace transform, yielding the following compact form

$$g(x,\tau;\xi) = \frac{e^{-\tau}}{2\sqrt{\pi\tau}}(e^{-\frac{(x-\xi)^2}{4\tau}} - e^{-\frac{(x+\xi)^2}{4\tau}}). \tag{8.94}$$

Following the procedure as described, we find up with another Green's function associated with (8.88): in the case of the problem for the equation in (8.88) subject to the initial condition of (8.89), specified as

$$\frac{\partial u(0,\tau)}{\partial x} - \gamma u(0,\tau) = 0, \quad \gamma \geq 0, \quad \text{and} \quad \lim_{x\to\infty}|u(x,\tau)| < \infty, \tag{8.95}$$

the Green's function is found as

$$\begin{aligned} g(x,\tau;\xi) = {} & \frac{e^{-\tau}}{2\sqrt{\pi\tau}}(e^{-\frac{(x-\xi)^2}{4\tau}} + e^{-\frac{(x+\xi)^2}{4\tau}}) \\ & + \gamma e^{\gamma(x+\xi)}e^{(\gamma^2-1)\tau}\operatorname{erfc}\left(\gamma\sqrt{\tau} + \frac{x+\xi}{2\sqrt{\tau}}\right). \end{aligned} \tag{8.96}$$

Note that for the specific case of $\gamma = 0$, which transforms the condition imposed for $x = 0$ in (8.95) to the Neumann type, the above expression for the Green's function reduces to

$$g(x,\tau;\xi) = \frac{e^{-\tau}}{2\sqrt{\pi\tau}}(e^{-\frac{(x-\xi)^2}{4\tau}} + e^{-\frac{(x+\xi)^2}{4\tau}}). \tag{8.97}$$

Now, let equation (8.88) be considered in a different domain $\Omega = (0 < x < b) \times (0 < \tau < \infty)$, with boundary conditions imposed as

$$u(0,\tau) = 0, \quad \text{and} \quad u(b,\tau) = 0. \tag{8.98}$$

We will discuss two alternative approaches to constructing the Green's function for the homogeneous initial-boundary-value problem corresponding to (8.88), (8.89) and (8.98).

In the first one, we expand the functions $u(x,\tau)$ and $f(x)$ in the Fourier sine series

$$u(x,\tau) = \sum_{n=1}^{\infty} u_n(\tau)\sin\nu x, \quad f(x) = \sum_{n=1}^{\infty} f_n \sin\nu x \tag{8.99}$$

with $\nu = n\pi/b$. It is evident that, with such a series formula for $u(x,\tau)$, the boundary conditions in (8.98) are satisfied and, upon substituting the above series into (8.88) and (8.89) we arrive at the initial value problem

$$\frac{du_n(\tau)}{d\tau} + (\nu^2+1)u_n(\tau) = 0,$$
$$u_n(0) = f_n$$

for the coefficients of the first series in (8.99). From this statement, it follows that

$$u_n(\tau) = f_n \mathrm{e}^{-(\nu^2+1)\tau}.$$

Expressing f_n by means of the Fourier–Euler formula [66] for the coefficients of the second series in (8.99), we obtain

$$u_n(\tau) = \left(\frac{2}{b}\int_0^b f(\xi)\sin\nu\xi d\xi\right)\mathrm{e}^{-(\nu^2+1)\tau}.$$

After substituting $u_n(\tau)$ into (8.99), the solution to (8.88),(8.89) and (8.98) is found as

$$u(x,\tau) = \int_0^b \frac{2}{b}\sum_{n=1}^{\infty} \mathrm{e}^{-(\nu^2+1)\tau}\sin\nu x\sin\nu\xi f(\xi)d\xi$$

revealing the following series representation

$$g(x,\xi;\tau) = \frac{2}{b}\sum_{n=1}^{\infty} \mathrm{e}^{-(\nu^2+1)\tau}\sin\nu x\sin\nu\xi \tag{8.100}$$

for the Green's function for the homogeneous initial-boundary-value problem corresponding to (8.88),(8.89) and (8.98).

We will show that we can express the sum of the series in (8.100) in terms of a special function. In doing so, recall the series expansion [1, 3, 27, 66]

$$\vartheta_3(\alpha,\beta) = 1 + 2\sum_{n=1}^{\infty} \mathrm{e}^{-n^2\pi^2\beta}\cos 2n\pi\alpha \tag{8.101}$$

of the so-called *Jacobi Theta function of the third kind* which we used earlier in our book (see, for example, Chapter 7).

In order to apply the above series expansion to (8.100), we transform the latter as

$$\begin{aligned} g(x,\xi;\tau) &= \frac{\mathrm{e}^{-\tau}}{b}\sum_{n=1}^{\infty} \mathrm{e}^{-\nu^2\tau}[\cos\nu(x-\xi) - \cos\nu(x+\xi)] \\ &= \frac{\mathrm{e}^{-\tau}}{b}\left(\sum_{n=1}^{\infty} \mathrm{e}^{-\nu^2\tau}\cos\nu(x-\xi) - \sum_{n=1}^{\infty} \mathrm{e}^{-\nu^2\tau}\cos\nu(x+\xi)\right) \end{aligned}$$

from which, in light of (8.101), it follows that

$$\begin{aligned} g(x,\xi;\tau) &= \frac{e^{-\tau}}{b}\left\{\left[\vartheta_3\left(\frac{x-\xi}{2b},\frac{\tau}{b^2}\right)-1\right]-\left[\vartheta_3\left(\frac{x+\xi}{2b},\frac{\tau}{b^2}\right)-1\right]\right\} \\ &= \frac{e^{-\tau}}{b}\left[\vartheta_3\left(\frac{x-\xi}{2b},\frac{\tau}{b^2}\right)-\vartheta_3\left(\frac{x+\xi}{2b},\frac{\tau}{b^2}\right)\right]. \end{aligned} \tag{8.102}$$

We can recommend this compact form of the Green's function for the problem in (8.88), (8.89) and (8.98) for practical implementations, given that most mathematical software available nowadays includes standard subroutines for the Jacobi Theta function.

To get a formula for the Green's function under consideration, alternative to (8.100) and (8.102), we will use the Laplace integral transform, which aids us in converting the problem in (8.88), (8.89) and (8.98) to

$$\begin{aligned} &\frac{d^2U(x,s)}{dx^2}-(s+1)U(x,s)=-f(x), \\ &U(0,s)=0, \quad U(b,s)=0 \end{aligned}$$

for the Laplace transform $U(x,s)$ of $u(x,\tau)$. Obtaining the Green's function $G(x,s;\xi)$ for the homogeneous problem corresponding to the above one is a routine procedure. Since the problem is in self-adjoint form, we show only the branch of $G(x,s;\xi)$ valid for $x\le\xi$:

$$G(x,s;\xi)=\frac{(e^{x\sqrt{s+1}}-e^{-x\sqrt{s+1}})(e^{(b-\xi)\sqrt{s+1}}-e^{(\xi-b)\sqrt{s+1}})}{2\sqrt{s+1}(e^{b\sqrt{s+1}}-e^{-b\sqrt{s+1}})}$$

whilst the other branch of $G(x,s;\xi)$, valid for $x\ge\xi$, can be obtained from the above one by exchanging x and ξ.

To facilitate the inverse transform of $G(x,s;\xi)$, we interpret the factor

$$\frac{1}{e^{b\sqrt{s+1}}-e^{-b\sqrt{s+1}}}=\frac{e^{-b\sqrt{s+1}}}{1-e^{-2b\sqrt{s+1}}}$$

as the sum of a geometric series, with first term $e^{-b\sqrt{s+1}}$, whereas its common ratio is $e^{-2b\sqrt{s+1}}<1$, yielding

$$\frac{1}{e^{b\sqrt{s+1}}-e^{-b\sqrt{s+1}}}=e^{-b\sqrt{s+1}}\sum_{n=0}^{\infty}e^{-2nb\sqrt{s+1}}.$$

This formula for one of the factors of $G(x,s;\xi)$ allows us to express it in series form

$$G(x,s;\xi) = \frac{1}{2\sqrt{s+1}} \sum_{n=0}^{\infty} (e^{-(\xi - x + 2nb)\sqrt{s+1}} - e^{-(\xi + x + 2nb)\sqrt{s+1}}$$
$$e^{-(2(n+1)b - \xi - x)\sqrt{s+1}} + e^{-(2(n+1)b + x - \xi)\sqrt{s+1}}).$$

Taking the inverse Laplace transform of the above, we obtain the Green's function $g(x,\tau;\xi)$ for the homogeneous initial-boundary-value problem corresponding to (8.88), (8.89) and (8.98)

$$g(x,\tau;\xi) = L^{-1}\left\{G(x,s;\xi)\right\},$$

which reads as

$$\frac{e^{-\tau}}{2\sqrt{\pi\tau}} \sum_{n=0}^{\infty} (e^{-\frac{(\xi - x + 2nb)^2}{4\tau}} - e^{-\frac{(\xi + x + 2nb)^2}{4\tau}} - e^{-\frac{(2(n+1)b - \xi - x)^2}{4\tau}} + e^{-\frac{(2(n+1)b - \xi + x)^2}{4\tau}}).$$

The above looks highly cumbersome. However, by combining the exponential functions and rearranging the summation, we arrive at its compact equivalent

$$g(x,\tau;\xi) = \frac{e^{-\tau}}{2\sqrt{\pi\tau}} \sum_{m=-\infty}^{\infty} (e^{-\frac{(\xi - x + 2mb)^2}{4\tau}} - e^{-\frac{(\xi + x + 2mb)^2}{4\tau}}). \tag{8.103}$$

Hence, (8.103) displays an alternative to (8.100) and it turns out that these two expressions are not equally suitable for computer implementation. In the section that follows, in order to illustrate this point, we will compare the computational potential of equations (8.100) and (8.103). In addition we will provide several numerical illustrations of the computer readiness of the Green's functions that we have derived in Sections 8.2 and 8.3 for a variety of terminal-boundary-value problems posed for the Black–Scholes equation.

8.4 Numerical Implementations

This section turns away from theoretical aspects; it intends to explore the computational potential of numerical procedures, which might use those Green's functions of the Black–Scholes equation that we have constructed earlier in this chapter, or those that could be constructed by following the various approaches described herein. Earlier in this book, we pointed out that Green's function-based numerical procedures should be highly efficient.

To justify the computational efficiency of the Green's function-based approach to terminal-boundary-value problems for the Black–Scholes equation, we note that, in

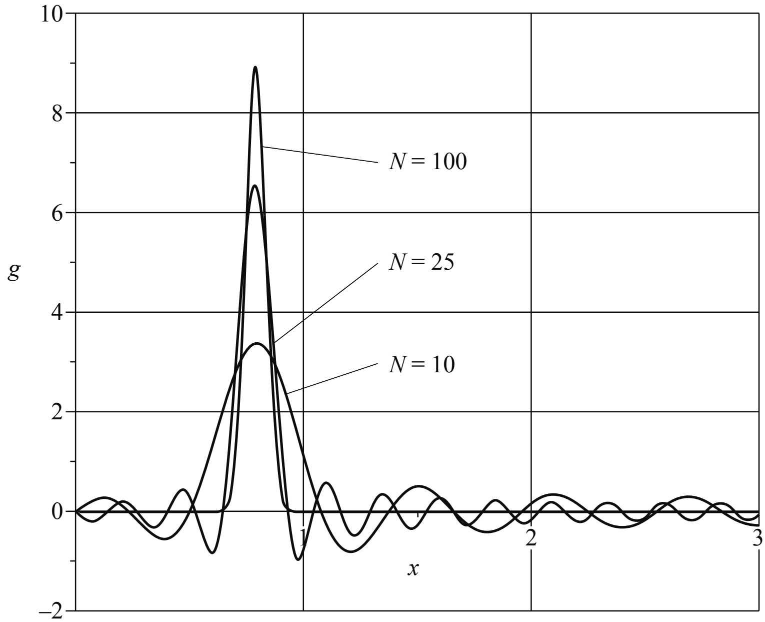

Figure 8.1. Convergence of the series in (8.100).

contrast to most of the purely numerical methods that are commonly utilized in the field (the finite difference methods, for example) [11, 34, 36, 58], the Green's function approach is well posed from a computational point of view. To support this point, we recall that the approach does not utilize numerical differentiation procedures, which are themselves quite expensive. Instead, it utilizes numerical integration routines, which are computer-friendly in nature.

Let us first recall two alternative formulas for the Green's function for the problem stated in (8.88), (8.89), and (8.98), obtained earlier in Section 8.3. These are the forms displayed in (8.100) and (8.103). It is evident that the convergence of the series representation in (8.100) strongly depends upon the variable τ, because the series converges at a slow rate for relatively *small* values of τ, implying that, in such cases, a high accuracy level is only attainable if a high order partial sum of the series is requested. To illustrate this point, we depict profiles of some Nth order partial sums of the series in (8.100) in Figure 8.1, with $b = \pi$, $\tau = 10^{-3}$ and $\xi = \pi/4$, for $N = 10, 25$, and 100.

To get a sense of the rate of convergence of the series in (8.103) and to compare it with that of the series in (8.100), we leave it as an exercise to the reader to study Figure 8.2, where we present convincing evidence for the efficiency of the formula in (8.103) for both relatively *small* and *large* values of τ. In Figure 8.2 we used the same defining parameters as in the case illustrated by Figure 8.1. That is, $b = \pi$ and

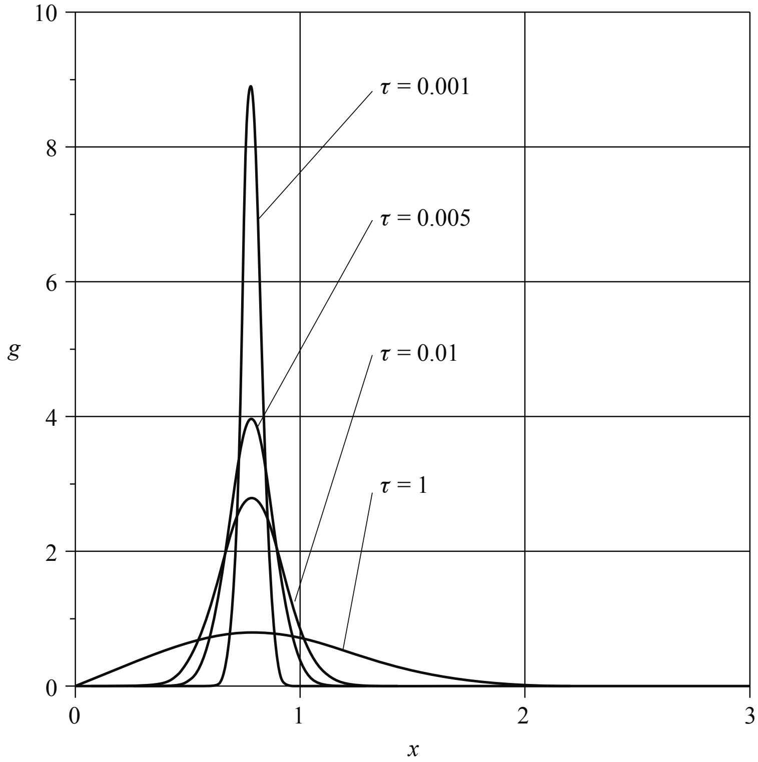

Figure 8.2. Convergence of the series in (8.103).

$\xi = \pi/4$, with the values of τ fixed as 10^{-3}, 5×10^{-3}, 10^{-2}, and 10^{-1}. The series was, in all cases, truncated to the 10th partial sum.

Hence, after analyzing the data displayed in Figures 8.1 and 8.2, we may conclude that of the two alternative representations for the Green's function to the problem posed in (8.88), (8.89), and (8.98), obtained in this section, the one in (8.103) is independent of the value of τ, whereas the formula in (8.100) is only recommended for relatively *large* values of τ.

In order to test the computability of the series representation for the Green's functions obtained in Section 8.2 (see the form in (8.42)), we turn to the next illustration, corresponding to the setting in (8.2)–(8.4). This is a terminal boundary-value problem stated for the Black–Scholes equation on the region $\Omega = (S_1 < S < S_2) \times (T > t > -\infty)$. As we showed earlier in Section 8.1, the solution to this problem is displayed in (8.5), with the component $v(S, t)$ expressed in the integral form of (8.9), the kernel $G(S, t; \eta)$ of which represents the Green's function to the homogeneous problem corresponding to that in (8.28)–(8.30) displayed in (8.42). It does not seem computer-friendly, because it is quite cumbersome and contains a series component.

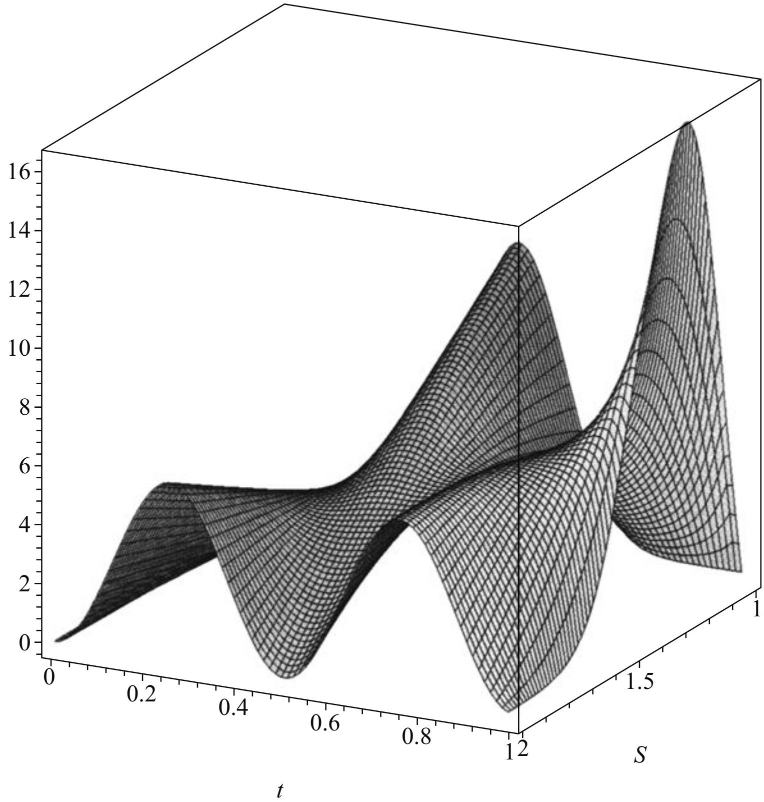

Figure 8.3. Solution of (8.2)–(8.4) with a non-smooth pay-off function.

However, with regard to the latter, the multi-parameter experiment, as described earlier in Section 8.2, reveals a high rate of convergence for the series. This promises that the formula in (8.42), as a whole, will be really computer-ready. The solution profile depicted in Figure 8.3 indeed confirms our high expectation. The parameters that specify the problem setting were chosen as: $r = 0.06$, $\sigma = 0.8$, $S_1 = 1.0$, $S_2 = 2.0$, and $T = 1.0$, the series component of $G(S, t; \eta)$ was truncated at $M = 5$, the right-hand sides $B_1(t)$ and $B_2(t)$ of the boundary conditions in (8.4) have been chosen as the following differentiable functions

$$B_1(t) = 10 \exp(-50(t - T/2)^2) \quad \text{and} \quad B_2(t) = 6 \sin^2(2\pi t/T),$$

whilst the right-hand side $F(S, t)$ of the governing equation in (8.2) and the pay-off function $\varphi(S)$ in (8.3) were chosen as

$$F(S, t) = S \;\text{ and }\; \varphi(S) = \begin{cases} 300(S - S_1)[(S_1 + S_2)/2 - S], & S \leq (S_1 + S_2)/2, \\ 0, & S > (S_1 + S_2)/2. \end{cases}$$

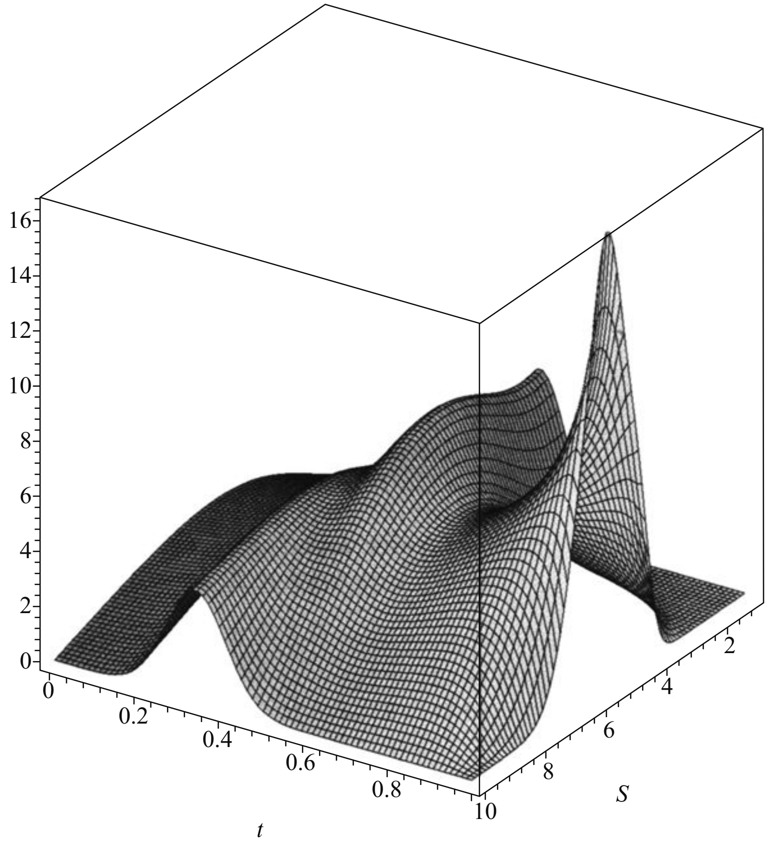

Figure 8.4. The case of a piecewise smooth pay-off function.

Notice that the chosen pay-off function loses its differentiability at the midpoint of the interval $[S_1, S_2]$, but the solution depicted in Figure 8.3 does not show any computational irregularity in the immediate vicinity of that point (like, for instance, abnormal oscillation), which would be unavoidable for a custom numerical procedure based on, for example, the method of finite differences. This is especially impressive given that the numerical integration for computing both the integrals in (8.9) was accomplished with the aid of the most primitive quadrature method (the standard trapezoid rule), where we used only ten quadrature nodes for both the variables of integration.

In the next illustration, we now also aim for a numerical solution of the terminal-boundary-value problem stated in (8.2)–(8.4). The region $\Omega = (S_1 < S < S_2) \times (T > t > -\infty)$ is, in this case, specified by $S_1 = 1$, $S_2 = 10$, and $T = 1.0$. The other parameters specifying the problem were chosen as: $r = 0.06$, $\sigma = 0.8$. The series component of $G(S, t; \eta)$ of (8.42) was truncated at $M = 5$. The right-hand sides $B_1(t)$ and $B_2(t)$ of the boundary conditions in (8.4) have been chosen as the following rapidly varying but still differentiable functions

$$B_1(t) = 6\sin^{24}(\pi t/T) \quad \text{and} \quad B_2(t) = 4\exp(-100(t - T/3)^2).$$

The right-hand side $F(S,t)$ of the governing equation in (8.2) and the pay-off function $\varphi(S)$ in (8.3) were chosen as

$$F(S,t) \equiv 0 \quad \text{and} \quad \varphi(S) = \begin{cases} 0, & S_1 \le S < E_1, \\ 20(S - E_1)/(E_2 - E_1), & E_1 \le S < E_2, \\ 20(S - E_3)/(E_2 - E_3), & E_2 \le S < E_3, \\ 0, & E_3 \le S < S_2, \end{cases} \quad (8.104)$$

where

$$E_1 = (2S_1 + S_2)/3, \quad E_2 = (S_1 + S_2)/2, \quad \text{and} \quad E_3 = (2S_2 + S_1)/3.$$

Clearly, the above problem setting looks even more challenging with regard to numerical treatment than the one depicted in Figure 8.3, because the pay-off function in (8.104) looses its differentiability at multiple points (E_1, E_2,and E_3) on the interval $[S_1, S_2]$. The solution profile depicted in Figure 8.4 is perfectly smooth and does not show any signs of oscillation generated by the irregularity of the pay-off function.

The case to which the solution is depicted in Figure 8.5 provides another illustration of high computational potential of the Green's function approach to terminal boundary-value problems, stated for the Black–Scholes equation. We turn again to the problem specified in (8.2)–(8.4) with $r = 0.06$, $\sigma = 0.8$, $S_1 = 10$, $S_2 = 50$, and $T = 1.0$. Additionally, we truncated the series component of the Green's function $G(S,t;\eta)$ in (8.42) at $M = 5$. The right-hand sides $B_1(t)$ and $B_2(t)$ of the boundary conditions in (8.4) have been chosen as the following quite rapidly varying but still differentiable functions

$$B_1(t) = 20\sin^{8}(\pi t/(2T)) \quad \text{and} \quad B_2(t) = 15t^2\sin^2(2\pi t/T).$$

The right-hand side $F(S,t)$ of the governing equation in (8.2) and the pay-off function $\varphi(S)$ in (8.3) were chosen as

$$F(S,t) \equiv 0 \quad \text{and} \quad \varphi(S) = \begin{cases} 20, & S_1 \le S < E_1 \\ 0, & E_1 \le S < S_2 \end{cases},$$

with E_1 being the midpoint of the interval $[S_1, S_2]$. Notice that this case is, without any doubt, the most challenging of all that we considered so far, because the pay-off function $\varphi(S)$ in this setup is discontinuous, but the smoothness of the function in Figure 8.5 is never affected by this discontinuity.

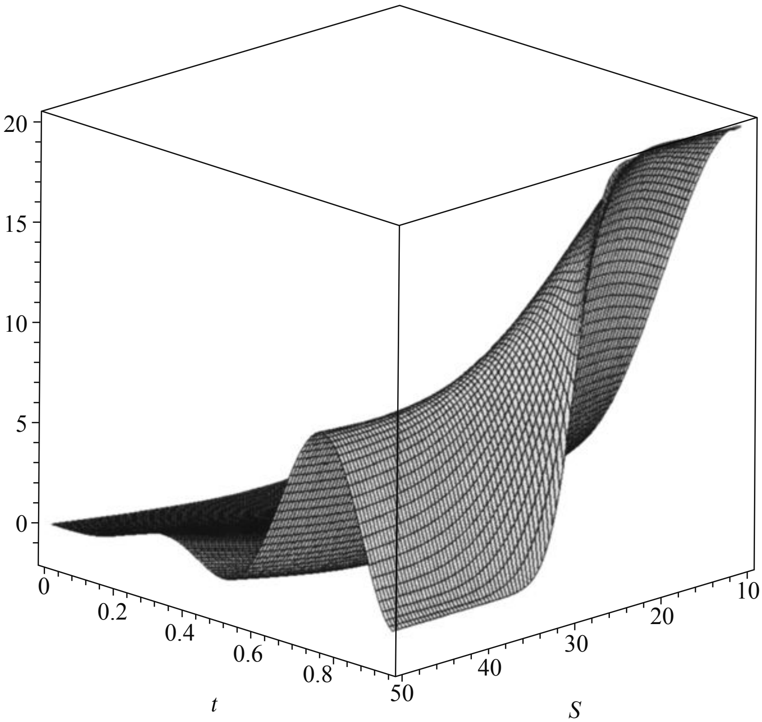

Figure 8.5. The case of a discontinuous pay-off function.

To further explore the computational potential of the Green's function method as applied to terminal-boundary-value problems with irregularities, stated for the Black–Scholes equation, we present the last of our illustrations, describing the problem

$$\frac{\partial v(S,t)}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 v(S,t)}{\partial S^2} + rS\frac{\partial v(S,t)}{\partial S} - rv(S,t) = 0, \tag{8.105}$$

$$v(S,T) = \varphi(S), \tag{8.106}$$

$$|v(0,t)| < \infty \quad \text{and} \quad \frac{\partial v(D,t)}{\partial S} = 0 \tag{8.107}$$

stated in S,t space for the region $\Omega = (0 < S < D) \times (T > t > -\infty)$, with D a positive constant.

The solution to the above problem is

$$v(S,t) = \int_0^D G(S,t;\eta)\varphi(\eta)d\eta, \tag{8.108}$$

with the kernel $G(S,t;\eta)$ representing the Green's function obtained earlier in Section 8.2 (see equation (8.56)).

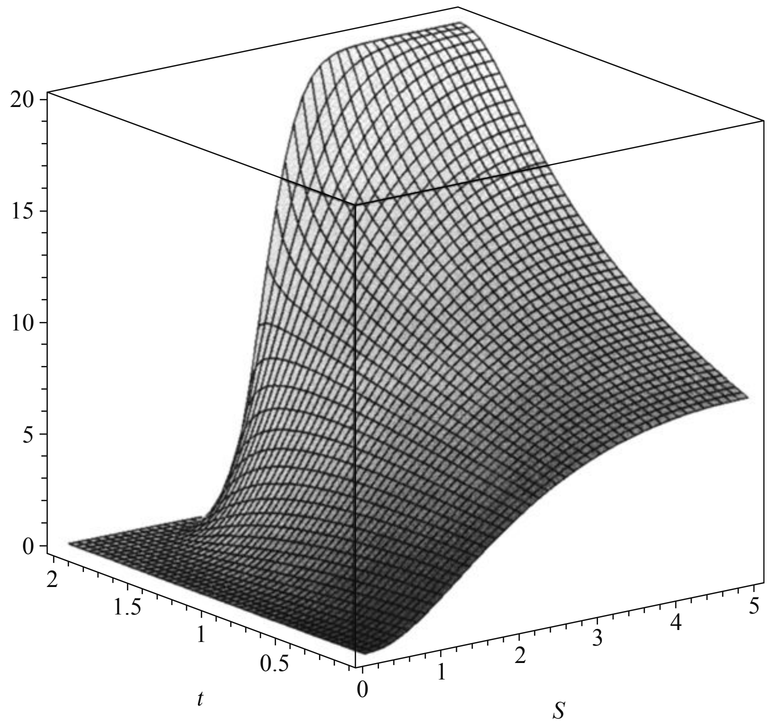

Figure 8.6. Solution profile of the problem in (8.105)–(8.107).

To be specific, we chose the following parameters for the problem in (8.105)–(8.107): $r = 0.06$, $\sigma = 0.8$, $D = 5$, and $T = 2$, whilst the terminal condition is imposed with the discontinuous pay-off function

$$\varphi(S) = \begin{cases} 0, & 0 \leq S < D/2, \\ 20, & D/2 \leq S < D, \end{cases}$$

having the jump of discontinuity at the midpoint of the interval $[0, D]$.

As usual, the standard trapezoid rule routine, with as little as ten quadrature nodes, was implemented in order to evaluate the integral in (8.108) accurately, for all points on Ω.

Note that similar to the case considered in the previous illustration and shown in Figure 8.5, the solution profile depicted in Figure 8.6 is also perfectly smooth, despite the sharp discontinuity of the pay-off function. Indeed, no visible oscillation or other irregularity is observed.

Hence, from the series of illustrations presented in this section, we can reasonably conclude that the Green's function-based method reveals quite promising computational potential for solving terminal-boundary-value problems for the Black–Scholes equation. The method provides the user with highly accurate numerical solutions, even for problem settings involving non-smooth and even discontinuous components. We must especially emphasize, that we applied no special tricks to suppress possible numerical oscillations of the solution in one way or the other. This reveals an evident regularizing feature of the Green's function method, which completely avoids numerical differentiation and only requires numerical integration in order to compute the solution.

8.5 Chapter Exercises

1. Use the approach developed in this chapter to derive the Green's function for the terminal-boundary-value problem stated in (8.44), (8.45) and (8.55).
2. Use the approach developed in this chapter to derive the Green's function for the terminal-boundary-value problem stated in (8.44), (8.45) and (8.57).
3. Conduct a multi-parameter numerical experiment to explore the convergence of the series formula for the Green's function, which we derived in (8.77).
4. Figure out the formula, to which the Green's function in (8.87) reduces, if the parameter γ in the first of the boundary conditions of (8.80) is either equal to zero or going to infinity.

Appendix
Answers to Chapter Exercises

Chapter 1

1. (a) Yes.
   (b) Yes.
   (c) Yes.
   (d) Yes.
   (e) Yes.
   (f) Yes.
   (g) No, because any function $y = \text{const}$ represents a solution.
   (h) No, because any linear function $y = Cx$, with C an arbitrary constant, is a solution.
   (i) Yes.
   (j) Yes.
   (k) Yes.
   (l) No, because any linear function $y = Cx + D$, with C and D arbitrary constants, is a solution.
2. (a) $g(x,s) = \begin{cases} -x, & \text{for } x \leq s, \\ -s, & \text{for } x \geq s. \end{cases}$

   (b) $g(x,s) = \dfrac{1}{1+ha} \begin{cases} x[h(s-a)-1], & \text{for } x \leq s, \\ s[h(x-a)-1], & \text{for } x \geq s. \end{cases}$

   For $h = 0$, the boundary conditions in this problem reduce to those of part a). This consequently reduces the above Green's function to that of part a).

   (c) $g(x,s) = \dfrac{1}{H} \begin{cases} (1+h_1x)[h_2(s-a)-1], & \text{for } x \leq s, \\ (1+h_1s)[h_2(x-a)-1], & \text{for } x \geq s, \end{cases}$

   with $H = h_2 + h_1(1 + h_2a)$. for $h_1 = 0$, the present Green's function reduces to that of Example 1.3 of Section 1.1.

   (d) $g(x,s) = \dfrac{1}{M} \begin{cases} \ln[(ms+p)/(ma+p)]\ln[(mx+p)/p], & x \leq s, \\ \ln[(mx+p)/(ma+p)]\ln[(ms+p)/p], & x \geq s, \end{cases}$

   with $M = m\ln[(ma+p)/p]$.

(e) $$g(x,s) = \frac{1}{\beta(e^{-\beta a}-1)} \begin{cases} (e^{-\beta x}-1)(e^{-\beta a}-e^{-\beta s}), & \text{for } x \le s, \\ (e^{-\beta s}-1)(e^{-\beta a}-e^{-\beta x}), & \text{for } x \ge s. \end{cases}$$

(f) $$g(x,s) = \frac{1}{\beta} \begin{cases} e^{-\beta x}-1, & \text{for } x \le s, \\ e^{-\beta s}-1, & \text{for } x \ge s. \end{cases}$$

(g) $$g(x,s) = \frac{1}{k\sin(ka)} \begin{cases} \sin k(a-s)\sin kx, & \text{for } x \le s, \\ \sin k(a-x)\sin ks, & \text{for } x \ge s. \end{cases}$$

(h) $$g(x,s) = \frac{1}{\Delta} \begin{cases} x^2[ks^2(3a-x)(3a-s)-2(3s-x)(6+ka^3)], \\ \qquad \text{for } x \le s, \\ s^2[kx^2(3a-s)(3a-x)-2(3x-s)(6+ka^3)], \\ \qquad \text{for } x \ge s, \end{cases}$$

with $\Delta = 12(3+ka^3)$.

(i) $$g(x,s) = \frac{1}{12a} \begin{cases} x^2(3s^2+2ax-6as), & \text{for } x \le s, \\ s^2(3x^2+2as-6ax), & \text{for } x \ge s. \end{cases}$$

(j) $$g(x,s) = \frac{1}{\Delta} \begin{cases} x^2[2(x-3s)(1+ka)+3ks^2], & \text{for } x \le s, \\ s^2[2(s-3x)(1+ka)+3kx^2], & \text{for } x \ge s, \end{cases}$$

with $\Delta = 12(1+ka)$.

(k) $$g(x,s) = \frac{1}{12a} \begin{cases} x^2(3s^2-6as+2ax)-1/k, & \text{for } x \le s, \\ s^2(3x^2-6ax+2as)-1/k, & \text{for } x \ge s. \end{cases}$$

(l) $$g(x,s) = \frac{1}{6a^3} \begin{cases} x^2(s-a)^2[2s(x-a)+a(x-s)], & \text{for } x \le s, \\ s^2(x-a)^2[2x(s-a)+a(s-x)], & \text{for } x \ge s. \end{cases}$$

3. (a) Yes.
   (b) Yes.
   (c) No, the condition for self-adjointness is not satisfied.
   (d) No, the condition for self-adjointness is not satisfied.
   (e) Yes.
4. (a) The integrating factor e^{-2x} reduces the given equation to the following self-adjoint form:

$$e^{-2x}y''(x) - 2e^{-2x}y'(x) + 4e^{-2x}y = 0.$$

   (b) The integrating factor $e^{x^2/2}$ reduces the given equation to the following self-adjoint form:

$$e^{x^2/2}y''(x) + xe^{x^2/2}y'(x) - x^2e^{x^2/2}y(x) = 0.$$

(c) The integrating factor x^{-3} reduces the given equation to the following self-adjoint form:

$$x^{-1}y''(x) - x^{-2}y'(x) + x^{-3}y(x) = 0.$$

(d) The integrating factor x^{-1} reduces the given equation to the following self-adjoint form:

$$xy''(x) + y'(x) - x^{-1}y(x) = 0.$$

5. (a) Yes.
   (b) Yes.
   (c) Yes.
   (d) Yes.
   (e) Yes.
   (f) No.
6. After applying the integrating factor e^{3x}, the original boundary-value problem, the Green's function of which

$$g(x,s) = \frac{1}{7}\begin{cases} e^{-(5x+2s)} - e^{2(x-s)}, & \text{for } x \leq s, \\ e^{-(5x+2s)} - e^{-5(x-s)}, & \text{for } x \geq s, \end{cases}$$

is asymmetric, and which is not self-adjoint, reduces to the following self-adjoint form

$$e^{3x}y''(x) + 3e^{3x}y'(x) - 10e^{3x}y(x) = 0, \quad y(0) = 0, \quad y(\infty) < \infty,$$

the Green's function of which

$$g(x,s) = \frac{1}{7}\begin{cases} e^{-2(x+s)} - e^{5x-2s}, & \text{for } x \leq s, \\ e^{-2(x+s)} - e^{5s-2x}, & \text{for } x \geq s, \end{cases}$$

turns out to be symmetric.
7. The Green's function is found as

$$g(x,s) = \frac{1}{k\cos k}\begin{cases} \sin kx \cos k(s-1), & \text{for } x \leq s, \\ \sin ks \cos k(x-1), & \text{for } x \geq s. \end{cases}$$

The construction procedure, which uses a *standard* fundamental set of solutions (in this case it is composed of the functions: $y_1 = \sin kx$ and $y_2 = \cos kx$), turns out to be more cumbersome in comparison with the procedure that uses the *special* fundamental set of solutions: $y_1(x) = \sin kx$ and $y_2(x) = \cos k(x-1)$. This occurs, because the latter procedure does not imply direct satisfaction of the boundary conditions as part of the construction of the Green's function. When the *special* fundamental set of solutions is obtained, the boundary conditions are taken care of in advance.

8. (a) $g(x,s) = \begin{cases} -\ln(s/a), & \text{for } x \le s, \\ -\ln(x/a), & \text{for } x \ge s. \end{cases}$

(b) $g(x,s) = \dfrac{1}{k\Delta} \begin{cases} (e^{kx} + \lambda e^{-kx}) \sinh k(s-a), & \text{for } x \le s, \\ (e^{ks} + \lambda e^{-ks}) \sinh k(x-a), & \text{for } x \ge s, \end{cases}$

with: $\lambda = (k-h)/(k+h)$ and $\Delta = e^{ka} + \lambda e^{-ka}$.

(c) $g(x,s) = \dfrac{1}{6a^3} \begin{cases} x^2(s-a)^2[2s(x-a) + a(x-s)], & \text{for } x \le s, \\ s^2(x-a)^2[2x(s-a) + a(s-x)], & \text{for } x \ge s. \end{cases}$

(d) $g(x,s) = \dfrac{1}{\Delta} \begin{cases} x[3s(s-2a)(2+kx) + 2x^2(1+ka)], & \text{for } x \le s, \\ s[3x(x-2a)(2+ks) + 2s^2(1+ka)], & \text{for } x \ge s, \end{cases}$

with $\Delta = 12(1+ka)$.

(e) $g(x,s) = \dfrac{1}{6} \begin{cases} (s-a)^2[2(x-a) + (x-s)], & \text{for } x \le s, \\ (x-a)^2[2(s-a) + (s-x)], & \text{for } x \ge s. \end{cases}$

(f) $g(x,s) = \dfrac{1}{\Delta} \begin{cases} x[ka(a-s)(x^2+s^2-2as) - 6s], & \text{for } x \le s, \\ s[ka(a-x)(x^2+s^2-2ax) - 6x], & \text{for } x \ge s, \end{cases}$

with $\Delta = 6ka^2$.

(g) $g(x,s) = \dfrac{1}{12a} \begin{cases} x^2(3s^2 - 6as + 2ax) - 1/k, & \text{for } x \le s, \\ s^2(3x^2 - 6ax + 2as) - 1/k, & \text{for } x \ge s. \end{cases}$

(h) $g(x,s) = \dfrac{1}{6} \begin{cases} x[(3s^2 + x^2 - 6as) - 6a/k], & \text{for } x \le s, \\ s[(3x^2 + s^2 - 6ax) - 6a/k], & \text{for } x \ge s. \end{cases}$

9. (a) $y(x) = \dfrac{2\cos a + \sin a}{2\sin a - \cos a} \sin x - \cos x + e^{2x}$.

(b) $y(x) = (a-1)(a-3)e^a \cosh x - 2x - 3$.

(c) $y(x) = \dfrac{\sin a}{a}(1+x)e^{a-x} - \cos x$.

(d) $y(x) = \dfrac{1}{120}x^2(x^4 - 4a^3x + 3a^4)$.

(e) $y(x) = 6x^5 - \dfrac{2a^2}{a+1}\left[2x^2(3a+5) - a^2(3a+7)\right]$.

(f) $y(x) = \cos(\dfrac{\pi x}{2a}) - \frac{\pi x^2}{96a^3}\left[2\pi^2 x - 3a(8+\pi^2)\right] - 1$.

(g) $y(x) = \dfrac{1}{2}(x^3 + 3x^2 + 6x + 6)e^{-x}$.

Chapter 2

In the answers to Exercises 6 through 9, for compactness, complex variable notation is used for the observation and the source point. These read $z = x + iy$ and $\zeta = \xi + i\eta$, respectively.

6. $$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{|1 - e^{\omega(z-\overline{\zeta})}||1 + e^{\omega(z-\zeta)}|}{|1 - e^{\omega(z-\zeta)}||1 + e^{\omega(z-\overline{\zeta})}|}, \quad \omega = \frac{\pi}{2b}.$$

7. $$G(x, y; \xi, \eta) = \frac{1}{2\pi}\left(\ln \frac{|1 - e^{\omega(z-\overline{\zeta})}||1 + e^{\omega(z-\zeta)}|}{|1 - e^{\omega(z-\zeta)}||1 + e^{\omega(z-\overline{\zeta})}|} + \ln \frac{|1 - e^{\omega(z+\overline{\zeta})}||1 + e^{\omega(z+\zeta)}|}{|1 - e^{\omega(z+\zeta)}||1 + e^{\omega(z+\overline{\zeta})}|}\right), \quad \omega = \frac{\pi}{2b}.$$

8. $$G(x, y; \xi, \eta) = \frac{1}{2\pi}\left(\ln \frac{|1 - e^{\omega(z-\overline{\zeta})}||1 + e^{\omega(z-\zeta)}|}{|1 - e^{\omega(z-\zeta)}||1 + e^{\omega(z-\overline{\zeta})}|} + \ln \frac{|1 + e^{\omega(z+\overline{\zeta})}||1 - e^{\omega(z+\zeta)}|}{|1 + e^{\omega(z+\zeta)}||1 - e^{\omega(z+\overline{\zeta})}|}\right), \quad \omega = \frac{\pi}{2b}.$$

9. $$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{|1 - e^{\omega(z-\overline{\zeta})}||1 - e^{\omega(z+\overline{\zeta})}|}{|1 - e^{\omega(z-\zeta)}||1 - e^{\omega(z+\zeta)}|} - \frac{2}{b} \sum_{n=1}^{\infty} \frac{(\beta - \nu) \sinh \nu x \sinh \nu \xi}{\nu e^{\nu a}(\beta \sinh \nu a + \nu \cosh \nu a)} \sin \nu y \sin \nu \eta,$$
with $\omega = \pi/b$ and $\nu = n\omega$.

In the answers to Exercises 10 through 14, we accept the complex variable notation $z = r(\cos\varphi + i \sin\varphi)$ and $\zeta = \varrho(\cos\psi + i \sin\psi)$ for the observation and the source point, respectively.

10. $$G(r, \varphi; \varrho, \psi) = \frac{1}{2\pi}\left\{ \ln \frac{|a^2 - z\overline{\zeta}||b^2 z - a^2\zeta|}{|z - \zeta||b^2 - z\overline{\zeta}|} + K_0(r, \varrho) + \sum_{n=1}^{\infty} K_n(r, \varrho) \cos n(\varphi - \psi)\right\}, \quad r \le \varrho,$$
with

$$K_0(r, \varrho) = -\ln r + \frac{1 + b\beta \ln(b/\varrho)}{1 + b\beta \ln(b/a)} \ln \frac{r}{a},$$

$$K_n(r, \varrho) = (r^{2n} - a^{2n}) \times \frac{na^{2n}(b^{2n} + \varrho^{2n}) + b\beta[\varrho^{2n}(b^{2n} - a^{2n}) + b^{2n}(\varrho^{2n} - a^{2n})]}{n(b^2 r\varrho)^2[n(b^{2n} + a^{2n}) + b\beta(b^{2n} - a^{2n})]},$$

and for $r \geq \varrho$, the variables r and ϱ must be exchanged in $K_0(r, \varrho)$ and $K_n(r, \varrho)$, whilst the factor $|b^2 z - a^2 \zeta|$ must, in this case be replaced by $|a^2 z - b^2 \zeta|$, in the logarithmic component of $G(r, \varphi; \varrho, \psi)$.

11. $$G(r, \varphi; \varrho, \psi) = \frac{1}{2\pi} \left\{ \ln \frac{|b^2 - z\overline{\zeta}| \left| b^2 z - a^2 \zeta \right|}{|z - \zeta| \, |a^2 - z\overline{\zeta}|} + K_0(r, \varrho) + \sum_{n=1}^{\infty} K_n(r, \varrho) \cos n(\varphi - \psi) \right\}, \quad r \leq \varrho,$$

with

$$K_0(r, \varrho) = \frac{a\beta \ln(b/r) \ln(\varrho/b)}{1 + a\beta \ln(b/a)},$$
$$K_n(r, \varrho) = a^{2n}(\varrho^{2n} - b^{2n}) \times \frac{n(r^{2n} + a^{2n}) + a\beta[(b^{2n} - a^{2n}) + (b^{2n} - r^{2n})]}{n(b^2 r\varrho)^2[n(b^{2n} + a^{2n}) + a\beta(b^{2n} - a^{2n})]},$$

and for $r \geq \varrho$, the variables r and ϱ must be exchanged in $K_0(r, \varrho)$ and $K_n(r, \varrho)$, whilst the factor $\left| b^2 z - a^2 \zeta \right|$ must, in this case, be replaced by $\left| a^2 z - b^2 \zeta \right|$, in the logarithmic component of $G(r, \varphi; \varrho, \psi)$.

12. $$G(r, \varphi; \varrho, \psi) = \frac{1}{2\pi} \left\{ \ln \frac{|z - \zeta| \, |a^2 - z\overline{\zeta}| |b^2 - z\overline{\zeta}| \left| b^2 z - a^2 \zeta \right|}{(b^2 r\varrho)^2} + K_0(r, \varrho) + 2 \sum_{n=1}^{\infty} [K_n(r, \varrho) + K_n^*(r, \varrho)] \cos n(\varphi - \psi) \right\}, \quad r \leq \varrho,$$

with

$$K_0(r, \varrho) = \frac{1 - b\beta \ln(\varrho/b)}{b\beta},$$
$$K_n^*(r, \varrho) = \frac{b\beta}{2n^3(b^2 r\varrho)^n} \left\{ n[(rb)^{2n} - (a\varrho)^{2n})] - [(r\varrho)^{2n} - (ab)^{2n})] \right\},$$
$$K_n(r, \varrho) = [(n - b\beta)a^{2n} - b^{2n+1}\beta] \times \frac{(r^{2n} + a^{2n})[(n + b\beta)b^{2n} + (n - b\beta)\varrho^{2n}]}{2n^2(b^2 r\varrho)^n[(n + b\beta)b^{2n} - (n - b\beta)a^{2n}]},$$

and for $r \geq \varrho$, the variables r and ϱ must be exchanged in $K_0(r, \varrho)$, $K_n(r, \varrho)$, and $K_n^*(r, \varrho)$, whilst the factor $|b^2 z - a^2 \zeta|$ must, in this case, be replaced by $|a^2 z - b^2 \zeta|$, in the logarithmic component of $G(r, \varphi; \varrho, \psi)$. It can be easily observed that the above expression for the Green's function becomes undefined if $\beta = 0$.

13. $$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left\{\ln\frac{|z-\zeta|\,|a^2-z\overline{\zeta}||b^2-z\overline{\zeta}|\,\left|b^2z-a^2\zeta\right|}{(b^2r\varrho)^2} +K_0(r,\varrho)+2\sum_{n=1}^{\infty}[K_n(r,\varrho)+K_n^*(r,\varrho)]\cos n(\varphi-\psi)\right\},\quad r\le\varrho,$$

with

$$K_0(r,\varrho) = \frac{1+a\beta\ln(r/a)}{a\beta},$$

$$K_n^*(r,\varrho) = \frac{a\beta}{2n^3(b^2r\varrho)^n}\left\{n[(rb)^{2n}-(a\varrho)^{2n})]-[(r\varrho)^{2n}-(ab)^{2n})]\right\},$$

$$K_n(r,\varrho) = [(n-a\beta)a^{2n}-ab^{2n}\beta] \times\frac{(b^{2n}+\varrho^{2n})[(n+a\beta)r^{2n}+(n-a\beta)a^{2n}]}{2n^2(b^2r\varrho)^n[(n+a\beta)b^{2n}-(n-a\beta)a^{2n}]},$$

and for $r\ge\varrho$, the variables r and ϱ must be exchanged in $K_0(r,\varrho)$, $K_n(r,\varrho)$, and $K_n^*(r,\varrho)$, whilst the factor $|b^2z-a^2\zeta|$ must, in this case, be replaced by $|a^2z-b^2\zeta|$, in the logarithmic component of $G(r,\varphi;\varrho,\psi)$.

14. $$G(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left\{\ln\frac{|z-\zeta|\,|a^2-z\overline{\zeta}||b^2-z\overline{\zeta}|\,\left|b^2z-a^2\zeta\right|}{(b^2r\varrho)^2} +K_0(r,\varrho)+2\textstyle\sum_{n=1}^{\infty}[K_n(r,\varrho)+K_n^*(r,\varrho)]\cos n(\varphi-\psi)\right\},\quad r\le\varrho,$$

with

$$K_0(r,\varrho) = \frac{[1+a\beta_1\ln(r/a)][1-b\beta_2\ln(\varrho/b)]}{a\beta_1+b\beta_2[1+a\beta_1\ln(b/a)]},$$

$$K_n(r,\varrho) = \frac{A(a,r,n,\beta_1)B(b,\varrho,n,\beta_2)}{2n^3(b^2r\varrho)^nD(a,b,n,\beta_1,\beta_2)}R(a,b,n,\beta_1,\beta_2),$$

$$K_n^*(r,\varrho) = \frac{1}{2n^3(b^2r\varrho)^n}\left\{n\left[(a\beta_1+b\beta_2)((rb)^{2n}-(a\varrho)^{2n}) +(a\beta_1-b\beta_2)((r\varrho)^{2n}-(ab)^{2n})\right]+ab\beta_1\beta_2(b^{2n}-\varrho^{2n})(r^{2n}-a^{2n})\right\},$$

and

$$A(a,r,n,\beta_1) = (n+a\beta_1)r^{2n}+(n-a\beta_1)a^{2n},$$

$$B(b,\varrho,n,\beta_2) = (n+b\beta_2)b^{2n}+(n-b\beta_2)\varrho^{2n},$$

$$D(a,b,n,\beta_1,\beta_2) = (n+a\beta_1)(n+b\beta_2)b^{2n}-(n-a\beta_1)(n-b\beta_2)a^{2n},$$

$$R(a,b,n,\beta_1,\beta_2) = (n-a\beta_1)(a-b\beta_2)a^{2n}-[n(a\beta_1+b\beta_2)+ab\beta_1\beta_2]b^{2n}.$$

Note that for $r \geq \varrho$, the variables r and ϱ must be exchanged in $K_0(r,\varrho)$, $K_n(r,\varrho)$, and $K_n^*(r,\varrho)$, whilst the factor $|b^2z - a^2\zeta|$ must, in this case, be replaced by $|a^2z - b^2\zeta|$ in the logarithmic component of $G(r,\varphi;\varrho,\psi)$.

Chapter 4

1. $G(x,y;\xi,\eta) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1+\nu|x-\xi|}{4\nu^3}\mathrm{e}^{-\nu|x-\xi|}\sin\nu y\sin\nu\eta, \quad \nu = \dfrac{n\pi}{b}.$

2. (b) $G(x,y;\xi,\eta) = \dfrac{1}{4b}\displaystyle\sum_{n=1}^{\infty}\frac{g_n(x,\xi)}{\nu^3(1+2\nu^2a^2-\cosh 2\nu a)}\sin\nu y\sin\nu\eta,$

$\nu = \dfrac{n\pi}{b},$

with

$$\begin{aligned}
g_n(x,\xi) = 2\sinh\nu x\Big\{&\mathrm{e}^{\nu\xi}\left[(\nu\xi-1)(1+2\nu a-\mathrm{e}^{-2\nu a})-2\nu^2a^2\right]\\
&-\mathrm{e}^{-\nu\xi}\left[(\nu\xi+1)(1-2\nu a-\mathrm{e}^{-2\nu a})+2\nu^2a^2\right]\Big\}\\
&+\nu x\mathrm{e}^{-\nu x}\Big\{\mathrm{e}^{\nu\xi}\left[1+2\nu a+4\nu^2a(a-\xi)\right]\\
&+\mathrm{e}^{-\nu\xi}\left[(2\nu\xi+1)(1-\mathrm{e}^{2\nu a})-2\nu a\right]\Big\}\\
&-\nu x\mathrm{e}^{\nu x}\Big\{\mathrm{e}^{\nu\xi}\left[(2\nu\xi-1)(1-\mathrm{e}^{-2\nu a})-2\nu a\right]\\
&-\mathrm{e}^{-\nu\xi}\left[1-2\nu a+4\nu^2a(a-\xi)-\mathrm{e}^{2\nu a}\right]\Big\},
\end{aligned}$$

valid for $x \leq \xi$, whereas the formula $g_n(x,\xi)$ is valid for $x \geq \xi$ can be obtained from that above by exchanging x and ξ.

3.
$$\begin{aligned}
w(x,y) &= -\frac{4Q_0}{abD}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\mu x\sin\nu y}{\mu\nu(\mu^2+\nu^2)^2}\\
&\quad\times(\cos\mu a_2-\cos\mu a_1)(\cos\nu b_2-\cos\nu b_1),\\
M_x(x,y) &= \frac{4Q_0}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\mu^2+\sigma\nu^2)\sin\mu x\sin\nu y}{\mu\nu(\mu^2+\nu^2)^2}\\
&\quad\times(\cos\mu a_2-\cos\mu a_1)(\cos\nu b_2-\cos\nu b_1),\\
M_y(x,y) &= \frac{4Q_0}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\nu^2+\sigma\mu^2)\sin\mu x\sin\nu y}{\mu\nu(\mu^2+\nu^2)^2}\\
&\quad\times(\cos\mu a_2-\cos\mu a_1)(\cos\nu b_2-\cos\nu b_1),
\end{aligned}$$

with $\mu = m\pi/a$ and $\nu = n\pi/b$.

4. (a) $$w(x,y) = -\frac{4Q_0}{abD}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\mu x\sin\nu y}{\mu\nu(\mu^2+\nu^2)^2}\left(1-\cos\frac{\nu b}{2}\right)(1-\cos\mu a).$$

(b) $$w(x,y) = -\frac{2Q_0}{abD}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\mu x\sin\nu y}{\mu^2\nu^2(\mu^2+\nu^2)^2}$$
$$\times\left(2\sin\frac{\mu a}{2}-\mu a\cos\frac{\mu a}{2}\right)(\sin\nu b-\nu b\cos\nu b),$$

with $\mu = m\pi/a$ and $\nu = n\pi/b$.

6. (a) $$w(x,y) = -\frac{4Q_0}{abD}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\mu x\sin\nu y}{\mu\nu\left[(\mu^2+\nu^2)^2+\lambda\right]}\left(1-\cos\frac{\mu a}{2}\right)(1-\cos\nu b).$$

(b) $$w(x,y) = -\frac{4Q_0}{abD}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\mu x\sin\nu y}{\mu\nu\left[(\mu^2+\nu^2)^2+\lambda\right]}$$
$$\times\left(1-\cos\frac{\mu a}{2}\right)\left(1-\cos\frac{\nu b}{2}\right).$$

(c) $$w(x,y) = -\frac{4Q_0}{abD}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\mu x\sin\nu y}{\mu^2\nu^2\left[(\mu^2+\nu^2)^2+\lambda\right]}$$
$$\times(\sin\mu a-\mu a\cos\mu a)(\sin\nu b-\nu b\cos\nu b),$$

with $\mu = m\pi/a$ and $\nu = n\pi/b$.

10. $$u(x,\varphi) = -\frac{48a^4(1-\sigma^2)Z_0}{Ehlb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\widetilde{\mu}^2(1+\sigma)-(\widetilde{\mu}^2+\nu^2)}{\nu\Delta}$$
$$\times\cos\mu x\sin\nu\varphi\left(1-\cos\frac{\mu l}{2}\right)(1-\cos\nu b),$$
$$v(x,\varphi) = -\frac{48a^4(1-\sigma^2)Z_0}{Ehlb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\widetilde{\mu}^2(1+\sigma)+(\widetilde{\mu}^2+\nu^2)}{\widetilde{\mu}\Delta}$$
$$\times\cos\mu x\sin\nu\varphi\left(1-\cos\frac{\mu l}{2}\right)(1-\cos\nu b),$$
$$w(x,\varphi) = \frac{48a^4(1-\sigma^2)Z_0}{Ehlb}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(\widetilde{\mu}^2+\nu^2)^2}{\widetilde{\mu}\nu\Delta}$$
$$\times\cos\mu x\sin\nu\varphi\left(1-\cos\frac{\mu l}{2}\right)(1-\cos\nu b),$$

with
$$\mu = \frac{m\pi}{l}, \quad \nu = \frac{n\pi}{b}, \quad \widetilde{\mu} = \mu a,$$
and
$$\Delta = 12\widetilde{\mu}^4 a^2(1-\sigma^2) + h^2(\widetilde{\mu}^2 + \nu^2)^4.$$

Chapter 5

1. (a) Yes.
   (b) Yes.
   (c) Yes.
2. (a) The elements $g_{ij}(x,\xi), (i,j = 1,2)$ of the matrix of Green's type are defined as
$$g_{11}(x,\xi) = \frac{1}{H}\begin{cases}(1-\lambda k\xi)(x+a) & \text{for } -a \le x \le \xi \le 0,\\ (1-\lambda kx)(\xi+a) & \text{for } -a \le \xi \le x \le 0,\end{cases}$$
$$g_{12}(x,\xi) = \frac{\lambda k(x+a)}{H}\mathrm{e}^{-k\xi} \quad \text{for } -a \le x \le 0, \quad 0 \le \xi < \infty,$$
$$g_{21}(x,\xi) = \frac{\lambda k(\xi+a)}{H}\mathrm{e}^{-kx} \quad \text{for } -a \le \xi \le 0, \quad 0 \le x < \infty,$$
$$g_{22}(x,\xi) = \frac{1}{2H}\begin{cases}H\mathrm{e}^{k(x-\xi)} - (1-\lambda ka)\mathrm{e}^{-k(x+\xi)} & \text{for } 0 \le x \le \xi < \infty,\\ H\mathrm{e}^{k(\xi-x)} - (1-\lambda ka)\mathrm{e}^{-k(x+\xi)} & \text{for } 0 \le \xi \le x < \infty,\end{cases}$$
with $H = 1 + \lambda ka$.
   (b) The elements $g_{ij}(x,\xi), (i,j = 1,2)$ of the matrix of Green's type are defined as
$$g_{11}(x,\xi) = \frac{1}{k\Delta^*}\begin{cases}\sinh k(x+a)(\cosh k\xi - \lambda\sinh k\xi)\\ \qquad \text{for } -a \le x \le \xi \le 0,\\ \sinh k(\xi+a)(\cosh kx - \lambda\sinh kx)\\ \qquad \text{for } -a \le \xi \le x \le 0,\end{cases}$$
$$g_{12}(x,\xi) = \frac{\lambda\mathrm{e}^{-k\xi}}{k\Delta^*}\sinh k(x+a) \quad \text{for } -a \le x \le 0 \le \xi < \infty,$$
$$g_{21}(x,\xi) = \frac{\mathrm{e}^{-kx}}{k\Delta^*}\sinh k(\xi+a) \quad \text{for } -a \le \xi \le 0 \le x < \infty,$$
$$g_{22}(x,\xi) = \frac{1}{k\Delta^*}\begin{cases}\mathrm{e}^{-k\xi}(\lambda\sinh ka\cosh kx + \cosh ka\sinh kx)\\ \qquad \text{for } 0 \le x \le \xi < \infty,\\ \mathrm{e}^{-kx}(\lambda\sinh ka\cosh k\xi + \cosh ka\sinh k\xi)\\ \qquad \text{for } 0 \le \xi \le x < \infty,\end{cases}$$
with $\Delta^* = \lambda\sinh ka + \cosh ka$.

(c) The elements $g_{ij}(x,\xi), (i,j = 1,2)$ of the matrix of Green's type are defined as

$$g_{11}(x,\xi) = \frac{1}{\Delta^*}\begin{cases} \sin k(x+a)(\lambda \sin k\xi \cos ka - \cos k\xi \sin ka) \\ \qquad \text{for } -a \le x \le \xi \le 0, \\ \sin k(\xi+a)(\lambda \sin ka \cos kx - \cos ka \sin kx) \\ \qquad \text{for } -a \le \xi \le x \le 0, \end{cases}$$

$$g_{12}(x,\xi) = \frac{\lambda}{\Delta^*}\sin k(x+a)\sin k(\xi - a), \quad -a \le x \le 0 \le \xi \le a,$$

$$g_{21}(x,\xi) = \frac{1}{\Delta^*}\sin k(\xi+a)\sin k(x - a), \quad -a \le \xi \le 0 \le x \le a,$$

$$g_{22}(x,\xi) = \frac{1}{\Delta^*}\begin{cases} \sin k(\xi-a)(\lambda \sin ka \cos kx + \cos ka \sin kx) \\ \qquad \text{for } 0 \le x \le \xi \le a, \\ \sin k(x-a)(\lambda \sin ka \cos k\xi + \cos ka \sin k\xi) \\ \qquad \text{for } 0 \le \xi \le x \le a, \end{cases}$$

with $\Delta^* = k(1+\lambda)\sinh ka \cosh ka$.

(d) The diagonal elements $g_{ii}(x,\xi)$ of the matrix of Green's type are defined as

$$g_{ii}(x,\xi) = \frac{1}{H}\begin{cases} e^{-k\xi}[h_1 \cosh kx + (h_2+h_3)\sinh kx] \\ \qquad \text{for } 0 \le x \le \xi < \infty, \\ e^{-kx}[h_1 \cosh k\xi + (h_2+h_3)\sinh k\xi] \\ \qquad \text{for } 0 \le \xi \le x < \infty, \end{cases}$$

with $i = 1,2,3$ and $H = k(h_1+h_2+h_3)$, while its peripheral elements $g_{ij}(x,\xi), (i \ne j)$ are defined as

$$g_{ij}(x,\xi) = -\frac{h_j}{H}e^{-k(x+\xi)}, \quad \text{for } 0 \le x, \xi < \infty.$$

3.
$$y_1(x) = \frac{1}{\pi}Hh_1(x-1) - \frac{1}{\pi^2}\sin \pi x,$$
$$y_2(x) = -\frac{1}{\pi}Hh_1(x-1),$$
$$y_3(x) = -\frac{1}{\pi}Hh_1(x-1)$$

with $H = (h_1+h_2+h_3)^{-1}$.

4. Yes, it is.

5. The elements $g_{ij}(x, \xi)$, $(i, j = 1, 2)$ of the influence matrix are found as

$$g_{11}(x,\xi) = -\frac{1}{6a^3R_1}\begin{cases} \xi(x+a)^2[2EI_1(\xi^2x - 3a^2x - 2a\xi^2) \\ +3EI_2\xi(2x\xi + 3ax - a\xi)], \quad -a \le x \le \xi \le 0, \\ x(\xi+a)^2[2EI_1(x^2\xi - 3a^2\xi - 2ax^2) \\ +3EI_2x(2x\xi + 3a\xi - ax)], \quad -a \le \xi \le x \le 0, \end{cases}$$

$$g_{12}(x,\xi) = \frac{1}{2a^3R_0}x\xi(x+a)^2(2a-\xi)(a-\xi), \quad -a \le x \le 0 \le \xi \le a,$$

$$g_{21}(x,\xi) = \frac{1}{2a^3R_0}x\xi(\xi+a)^2(2a-x)(a-x), \quad -a \le \xi \le 0 \le x \le a,$$

$$g_{22}(x,\xi) = \frac{1}{6a^3R_2}\begin{cases} x(\xi+a)^2[2EI_1x(2ax(a+\xi) - \xi(x\xi + 6a^2x - 3a\xi) \\ +3EI_2a^2(x^2 - 2a\xi + \xi^2)], \quad 0 \le x \le \xi \le a, \\ \xi(x+a)^2[2EI_1\xi(2a\xi(a+x) - x(x\xi + 6a^2\xi - 3ax) \\ +3EI_2a^2(\xi^2 - 2ax - x^2)], \quad 0 \le \xi \le x \le a, \end{cases}$$

where R_0, R_1, and R_2 are introduced as

$$R_0 = 4EI_1 + 3EI_2, R_1 = EI_1R_0, \quad \text{and} \quad R_2 = EI_2R_0.$$

6. The elements $g_{ij}(x, \xi)$, $(i, j = 1, 2, 3)$ of the influence matrix are found as:

$$g_{11}(x,\xi) = \frac{1}{12a^3EI_1}\begin{cases} x^2(a-\xi)[\xi(x-3a)(\xi-2a) - 2a^2x], \\ \qquad 0 \le x \le \xi \le a, \\ \xi^2(a-x)[x(\xi-3a)(x-2a) - 2a^2\xi], \\ \qquad 0 \le \xi \le x \le a, \end{cases}$$

$$g_{12}(x,\xi) = \frac{1}{4aEI_1}x^2(a-\xi)(a-x), \quad 0 \le x \le a \le \xi \le 2a,$$

$$g_{13}(x,\xi) = \frac{1}{4aEI_1}x^2(a-\xi)(a-x), \quad 0 \le x \le a, 2a \le \xi \le 3a,$$

$$g_{21}(x,\xi) = \frac{1}{4aEI_1}\xi^2(a-\xi)(a-x), \quad 0 \le \xi \le a \le x \le 2a,$$

$$g_{22}(x,\xi) = \frac{1}{12EI_1}\begin{cases} (a-x)[(2x-a)(a-3\xi) + 2x^2], \quad a \le x \le \xi \le 2a, \\ (a-\xi)[(2\xi-a)(a-3x) + 2\xi^2], \quad a \le \xi \le x \le 2a, \end{cases}$$

$$g_{23}(x,\xi) = \frac{1}{12EI_1}(a-x)[(2x-a)(a-3\xi) + 2x^2], \quad a \le x \le 2a \le \xi \le 3a,$$

$$g_{31}(x,\xi) = \frac{1}{4aEI_1}\xi^2(a-\xi)(a-x), \quad 0 \le \xi \le a, 2a \le x \le 3a,$$

$$g_{32}(x,\xi) = \frac{1}{12EI_1}(a-\xi)[(2\xi-a)(a-3x)+2\xi^2], \quad a \le \xi \le 2a \le x \le 3a,$$

$$g_{33}(x,\xi) = \frac{1}{12EI_1EI_2}\begin{cases} 2EI_1[x^2(x-3\xi)+12x(x\xi-ax-a\xi)+16a^3] \\ +3aEI_2[7a(x+\xi)-5x\xi], \quad 2a \le x \le \xi \le 3a, \\ 2EI_1[\xi^2(\xi-3x)+12\xi(x\xi-ax-a\xi)+16a^3] \\ +3aEI_2[7a(x+\xi)-5x\xi], \quad 2a \le \xi \le x \le 3a. \end{cases}$$

7. The elements $g_{ij}(x,\xi), (i,j=1,2,3)$ of the influence matrix are found as:

$$g_{11}(x,\xi) = \frac{1}{6EI}\begin{cases} (\xi-a)(\xi^2+a\xi-4a^2+5ax-3x\xi), & 0 \le x \le \xi \le a, \\ (x-a)(x^2+ax-4a^2+5a\xi-3x\xi), & 0 \le \xi \le x \le a, \end{cases}$$

$$g_{12}(x,\xi) = \frac{1}{6aEI}(a-\xi)(2a-\xi)(3a-\xi)(a-x), \quad 0 \le x \le a \le \xi \le 2a,$$

$$g_{13}(x,\xi) = \frac{a}{6EI}(2a-\xi)(x-a), \quad 0 \le x \le a, \quad 2a \le \xi \le 3a,$$

$$g_{21}(x,\xi) = \frac{1}{6aEI}(a-\xi)(2a-x)(3a-x)(a-x), 0 \le \xi \le a \le x \le 2a,$$

$$g_{22}(x,\xi) = \frac{1}{6aEI}\begin{cases} (a-x)(2a-\xi)[(x+\xi-2a)^2+2x(a-\xi)] \\ \qquad \text{for } a \le x \le \xi \le 2a, \\ (a-\xi)(2a-x)[(x+\xi-2a)^2+2\xi(a-x)] \\ \qquad \text{for } a \le \xi \le x \le 2a, \end{cases}$$

$$g_{23}(x,\xi) = \frac{x}{6aEI}(a-x)(x-2a)(2a-\xi), \quad a \le x \le 2a \le \xi \le 3a,$$

$$g_{31}(x,\xi) = \frac{a}{6EI}(a-\xi)(x-2a), \quad 2a \le x \le 3a, \quad 0 \le \xi \le a,$$

$$g_{32}(x,\xi) = \frac{\xi}{6aEI}(a-\xi)(2a-\xi)(x-2a), \quad a \le \xi \le 2a \le x \le 3a,$$

$$g_{33}(x,\xi) = \frac{1}{6EI}\begin{cases} (2a-x)(x^2+2ax-3x\xi+4a\xi-4a^2) \\ \qquad \text{for } 2a \le x \le \xi \le 3a, \\ (2a-\xi)(\xi^2+2a\xi-3x\xi+4ax-4a^2) \\ \qquad \text{for } 2a \le \xi \le x \le 3a. \end{cases}$$

8. Yes, it is.

Chapter 6

1. The elements $G_{ij}(x,y;\xi,\eta)$, $i,j=1,2$, of the matrix of Green's type are defined with the observation point $(x,y) \in \Omega_i$ and the source point $(\xi,\eta) \in \Omega_j$.

They are found in compact closed form:

$$G_{11}(x,y;\xi,\eta) = \frac{1}{2\pi}\left[\ln\frac{E_1(z-\zeta)E_2(z-\overline{\zeta})}{E_2(z-\zeta)E_1(z-\overline{\zeta})} + \frac{\lambda-1}{\lambda+1}\ln\frac{E_1(z+\zeta)E_2(z+\overline{\zeta})}{E_2(z+\zeta)E_1(z+\overline{\zeta})}\right],$$

$$G_{12}(x,y;\xi,\eta) = \frac{\lambda}{\pi(\lambda+1)}\ln\frac{E_1(z-\zeta)E_2(z-\overline{\zeta})}{E_2(z-\zeta)E_1(z-\overline{\zeta})},$$

$$G_{21}(x,y;\xi,\eta) = \frac{1}{\pi(\lambda+1)}\ln\frac{E_1(z-\zeta)E_2(z-\overline{\zeta})}{E_2(z-\zeta)E_1(z-\overline{\zeta})},$$

$$G_{22}(x,y;\xi,\eta) = \frac{1}{2\pi}\left[\ln\frac{E_1(z-\zeta)E_2(z-\overline{\zeta})}{E_2(z-\zeta)E_1(z-\overline{\zeta})} - \frac{\lambda-1}{\lambda+1}\ln\frac{E_1(z+\zeta)E_2(z+\overline{\zeta})}{E_2(z+\zeta)E_1(z+\overline{\zeta})}\right],$$

where complex variable notation is used for the observation $z = x + iy$ and the source $\zeta = \xi + i\eta$ points. The parameter λ represents the ratio $\lambda = \lambda_2/\lambda_1$ of the conductivities of the materials filling the segments of Ω, and, for compactness, we introduce the real-valued functions of a complex variable E_1 and E_2:

$$E_1(p) = \left|1 + \exp\left(\frac{\pi p}{2b}\right)\right| \quad \text{and} \quad E_2(p) = \left|1 - \exp\left(\frac{\pi p}{2b}\right)\right|.$$

2. The elements $G_{ij}(x,y;\xi,\eta)$, $(i,j = 1,2)$ of the matrix of Green's type are defined with the observation point $(x,y) \in \Omega_i$ and the source point $(\xi,\eta) \in \Omega_j$. The diagonal elements $G_{11}(x,y;\xi,\eta)$ and $G_{22}(x,y;\xi,\eta)$ contain the logarithmic singularity. After splitting it off, all elements are found as:

$$G_{11}(x,y;\xi,\eta) = \frac{1}{2\pi}\left[\ln\frac{E(z-\overline{\zeta})[E(z+\zeta)]^k}{E(z-\zeta)[E(z+\overline{\zeta})]^k} - 8\sum_{n=1}^{\infty} g_{11}^n(x,\xi)\sin\nu y\sin\nu\eta\right],$$

$$G_{12}(x,y;\xi,\eta) = \frac{4}{\pi}\sum_{n=1}^{\infty} g_{12}^n(x,\xi)\sin\nu y\sin\nu\eta,$$

$$G_{21}(x,y;\xi,\eta) = \frac{4}{\pi}\sum_{n=1}^{\infty} g_{21}^n(x,\xi)\sin\nu y\sin\nu\eta,$$

$$G_{22}(x, y; \xi, \eta)$$
$$= \frac{1}{2\pi}\left[\ln \frac{E(z-\overline{\zeta})[E(z+\overline{\zeta})]^k}{E(z-\zeta)[E(z+\zeta)]^k} - 8\sum_{n=1}^{\infty} g_{22}^n(x, \xi) \sin \nu y \sin \nu \eta\right],$$

containing only uniformly convergent series. This makes the above expressions quite computer-friendly. Complex variable notation is used in the above for the observation point $z = x + iy$ and the source point $\zeta = \xi + i\eta$. The parameters λ and k and ν are introduced as: $\lambda = \lambda_2/\lambda_1$, $k = (1-\lambda)/(1+\lambda)$, and $\nu = n\pi/b$, whilst, for compactness, the real-valued function of a complex variable E is introduced as

$$E(p) = \left|1 - \exp\left(\frac{\pi p}{b}\right)\right|.$$

The coefficients $g_{ij}^n(x, \xi)$ of the series components of $G_{ij}(x, y; \xi, \eta)$ are defined as

$$g_{11}(x, \xi) = \frac{\sinh \nu a \cosh \nu x \cosh \nu \xi + \lambda \cosh \nu a \sinh \nu x \sinh \nu \xi}{(1+\lambda) n e^{\nu a} \sinh 2\nu a},$$

valid for $-a \le x \le \xi \le 0$, whereas, in the above, the variables x and ξ must be exchanged with $-a \le \xi \le x \le 0$;

$$g_{12}(x, \xi) = \lambda \frac{\sinh v(a+x) \sinh \nu(a-\xi)}{(1+\lambda) n \sinh 2\nu a},$$

valid for $-a \le x \le 0 \le \xi \le a$,

$$g_{21}(x, \xi) = \frac{\sinh v(a+\xi) \sinh \nu(a-x)}{(1+\lambda) n \sinh 2\nu a},$$

valid for $-a \le \xi \le 0 \le x \le a$, and

$$g_{22}(x, \xi) = \frac{\sinh \nu x \cosh \nu a \sinh \nu \xi + \lambda \cosh \nu x \sinh \nu a \cosh \nu \xi}{(1+\lambda) n e^{\nu a} \sinh 2\nu a},$$

valid for $0 \le x \le \xi \le a$, whereas, in the above, the variables x and ξ must be exchanged with $-a \le \xi \le x \le 0$.

3. After splitting off the singular components in the diagonal elements $G_{11}(r, \varphi; \varrho, \psi)$ and $G_{22}(r, \varphi; \varrho, \psi)$ of the matrix of Green's type in question,

its elements are expressed in the form:

$$G_{11}(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left[\ln\frac{|z-\overline{\zeta}|}{|z-\zeta|} + 4\sum_{n=1}^{\infty} g_{11}^{n}(r,\varrho)\sin n\varphi\sin n\psi\right],$$

$$G_{12}(r,\varphi;\varrho,\psi) = \frac{2}{\pi}\sum_{n=1}^{\infty} g_{12}^{n}(r,\varrho)\sin n\varphi\sin n\psi,$$

$$G_{21}(r,\varphi;\varrho,\psi) = \frac{2}{\pi}\sum_{n=1}^{\infty} g_{21}^{n}(r,\varrho)\sin n\varphi\sin n\psi,$$

$$G_{22}(r,\varphi;\varrho,\psi) = \frac{1}{2\pi}\left[\ln\frac{|z-\overline{\zeta}|}{|z-\zeta|} + 4\sum_{n=1}^{\infty} g_{22}^{n}(r,\varrho)\sin n\varphi\sin n\psi\right],$$

containing only uniformly convergent series. This makes the above representations convenient for immediate computer implementation. Note that the observation point and the source point in the expressions for $G_{ij}(r,\varphi;\varrho,\psi)$ belong to individual regions, i.e. $(r,\varphi)\in\Omega_i$ and $(\varrho,\psi)\in\Omega_j$. Complex variable notation, with $z = r(\cos\varphi + i\sin\varphi)$ and $\zeta = \varrho(\cos\psi + i\sin\psi)$ is used for compactness. The coefficients $g_{ij}^{n}(r,\varrho)$ of the series components of $G_{ij}(r,\varphi;\varrho,\psi)$ are defined as:

$$g_{11}(r,\varrho) = -\frac{(r\varrho)^n}{2n\Delta^*(ab)^{2n}}\left[(1+\lambda)a^{2n} + (1-\lambda)b^{2n}\right], 0\le r,\varrho\le a,$$

$$g_{12}(r,\varrho) = \frac{\lambda}{n\Delta^*}\left[\left(\frac{r}{\varrho}\right)^n + \left(\frac{r\varrho}{b^2}\right)^n\right], \quad 0\le r\le a\le\varrho\le b,$$

$$g_{21}(r,\varrho) = \frac{1}{n\Delta^*}\left[\left(\frac{\varrho}{r}\right)^n + \left(\frac{r\varrho}{b^2}\right)^n\right], \quad 0\le\varrho\le a\le r\le b,$$

and

$$g_{22}(r,\varrho) = \frac{1}{2n\Delta^*}\left[(1-\lambda)\frac{a^{2n}(b^{2n}-r^{2n}-\varrho^{2n})}{(r\varrho b^2)^n} - (1+\lambda)\left(\frac{r\varrho}{b^2}\right)^n\right],$$
$$a\le r,\varrho\le b.$$

4. The boundary-contact-value problem modeling the sought-after potential field in the composite structure is stated as:

$$\frac{1}{\sin\varphi_1}\frac{\partial}{\partial\varphi_1}\left(\sin\varphi_1\frac{\partial u_1(\varphi_1,\vartheta_1)}{\partial\varphi_1}\right) + \frac{1}{\sin^2\varphi_1}\frac{\partial^2 u_1(\varphi_1,\vartheta_1)}{\partial\vartheta_1^2} = 0,$$
$$(\varphi_1,\vartheta_1)\in\Omega_1,$$

$$\frac{1}{\sin\varphi_2}\frac{\partial}{\partial\varphi_2}\left(\sin\varphi_2\frac{\partial u_2(\varphi_2,\vartheta_2)}{\partial\varphi_2}\right)+\frac{1}{\sin^2\varphi_2}\frac{\partial^2 u_2(\varphi_2,\vartheta_2)}{\partial\vartheta_2^2}=0,$$
$$(\varphi_2,\vartheta_2)\in\Omega_2,$$
$$\lim_{\varphi_1\to 0}|u_1(\varphi_1,\vartheta_1)|<\infty,\quad u_1\left(\frac{\pi}{2},\vartheta_1\right)=u_2(\varphi_0,\vartheta_2)$$

and

$$\frac{\partial u_1(\pi/2,\vartheta_1)}{\partial\varphi_1}=\lambda\frac{\partial u_2(\varphi_0,\vartheta_2)}{\partial\varphi_2},\quad u_2\left(\frac{\pi}{2},\vartheta_2\right)=0,\quad \lambda=\frac{\lambda_2}{\lambda_1}.$$

The elements of the matrix of Green's type for the above problem are found, for $\lambda/\sin\varphi_0=1$, in the compact closed form:

$$G_{11}(\varphi_1,\vartheta_1;\psi_1,\tau_1)=\frac{1}{4\pi}\ln\frac{\Phi_0^{-2}-2\Phi_1\Psi_1\Theta_{11}+(\Phi_1\Psi_1\Phi_0)^2}{\Phi_1^{-2}-2\Phi_1\Psi_1\Theta_{11}+\Psi_1^2},$$
$$G_{12}(\varphi_1,\vartheta_1;\psi_2,\tau_2)=\frac{1}{4\pi}\ln\frac{1-2\Phi_1\Psi_2\Phi_0\Theta_{12}+(\Phi_1\Psi_2\Phi_0)^2}{\Psi_2^2-2\Phi_1\Psi_2\Phi_0\Theta_{12}+(\Phi_1\Phi_0)^2},$$
$$G_{21}(\varphi_2,\vartheta_2;\psi_1,\tau_1)=\frac{1}{4\pi}\ln\frac{1-2\Phi_2\Psi_1\Phi_0\Theta_{21}+(\Phi_2\Psi_1\Phi_0)^2}{\Phi_2^2-2\Phi_2\Psi_1\Phi_0\Theta_{21}+(\Psi_1\Phi_0)^2},$$

and

$$G_{22}(\varphi_2,\vartheta_2;\psi_2,\tau_2)=\frac{1}{4\pi}\ln\frac{1-2\Phi_2\Psi_2\Theta_{22}+(\Phi_2\Psi_2)^2}{\Phi_2^2-2\Phi_2\Psi_2\Theta_{22}+\Psi_2^2},$$

with $\Phi_0=\tan\varphi_0/2$, whilst the functional components Φ_k,Ψ_k,and $\Theta_{kl}, k,l=1,2$, in terms of the independent variables, read as:

$$\Phi_k=\tan\frac{\varphi_k}{2},\quad \Psi_k=\tan\frac{\psi_k}{2},\quad\text{and}\quad \Theta_{kl}=\cos(\vartheta_k-\tau_l).$$

Chapter 7

Some of the answers to the exercises in Chapter 7 are expressed in terms of the Jacobi Theta function of the third kind $\vartheta_3(p,r)$. We recall that a series representation of this function reads

$$\vartheta_3(p,r)=1+2\sum_{n=1}^{\infty}e^{-n^2\pi^2 r}\cos 2n\pi p.$$

2. (a)
$$g(x,t;\xi)=\frac{1}{2\sqrt{\pi t}}\sum_{n=0}^{\infty}(-1)^n\left\{e^{-\frac{(|x-\xi|+2an)^2}{4t}}+e^{-\frac{(x+\xi+2an)^2}{4t}}\right.$$
$$\left.-e^{-\frac{[x+\xi-2a(n+1)]^2}{4t}}-e^{-\frac{[|x-\xi|-2a(n+1)]^2}{4t}}\right\}.$$

3.
$$G(x,y,t;\xi,\eta)=\frac{e^{-(\frac{(x-\xi)^2}{4\varkappa t}+\beta t)}}{4b\sqrt{\varkappa\pi t}}\left[\vartheta_3\left(\frac{y-\eta}{2b},\frac{\varkappa t}{b^2}\right)+\vartheta_3\left(\frac{y+\eta}{2b},\frac{\varkappa t}{b^2}\right)\right].$$

4. (b)
$$G(x,y,t;\xi,\eta)=\frac{e^{-\beta t}}{4b\sqrt{\varkappa\pi t}}[e^{-\frac{(x-\xi)^2}{4\varkappa t}}-e^{-\frac{(x+\xi)^2}{4\varkappa t}}]\left[\vartheta_3\left(\frac{y-\eta}{4b},\frac{\varkappa t}{b^2}\right)\right.$$
$$-\vartheta_3\left(\frac{y+\eta}{4b},\frac{\varkappa t}{b^2}\right)-\vartheta_3\left(\frac{y-\eta}{2b},\frac{4\varkappa t}{b^2}\right)$$
$$\left.+\vartheta_3\left(\frac{y+\eta}{2b},\frac{4\varkappa t}{b^2}\right)\right].$$

(c)
$$G(x,y,t;\xi,\eta)=\frac{e^{-\beta t}}{4b\sqrt{\varkappa\pi t}}\left[-2\gamma\sqrt{\varkappa\pi t}e^{\gamma(x+\xi+\gamma\varkappa t)}\right.$$
$$\left.\times\mathrm{erfc}\left(\gamma\sqrt{\varkappa t}+\frac{x+\xi}{2\sqrt{\varkappa t}}\right)+e^{-\frac{(x-\xi)^2}{4\varkappa t}}+e^{-\frac{(x+\xi)^2}{4\varkappa t}}\right]$$
$$\left[\vartheta_3\left(\frac{y-\eta}{4b},\frac{\varkappa t}{b^2}\right)-\vartheta_3\left(\frac{y+\eta}{4b},\frac{\varkappa t}{b^2}\right)\right.$$
$$\left.-\vartheta_3\left(\frac{y-\eta}{2b},\frac{4\varkappa t}{b^2}\right)+\vartheta_3\left(\frac{y+\eta}{2b},\frac{4\varkappa t}{b^2}\right)\right].$$

(d)
$$G(x,y,t;\xi,\eta)=\frac{e^{-\beta t}}{4b\sqrt{\varkappa\pi t}}[e^{-\frac{(x-\xi)^2}{4\varkappa t}}+e^{-\frac{(x+\xi)^2}{4\varkappa t}}]\left[\vartheta_3\left(\frac{y-\eta}{4b},\frac{\varkappa t}{b^2}\right)\right.$$
$$-\vartheta_3\left(\frac{y+\eta}{4b},\frac{\varkappa t}{b^2}\right)-\vartheta_3\left(\frac{y-\eta}{2b},\frac{4\varkappa t}{b^2}\right)$$
$$\left.+\vartheta_3\left(\frac{y+\eta}{2b},\frac{4\varkappa t}{b^2}\right)\right].$$

(e)
$$G(x,y,t;\xi,\eta)=\frac{e^{-\beta t}}{4b\sqrt{\varkappa\pi t}}[e^{-\frac{(x-\xi)^2}{4\varkappa t}}+e^{-\frac{(x+\xi)^2}{4\varkappa t}}]$$
$$\times\left[\vartheta_3\left(\frac{y-\eta}{2b},\frac{\varkappa t}{b^2}\right)+\vartheta_3\left(\frac{y+\eta}{2b},\frac{\varkappa t}{b^2}\right)\right].$$

5. $$G(x,y,t;\xi,\eta) = \frac{e^{-\beta t}}{b\sqrt{\varkappa\pi t}}\sum_{k=0}^{\infty}[e^{-\frac{[2ak+(x+\xi)]^2}{4\varkappa t}} - e^{-\frac{[2a(k+1)-(x+\xi)]^2}{4\varkappa t}}$$
$$+ e^{-\frac{[2ak+|x-\xi|]^2}{4\varkappa t}} - e^{-\frac{[2a(k+1)-|x-\xi|]^2}{4\varkappa t}}]$$
$$\times \sum_{n=1}^{\infty} e^{-\frac{\varkappa(n\pi)^2 t}{b^2}} \sin\frac{n\pi y}{b} \sin\frac{n\pi\eta}{b}.$$

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