

## 1-D Heat Equation. Green's Function.

### Infinite Bar

Consider the initial value problem

$$(*) \begin{cases} u_t - ku_{xx} = 0 & -\infty < x < \infty, t > 0, \\ |u(x, t)| \text{ remains bounded as } x \rightarrow \pm\infty & t > 0, \\ u(x, 0) = f(x) & -\infty < x < \infty. \end{cases}$$

*The Heat Kernel*

$$G(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/(4kt)} \quad -\infty < x < \infty, t > 0,$$

is the solution of (\*) with singular initial data  $\delta(x)$ :

$$\lim_{t \downarrow 0} G(x, t) = \delta(x).$$

More precisely,  $G(x, t)$  satisfies the heat equation  $G_t = kG_{xx}$  and

$$\lim_{t \downarrow 0} \int_{-\infty}^{\infty} G(x, t) f(x) dx = f(0),$$

for any bounded smooth function  $f$ .

*The Green's function*  $G(x, t; \xi)$  for Problem (\*) is the solution of (\*) with initial data  $\delta(x - \xi)$ , where  $\xi$  is a point in  $(-\infty, \infty)$ . The singular initial data  $\delta(x - \xi)$  is a translation of  $\delta(x)$  in space. Thus,

$$G(x, t; \xi) = G(x - \xi, t) = \frac{1}{\sqrt{4\pi kt}} e^{-(x-\xi)^2/(4kt)} \quad -\infty < x < \infty, t > 0,$$

satisfies

$$\begin{cases} G_t(x, t; \xi) - kG_{xx}(x, t; \xi) = 0 & -\infty < x < \infty, t > 0, \\ \lim_{t \downarrow 0} G(x, t; \xi) = \delta(x - \xi) & -\infty < x < \infty. \end{cases}$$

**SOLUTION OF THE INITIAL VALUE PROBLEM (\*)** is given by

$$u(x, t) = \int_{-\infty}^{\infty} G(x, t; \xi) f(\xi) d\xi = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-(x-\xi)^2/(4kt)} f(\xi) d\xi \quad -\infty < x < \infty, t > 0.$$

**1-D Heat Equation. Green's Function.**  
**Semi-Infinite Bar with Fixed End Temperature**

Consider the initial-boundary value problem

$$(**) \begin{cases} u_t - ku_{xx} = 0 & x > 0, t > 0, \\ u(0, t) = 0 & t > 0, \\ |u(x, t)| \text{ remains bounded as } x \rightarrow \infty & t > 0, \\ u(x, 0) = f(x) & x \geq 0. \end{cases}$$

The Green's function  $G(x, t; \xi)$  for Problem (\*\*) is the solution of (\*\*) with singular initial data  $\delta(x - \xi)$ , where  $\xi$  is a point in  $(0, \infty)$ ; in other words, it should satisfy

$$\begin{cases} G_t(x, t; \xi) - kG_{xx}(x, t; \xi) = 0 & x > 0, t > 0, \\ G(0, t; \xi) = 0 & t > 0, \\ G(x, 0; \xi) = \delta(x - \xi) & x > 0. \end{cases}$$

To solve this, we make the odd extension of  $G(x, t; \xi)$  with respect to the space variable  $x$ , denoted by  $G_o(x, t; \xi)$ . The new function  $G_o(x, t; \xi)$  is defined on the whole real line of  $x$  and satisfies

$$\begin{cases} \frac{\partial G_o}{\partial t}(x, t; \xi) - k\frac{\partial^2 G_o}{\partial x^2}(x, t; \xi) = 0 & -\infty < x < \infty, t > 0, \\ G_o(x, 0; \xi) = \delta(x - \xi) - \delta(x + \xi) & -\infty < x < \infty. \end{cases}$$

Notice that the equation for the initial condition of  $G_o$  is constructed from the odd extension of  $\delta(x - \xi)$  with respect to  $x$ . The solution of this problem for  $G_o$  is given by a linear superposition of the heat kernel:

$$\begin{aligned} G_o(x, t; \xi) &= G(x - \xi, t) - G(x + \xi, t) \\ &= \frac{1}{\sqrt{4\pi kt}} e^{-(x-\xi)^2/(4kt)} - \frac{1}{\sqrt{4\pi kt}} e^{-(x+\xi)^2/(4kt)} \quad -\infty < x < \infty, t > 0. \end{aligned}$$

Therefore, the Green's function for problem (\*\*) is

$$\begin{aligned} G(x, t; \xi) &= G(x - \xi, t) - G(x + \xi, t) \\ &= \frac{1}{\sqrt{4\pi kt}} e^{-(x-\xi)^2/(4kt)} - \frac{1}{\sqrt{4\pi kt}} e^{-(x+\xi)^2/(4kt)} \quad x \geq 0, t > 0. \end{aligned}$$

**SOLUTION OF THE INITIAL-BOUNDARY VALUE PROBLEM (\*\*) can be obtained using the Green's function :**

$$\begin{aligned} u(x, t) &= \int_0^\infty G(x, t; \xi) f(\xi) d\xi \\ &= \int_0^\infty \frac{1}{\sqrt{4\pi kt}} \left[ e^{-(x-\xi)^2/(4kt)} - e^{-(x+\xi)^2/(4kt)} \right] f(\xi) d\xi \quad x \geq 0, t > 0. \end{aligned}$$

## EXERCISES

[1] Construct the Green's function and then find the solution formula for the heat conduction on a semi-infinite bar with an insulated end point.

[2] Use the Green's function to find the solution of (\*) on infinite bar with initial data

$$f(x) = \begin{cases} T_0 & x \leq 0, \\ T_1 & x > 0. \end{cases}$$

[3] Find a solution of

$$\begin{cases} u_t - u_{xx} = 0 & -\infty < x < \infty, t > 0, \\ u(x, 0) = 3\delta(x - 2) - \delta(x - 5) & -\infty < x < \infty. \end{cases}$$

[4] Use the Green's function to find the solution of (\*\*) on semi-infinite bar with initial data

$$f(x) = \begin{cases} 0 & 0 \leq x < a, \\ T_0 & a < x < b, \\ 0 & b \leq x < \infty, \end{cases}$$

where  $0 < a < b$  are constants.

[5] Find a solution of

$$\begin{cases} u_t - u_{xx} = 0 & x > 0, t > 0, \\ u(0, t) = 0 & t > 0, \\ u(x, 0) = 3\delta(x - 2) - \delta(x - 5) & x \geq 0. \end{cases}$$

(See next page for the answers)

## ANSWERS

[1] The initial-boundary value problem

$$\begin{cases} u_t - ku_{xx} = 0 & x > 0, t > 0, \\ u_x(0, t) = 0 & t > 0, \\ |u(x, t)| \text{ remains bounded as } x \rightarrow \infty & t > 0, \\ u(x, 0) = f(x) & x \geq 0. \end{cases}$$

The Green's function is

$$\begin{aligned} G(x, t; \xi) &= G(x - \xi, t) + G(x + \xi, t) \\ &= \frac{1}{\sqrt{4\pi kt}} e^{-(x-\xi)^2/(4kt)} + \frac{1}{\sqrt{4\pi kt}} e^{-(x+\xi)^2/(4kt)} \quad x \geq 0, t > 0. \end{aligned}$$

The solution of the initial-boundary value problem is:

$$\begin{aligned} u(x, t) &= \int_0^\infty G(x, t; \xi) f(\xi) d\xi \\ &= \int_0^\infty \frac{1}{\sqrt{4\pi kt}} \left[ e^{-(x-\xi)^2/(4kt)} + e^{-(x+\xi)^2/(4kt)} \right] f(\xi) d\xi \quad x \geq 0, t > 0. \end{aligned}$$

[2]  $u(x, t) = \frac{T_0}{2} \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right) + \frac{T_1}{2} \operatorname{erfc}\left(-\frac{x}{2\sqrt{kt}}\right)$   $-\infty < x < \infty, t > 0,$   
 where  $\operatorname{erfc}$  is the complementary error function:

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-y^2} dy \quad -\infty < z < \infty.$$

[3]  $u(x, t) = \frac{3}{2\sqrt{\pi t}} e^{-(x-2)^2/(4t)} - \frac{1}{2\sqrt{\pi kt}} e^{-(x-5)^2/(4t)}$   $-\infty < x < \infty, t > 0.$

[4]

$$u(x, t) = \frac{T_0}{2} \left[ \operatorname{erf}\left(\frac{x-a}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{x-b}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{x+b}{2\sqrt{kt}}\right) + \operatorname{erf}\left(\frac{x+a}{2\sqrt{kt}}\right) \right] \quad x \geq 0, t > 0,$$

where  $\operatorname{erf}$  is the error function:  $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} dy$  for  $-\infty < z < \infty$ .

[5]

$$\begin{aligned} u(x, t) &= \frac{3}{2\sqrt{\pi t}} e^{-(x-2)^2/(4t)} - \frac{1}{2\sqrt{\pi t}} e^{-(x-5)^2/(4t)} \\ &\quad - \frac{3}{2\sqrt{\pi t}} e^{-(x+2)^2/(4t)} + \frac{1}{2\sqrt{\pi t}} e^{-(x+5)^2/(4t)} \quad x \geq 0, t > 0. \end{aligned}$$