IV. GREEN'S FUNCTIONS FOR TIME-DEPENDENT PROBLEMS

1. Green's functions for the heat equation.

Suppose we measure the temperature $u(x,t)$ at each point x in a bounded region V in $\mathbb{R}^n$ (for us, usually $n = 1, 2,$ or 3), and at each time t. The region is subjected to heat sources $f(x,t)$, and at its boundary, it is held at temperature $g(x,t)$ ($x \in \partial V$). Initially (at time $t = 0$, say), the temperature distribution in V is $u_0(x)$. The following initial-boundary value problem for the **heat equation** describes the temperature distrubution $u(x,t)$ at later times $t > 0$:

$$\begin{cases} \text{heat equation:} & \frac{\partial u}{\partial t} - D\Delta u = f(x,t), \quad x \in V, \ \ t > 0 \\ \text{boundary value:} & u(x,t) = g(x,t), \ \ x \in \partial V, \ \ t > 0 \\ \text{initial condition:} & u(x,0) = u_0(x), \ \ x \in V \end{cases}, \tag{23}$$

where $D > 0$ is the *diffusion rate* constant.

We would like to represent the solution to this problem using a Green's function. The first observation is that the differential operator appearing in the equation is *not* self-adjoint:

$$L := \frac{\partial}{\partial t} - D\Delta u \qquad \Longrightarrow \qquad L^* = -\frac{\partial}{\partial t} - D\Delta u.$$

Next, some notation. We fix a time interval $[0,T]$ (some $T > 0$), and let C_T denote the space-time cylinder $C_T = V \times [0,T]$.

As before, a Green's function for our problem should be a function of two sets of variables: $G(x,t;y,\tau)$, $x, y \in V$, $t, \tau \geq 0$. To determine the problem that G should solve, we suppose $u(y,\tau)$ solves problem (23), and integrate G against Lu over the space-time cylinder, and integrate by parts:

$$\begin{aligned} \int_{C_T} G \ Lu \ dyd\tau &= \int_{C_T} G(u_\tau - D\Delta u) dyd\tau \\ &= \int_{C_T} L^*G \ u \ dyd\tau + \int_0^T \int_{\partial V} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) dS(y)d\tau \\ &\quad + \int_V \left(Gu|_{\tau=T} - Gu|_{\tau=0} \right) dy. \end{aligned}$$

So supposing we demand that our Green's function $G(x,t;y,\tau)$ solves

$$\begin{cases} -\frac{\partial G}{\partial \tau} - D\Delta G = \delta(y-x)\delta(\tau - t) \\ G \equiv 0 \text{ for } y \in \partial V \\ G \equiv 0 \text{ for } \tau > t \quad (\text{“causality”}) \end{cases}, \tag{24}$$

we arrive at a representation formula for $u(x,t)$, $x \in V$, $0 \leq t < T$,

$$
\begin{aligned}
u(x,t) = \int_0^T \int_V G(x,t;y,\tau) f(y,\tau) dy d\tau + \int_V G(x,t;y,0) u_0(y) dy \\
- \int_0^T \int_{\partial V} \frac{\partial G}{\partial n} g(y,\tau) dS(y) d\tau.
\end{aligned}
\tag{25}
$$

Note that the above "causality" condition implies that the solution at time t should not depend on any of its values at later times – i.e. we are solving forward in time.

Problem (24) for the Green's function looks a little odd. It is a *backwards* heat equation, which would be nasty, except that is also solved backwards in time (starting from time $\tau = t$ and going down to $\tau = 0$). So, to straighten it out, it is useful to change the time variable from τ to $\sigma := t - \tau$. Problem (24) then becomes

$$
\begin{cases}
\frac{\partial G}{\partial \sigma} - D\Delta G = \delta(y-x)\delta(\sigma) \\
G = 0 \text{ for } y \in \partial V \\
G = 0 \text{ for } \sigma < 0
\end{cases}
. \tag{26}
$$

This is the problem we will try to solve in various situations. The simplest case is when there is no boundary – the *free-space* case.

2. Free-space Green's function for the heat equation.

We will find the free-space Green's function for the heat equation by using the *Fourier transform*, so let us first recall the difinition and some key properties of it.

Definition: Let f be an integrable function on $\mathbb{R}^n$ (that means $\int_{\mathbb{R}^n} |f(x)|dx < \infty$). The **Fourier transform** of f is another function, $\hat{f}$, defined by

$$\hat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx.$$

Here are some useful properties of the Fourier transform. Let f, g be smooth functions with rapid decay at ∞.

1. F.T. is linear: $(\widehat{\alpha f + \beta g})(\xi) = \alpha \hat{f}(\xi) + \beta \hat{g}(\xi)$

2. F.T. is (almost) its own inverse: $\check{\hat{f}} = f$, where

   $$\check{g}(x) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\xi\cdot x} g(\xi)d\xi$$

   is the **inverse Fourier transform**.

3. F.T. is *unitary* (its inverse is its adjoint): $(\hat{f},\ g) = (f,\ \check{g})$, and in particular (taking $g = \hat{f}$), preserves the "L^2-norm":

   $$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

4. F.T. "interchanges differentiation and coordinate multiplication":

   $$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = (i\xi_j)\,\hat{f}(\xi), \qquad (\widehat{x_j f(x)})(\xi) = i\frac{\partial}{\partial \xi}\hat{f}(\xi).$$

5. F.T "interchanges convolution and multiplication":

   $$\widehat{f * g}(\xi) = (2\pi)^n \hat{f}(\xi)\hat{g}(\xi), \qquad (f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy.$$

6. F.T interchanges coordinate translation and multiplication by an exponential: for $a \in \mathbb{R}^n$,

   $$f\widehat{(x-a)}(\xi) = e^{-ia\cdot\xi}\hat{f}(\xi).$$

7. F.T maps Gaussians to Gaussians: for $a > 0$,

   $$\widehat{e^{-\frac{a|x|^2}{2}}}(\xi) = a^{-n/2} e^{-\frac{|\xi|^2}{2a}}.$$

It is property 4 which makes the Fourier transform so useful for differential equations – it converts differential equations into algebraic ones.

Property 2 is deep, and difficult to prove. See an analysis textbook for this. The other properties are easy to show. We will do 4 and 7 – two that we will use shortly.

Proof of property 4: integrating by parts,

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \frac{\partial f}{\partial x_j}(x)dx = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} (-i\xi_j) e^{-ix\cdot\xi} f(x)dx = i\xi_j \hat{f}(\xi).$$

And for the other one,

$$i\frac{\partial}{\partial \xi_j}\hat{f}(\xi) = i\frac{\partial}{\partial \xi_j}(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx = i(2\pi)^{-n/2} \int_{\mathbb{R}^n} (-ix_j) e^{-ix\cdot\xi} f(x)dx = (\widehat{x_j f(x)})(\xi).$$

Proof of property 7: the higher-dimensional cases follow easily from the case $n = 1$, so we'll do that one. Completing the square,

$$\widehat{e^{-\frac{a}{2}|x|^2}}(\xi) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\cdot\xi} e^{-\frac{a}{2}x^2} dx = e^{-\frac{\xi^2}{2a}} (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-\frac{a}{2}(x+i\xi/a)^2} dx.$$

The last integral is the integral of the entire complex function $f(z) = e^{-az^2/2}$ along the contour $z = x + i\xi/a$, $-\infty < x < \infty$ in the complex plane. We can "shift" the contour to the real axis using Cauchy's theorem.

$$\int_{-\infty}^{\infty} e^{-\frac{a}{2}(x+i\xi/a)^2} dx = \lim_{R\to\infty} \int_{[-R,R]+i\xi/a} f(z)dz = \lim_{R\to\infty} \left[\int_{[-R,R]} f(z)dz + \int_{A_R \cup B_R} f(z)dz \right]$$

where A_R denotes the contour $z = -R + iy$, $y : e^{i\xi/a} \to 0$, and B_R denotes the contour $z = R + iy$, $y : 0 \to e^{i\xi/a}$ (draw a picture!). Along A_R and B_R, we have

$$|f(z)| = e^{-\frac{a}{2}\operatorname{Re}(z^2)} \le e^{\frac{a}{2}(R^2 - \xi^2/a^2)}$$

and so

$$\left| \int_{A_R \cup B_R} f(z)dz \right| \le 2e^{\frac{\xi^2}{2a}} e^{-\frac{a}{2}R^2} \to 0 \text{ as } R \to \infty.$$

Hence

$$\int_{-\infty}^{\infty} e^{-\frac{a}{2}(x+i\xi/a)^2} dx = \int_{-\infty}^{\infty} e^{-\frac{a}{2}x^2} dx = \sqrt{\frac{2\pi}{a}}$$

(the last equality is a standard fact which can be proved, for example, by squaring the integral, interpreting it as a two-dimensional integral, and changing to polar coordinates). Finally, then, we arrive at

$$\widehat{e^{-\frac{a}{2}|x|^2}}(\xi) = a^{-1/2} e^{-\frac{\xi^2}{2a}}$$

as needed.

Now let's return to the problem of finding the free-space Green's function for the heat equation. That is, solving (26) when $V = \mathbb{R}^n$. Let $\hat{G}(x,t;\xi,\sigma)$ denote the Fourier transform of $G(x,t;y,\sigma)$ in the variable y. Property 4 of the Fourier transform shows that Δ in the variable y corresponds to multiplication by $-|\xi|^2$. Also, note that

$$\widehat{\delta_x(y)}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-iy\cdot\xi}\delta(y-x)dy = (2\pi)^{-n/2}e^{-ix\cdot\xi},$$

and so we have to solve

$$\frac{\partial}{\partial\sigma}\hat{G} + D|\xi|^2\hat{G} = (2\pi)^{-n/2}e^{-ix\cdot\xi}\delta(\sigma).$$

For $\sigma > 0$, this is $\hat{G}_\sigma + D|\xi|^2\hat{G} = 0$, an ODE which is easily solved to find

$$\hat{G} = Ce^{-D|\xi|^2\sigma},$$

where $C = C(x,t,\xi)$ can be found by a jump condition:

$$1 = \int_{0-}^{0+}\delta(\sigma)d\sigma = (2\pi)^{n/2}e^{ix\cdot\xi}\int_{0-}^{0+}\left[\hat{G}_\sigma + D|\xi|^2\hat{G}\right] = (2\pi)^{n/2}e^{ix\cdot\xi}\hat{G}\Big|_{\sigma=0-}^{\sigma=0+} = (2\pi)^{n/2}e^{ix\cdot\xi}C$$

where we used the fact that $\hat{G}$ is bounded, so that the second term in the integral contributes nothing, and the "causality" condition that $\hat{G} = 0$ for $\sigma < 0$. Thus we arrive at

$$\hat{G} = (2\pi)^{-n/2}e^{-ix\cdot\xi}e^{-D|\xi|^2\sigma}.$$

Finally, then, inverting the Fourier tranform, and using properties 7 and 6, we find for $\tau > 0$,

$$G(x,t;y,\sigma) = (4\pi D\sigma)^{-n/2}e^{-\frac{|y-x|^2}{4D\sigma}}$$

and so, changing back to $\tau = t-\sigma$, our free-space Green's function (also called the **fundamental solution**) of the heat equation in $\mathbb{R}^n$ is

$$G(x,t;y,\tau) = \begin{cases} (4\pi D(t-\tau))^{-n/2}e^{-\frac{|y-x|^2}{4D(t-\tau)}} & \tau < t \\ 0 & \tau > t \end{cases}.$$

We obtain a solution formula for the heat equation in $\mathbb{R}^n$,

$$\begin{cases} \frac{\partial u}{\partial t} - D\Delta u = f(x,t), & x \in \mathbb{R}^n,\ t > 0 \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n \end{cases}, \tag{27}$$

by substituting our expression for G into (25):

$$u(x,t) = \int_0^t \int_V (4\pi D(t-\tau))^{-n/2} e^{-\frac{|y-x|^2}{4\pi D(t-\tau)}} f(y,\tau) dy\, d\tau + (4\pi Dt)^{-n/2} \int_V e^{-\frac{|y-x|^2}{4Dt}} u_0(y) dy. \tag{28}$$

It is not hard to prove the following (though we will not do it here):

Theorem: Suppose $f(x,t)$ is continuously differentiable and bounded on $\mathbb{R}^n \times [0,T]$, and $u_0(x)$ is continuous on $\mathbb{R}^n$. Then for $0 < t < T$, $u(x,t)$ given by formula (28) is continuously differentiable in t and twice continuously differentiable in x, and solves the heat equation in (27). Furthmore, for any $x \in \mathbb{R}^n$, $\lim_{t \downarrow 0} u(x,t) = u_0(x)$.

Based on our free-space solution, we can infer a couple of important **properties of diffusion**:

- **Instantaneous smoothing**: in the absence of sources, solutions of the heat equation become instantaneously smooth (even if the initial data is not). In the free-space case, this property is reflected in the fact that the fundamental solution $(4\pi Dt)^{-n/2} e^{-|x|^2/4Dt}$ is infinitely differentiable for $t > 0$ (though it is a delta function at $t = 0$!), and in the second integral of (28) (the one containing the intial data $u_0(x)$), derivatives in x and t will fall on the fundamental solution.

- **Infinite propagation speed**: suppose $f \equiv 0$ (no sources), and $u_0(x) \geq 0$ is positive in a ball of radius 1 about the origin, and vanishes outside a ball of radius 2. Then for any $t > 0$, and any $x \in \mathbb{R}^n$,

$$u(x,t) = (4\pi Dt)^{-n/2} \int_{\mathbb{R}^n} e^{-|y-x|^2/(4Dt)} u_0(x) dx > 0.$$

  That is, the solution becomes positive at *all* points in space intantaneously for $t > 0$ – hence the intial, localized disturbance is propagated with infinite speed.

3. Maximum principle for the heat equation.

We have already seen that the (elliptic) maximum principle is a powerful tool for analysing solutions of Laplace and Poisson equations. An analogous (parabolic) maximum principle plays the same role for heat equations.

Theorem: [**Maximum principle for the heat equation**]. Let V be a bounded (and open, and connected) region in $\mathbb{R}^n$, and let $T > 0$. Suppose $u(x,t)$ is continuous on the closed cylinder $\bar{V} \times [0,T]$, and continously differentiable (once in t, twice in x) and solving the heat equation $\partial u/\partial t = D\Delta u$ in $V \times (0,T]$. Then the maximum (and the minimum) of u over $\bar{V} \times [0,T]$ is attained either initially (at $t = 0$) or on the spatial boundary ($x \in \partial V$).

Remark: Just as in the "elliptic" case, there is also a "strong maximum principle" which says that *if* the max. (or min.) of u is also attained at an interior point (x_0, t_0) ($x_0 \in V,\ t_0 > 0$), then $u \equiv$ const. for $t \leq t_0$. (Notice it doesn't say anything for $t > t_0$.) We will not prove this here.

Proof of the maximum principle: the same argument as for harmonic functions – i.e. considering the function

$$v(x,t) := u(x,t) + \epsilon |x|^2$$

for $\epsilon > 0$ – shows that the max. (and min.) is attained somewhere on the boundary of the cylinder (since $\partial u/\partial t = 0$ at an interior max. or min.). We only have to show that the max. (or min.) is attained somwhere on the boundary *other than* at the "final time" $t = T$. So suppose v has a max. (say) at (x_0, T), for some $x_0 \in V$. Then

$$\Delta v(x_0, T) \leq 0,$$

and

$$\frac{\partial v}{\partial t}(x_0, T) = \lim_{h \to 0+} \frac{v(x_0,T) - v(x_0, T-h)}{h} \geq 0.$$

Hence

$$0 \leq \frac{\partial v}{\partial t}(x_0, T) - D\Delta v(x_0, T) = -2D\epsilon < 0,$$

a contradiction. So for any $(x,t) \in \bar{V} \times [0,T]$, we have

$$u(x,t) = v(x,t) - \epsilon|x|^2 \leq v(x,t) \leq \max_{\partial C_T \setminus \{t=T\}} v \leq \max_{\partial C_T \setminus \{t=T\}} u + (\text{const.})\epsilon,$$

and letting $\epsilon \downarrow 0$, we find

$$u(x,t) \leq \max_{\partial C_T \setminus \{t=T\}} u$$

as desired. □

Just as for the Laplace/Poisson equation, an immediate consequence of the maximum principle is the *uniqueness* of solutions of the initial-boundary-value problem for the heat equation.

As usual let V be a bounded (open, connected) domain in $\mathbb{R}^n$. Fix any $T > 0$. Let $f(x,t)$, $g(x,t)$, and $u_0(x)$ be continuous functions (on $V \times (0,T)$, $\partial V \times [0,T]$, and V, respectively), and consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} - D\Delta u = f(x,t), & x \in V,\ \ 0 < t \le T \\ u(x,t) = g(x,t), & x \in \partial V,\ \ 0 \le t \le T \\ u(x,0) = u_0(x), & x \in V \end{cases} . \tag{29}$$

Theorem: There is at most one function $u(x,t)$ which is continuous on $\bar{V} \times [0,T]$, continuously differentiable (once in t, twice in x) in $V \times (0,T]$, and solves problem (29).

Proof: if there are 2 solutions, their difference $w(x,t)$ satisfies

$$\begin{cases} \frac{\partial w}{\partial t} - D\Delta w = 0, & x \in V,\ \ t > 0 \\ w(x,t) = 0, & x \in \partial V,\ \ t > 0 \\ w(x,0) = 0, & x \in V \end{cases} .$$

Applying the (parabolic) maximum principle, we conclude that the max. and min. values of w on the cylinder are 0. Hence $w \equiv 0$. $\square$

4. Methods of images and eigenfunction expansion.

Notation: to make the writing easier, we will often denote derivatives using subscripts, eg. $u_t = \frac{\partial u}{\partial t}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc.

Example: (diffusion on the 1/2-line). Solve

$$\begin{cases} u_t - u_{xx} = f(x,t) \quad x > 0,\ t > 0 \\ u(0,t) = 0 \quad t > 0 \\ u(x,0) = u_0(x) \quad x > 0 \end{cases} \tag{30}$$

by finding the Green's function.

Recalling our notation $\sigma = t - \tau$, the problem for the Green's function $G(x,t;y,\sigma)$ is: for $x > 0$, $t > 0$,

$$\begin{cases} G_\sigma - G_{yy} = \delta(y-x)\delta(\sigma) \quad y > 0,\ \sigma \geq 0 \\ G|_{y=0} = 0 \\ G = 0 \quad \sigma < 0 \end{cases}$$

Since the geometry is so simple, let's try the method of images. We know that the free-space Green's function (in one space dimension) with singularity at x is

$$G^f(x,t;y,\sigma) = \frac{1}{\sqrt{4\pi\sigma}} e^{-\frac{(y-x)^2}{4\sigma}}.$$

If we put our "image charge" at $-x$ (outside our domain!), we can also satisfy the boundary conditions: set, for $\sigma > 0$,

$$G(x,t;y,\sigma) := G^f(x,t;y,\sigma) - G^f(-x,t;y,\sigma) = \frac{1}{\sqrt{4\pi\sigma}} \left[e^{-\frac{(y-x)^2}{4\sigma}} - e^{-\frac{(y+x)^2}{4\sigma}} \right].$$

Then

$$G_\sigma - G_{yy} = \delta(y-x)\delta(\sigma) - \delta(y+x)\delta(\sigma) = \delta(y-x)\delta(\sigma)$$

(since $x > 0$ and $y > 0$), and furthermore

$$G|_{y=0} = \frac{1}{\sqrt{4\pi\sigma}} \left[e^{-\frac{y^2}{4\sigma}} - e^{-\frac{y^2}{4\sigma}} \right] = 0$$

so we are in business! Replacing σ by $t - \tau$, our Green's function is, for $\tau < t$ (recall it is zero for $\tau > t$),

$$G(x,t;y,\tau) = \frac{1}{\sqrt{4\pi(t-\tau)}} \left[e^{-\frac{(y-x)^2}{4(t-\tau)}} - e^{-\frac{(y+x)^2}{4(t-\tau)}} \right]$$

and the resulting solution formula for our half-line problem (30) is

$$u(x,t) = \int_0^t \int_0^\infty \frac{1}{\sqrt{4\pi(t-\tau)}} \left[e^{-\frac{(y-x)^2}{4(t-\tau)}} - e^{-\frac{(y+x)^2}{4(t-\tau)}} \right] f(y,\tau) dy\, d\tau$$
$$+ \frac{1}{\sqrt{4\pi t}} \int_0^\infty \left[e^{-\frac{(y-x)^2}{4t}} - e^{-\frac{(y+x)^2}{4t}} \right] u_0(y) dy.$$

It can be checked rigorously that for reasonable functions f and u_0 (precisely: bounded and continuous, with f also continuously differentiable), this formula gives a continuously differentiable (once in t twice in x) function u which solves (30) – but we won't do it here.

Example: (diffusion on a rod). Solve

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < L,\ t > 0 \\ u(0,t) = u(L,t) = 0 & t > 0 \\ u(x,0) = u_0(x) & 0 \le x \le L \end{cases} \tag{31}$$

by finding the Green's function.

The method of images will not work so well here, so we try an eigenfunction expansion instead. The eigenfunctions of the spatial part of the differential operator (d^2/dx^2) with zero BCs at $x = 0$ and $x = L$ are $\sin(n\pi x/L)$, $n = 1, 2, 3, \ldots$, and so we seek our Green's function in the form

$$G(x,t;y,\sigma) = \sum_{n=1}^{\infty} g_n(x,t;\sigma) \sin(n\pi y/L)$$

for $\sigma > 0$. We require $G_\sigma - G_{yy} = \delta(y-x)\delta(\sigma)$, thus

$$\sum_{n=1}^{\infty} \left[(g_n)_\sigma + \frac{n^2\pi^2}{L^2} g_n \right] \sin(n\pi y/L) = \delta(y-x)\delta(\sigma)$$

and so

$$(g_n)_\sigma + \frac{n^2\pi^2}{L^2} g_n = \frac{2}{L} \int_0^L \sin(n\pi y/L)\delta(y-x)\delta(\sigma) dy = \frac{2}{L} \sin(n\pi x/L)\delta(\sigma).$$

For $\sigma > 0$, we have $(g_n)_\sigma + (n^2\pi^2/L^2) g_n = 0$, and so

$$g_n = Ce^{-(n^2\pi^2\sigma/L^2)}$$

and we determine the constant C from a jump condition:

$$\begin{aligned} 1 &= \int_{0-}^{0+} \delta(\sigma) d\sigma = \frac{L}{2\sin(n\pi x/L)} \int_{0-}^{0+} \left[(g_n)_\sigma + (n^2\pi^2/L^2) g_n \right] d\sigma \\ &= \frac{L}{2\sin(n\pi x/L)} g_n|_{\sigma=0-}^{\sigma=0+} = \frac{L}{2\sin(n\pi x/L)} C \end{aligned}$$

(where we used $G = 0$ for $\sigma < 0$). Hence $C = 2\sin(n\pi x/L)/L$, and we have an expression for our Green's function

$$G = \frac{2}{L} \sum_{n=1}^{\infty} e^{-(n^2\pi^2\sigma/L^2)} \sin(n\pi x/L) \sin(n\pi y/L)$$

(which indeed satisfies the zero boundary conditions at $y = 0$ and $y = L$). The corresponding formula for the solution of problem (31) is

$$u(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-(n^2\pi^2 t/L^2)} \sin(n\pi x/L) \int_0^L \sin(n\pi y/L) u_0(y) dy.$$

5. Green's function for the 1D wave equation.

Suppose we measure the displacement $u(x,t)$ from equilibrium, at each point x in a bounded region V in $\mathbb{R}^n$ (representing an elastic string ($n=1$), membrane ($n=2$), or solid ($n=3$)) , and at each time t. The string/membrane/solid is subjected to forces $f(x,t)$, and at its boundary, it is held at fixed displacement $g(x,t)$ ($x \in \partial V$). At time $t=0$, at each $x \in V$, the initial displacement is $u_0(x)$, and the initial velocity is $v_0(x)$. Assuming the displacements are small, the following initial-boundary value problem for the **wave equation** is a reasonable description of the displacement $u(x,t)$ at later times $t>0$:

$$\left\{ \begin{array}{lll} \text{wave equation:} & \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = f(x,t), & x \in V, \;\; t>0 \\ \text{boundary value:} & u(x,t) = g(x,t), \;\; x \in \partial V, \;\; t>0 & \\ \text{initial conditions:} & u(x,0) = u_0(x), \;\; \frac{\partial u}{\partial t}(x,0) = v_0(x), & x \in V \end{array} \right. , \qquad (32)$$

where $c>0$ is the *wave speed.* The wave equation also describes the propagation of other waves, such as sound and light.

Just as for the heat equation, we would like to solve problems like this one via Green's functions. We will jump right in, by looking immediately for the **free-space Green's functions** (or **fundamental solutions**) in dimensions one, two, and three (unlike for the heat equation, the form of the Green's function depends significantly on the dimension).

1D free-space Green's function for the wave equation:
As with the heat equation, the Greens function should be a function of two sets of space-time variables: $x,t;y,\tau$. And as for the heat equation, it is sometimes convenient to work with $\sigma := t-\tau$, rather than τ. Hence, for $x,\ t,\ y,\ \sigma \in \mathbb{R}$, we seek $G(x,t;y,\sigma)$ solving

$$\left\{ \begin{array}{c} G_{\sigma\sigma} - c^2 G_{yy} = \delta(y-x)\delta(\sigma), \qquad -\infty < y < \infty, \;\; \sigma \geq 0 \\ G \equiv 0, \qquad \sigma < 0 \quad \text{(causality)} \end{array} \right. . \qquad (33)$$

You can think of G as the signal (eg. sound or light) emitted by a unit point source at spatial point x and at time 0.

For the sake of variety, we will solve this problem using the **Laplace transform**. Recall that the Laplace transform of a (say, bounded, continuous) function f defined on $[0,\infty)$ is another function defined on $[0,\infty)$:

$$\tilde{f}(s) = \mathcal{L}(f)(s) := \int_0^\infty e^{-st} f(t) dt.,$$

and recall the key property ("Laplace transform turns differentiation into coordinate multiplication")

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0),$$

which is easily verified by integration by parts. Thus if we let $\tilde{G}(x,t;y,s)$ be the Laplace tranform of G in the variable σ, and use $\mathcal{L}(\delta) = 1$, and the causality condition, we find

$$s^2\tilde{G} - c^2\tilde{G}_{yy} = \delta(y-x).$$

Solving this ODE (in y) to the left and right of x, and imposing the conditions that $\tilde{G}$ decay as $y \to \pm\infty$, and that is be continuous at $y = x$, yields

$$\tilde{G} = Ae^{-\frac{s}{c}|y-x|}.$$

As usual, the remaining constant A is determined by the jump condition:

$$1 = \int_{x-}^{x+} \delta(y-x)dy = \int_{x-}^{x+} \left[s^2\tilde{G} - c^2 G_{yy} \right] dx = -c^2 G_y|_{y=x-}^{y=x+} = 2scA$$

hence $A = (2sc)^{-1}$, and

$$\tilde{G}(x;y,s) = \frac{1}{2c}\frac{e^{-(|y-x|/c)s}}{s}.$$

Now for any $r \geq 0$, the Laplace transform of the *Heavyside function*

$$H(t) := \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad \Longrightarrow \quad \mathcal{L}(H(t-r)) = \int_r^\infty e^{-st}dt = \frac{e^{-rs}}{s},$$

so by comparison, we must have

$$G = \frac{1}{2c} H\left(\sigma - \frac{1}{c}|y-x|\right).$$

Restoring $\tau = t - \sigma$, we arrive at

$$G(x,t;y,\tau) = \frac{1}{2c} H\left(t - \tau - \frac{1}{c}|y-x|\right),$$

our $1D$ free-space Green's function. It is instructive to sketch a space-time graph (i.e. in the y-τ plane) of G!

Knowing the Green's function, we can find the solution to the initial value problem for the wave equation on the line:

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty,\ t > 0 \\ u(x,0) = u_0(x), & u_t(x,0) = v_0(x) \end{cases}, \tag{34}$$

which is sometimes called the *Cauchy problem* for the wave equation.

Integrating by parts, we have, for any $T > t$,

$$
\begin{aligned}
u(x,t) &= \int_0^T \int_{-\infty}^{\infty} u(y,\tau)\delta(y-x)\delta(\tau-t)dyd\tau = \int_0^T \int_{-\infty}^{\infty} u(y,\tau)(G_{\tau\tau} - c^2 G_{yy})dyd\tau \\
&= \int_0^T \int_{-\infty}^{\infty} (u_{\tau\tau} - c^2 u_{yy})Gdyd\tau + \int_{-\infty}^{\infty} (uG_\tau - u_\tau G)|_{\tau=0}^{\tau=T} dy \\
&= \int_{-\infty}^{\infty} (v_0(y)G(x,t;y,0) - u_0(y)G_\tau(x,t;y,0))dy
\end{aligned}
$$

where we used the causality condition $G \equiv G_\tau \equiv 0$ for $\tau > t$. Now, since $H' = \delta$,

$$
G_\tau = -G_\sigma = -\frac{1}{2c}\delta(\sigma - \frac{1}{c}|y-x|) = -\frac{1}{2}\left[\delta(y-(x+ct)) + \delta(y-(x-ct))\right],
$$

and so our solution to (34) is

$$
u(x,t) = \frac{1}{2c}\left[u_0(x+ct) + u_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)dy,
$$

which is known as **D'Alembert's formula**.

Remark:

- Notice that D'Alembert's formula represents the sum of two waves, one moving to the left with speed c, one moving to the right with speed c:

$$
u(x,t) = f^+(x+ct) + f^-(x-ct), \qquad f^\pm(z) := \frac{1}{2}u_0(z) \pm \frac{1}{2c}\int_0^z v_0(z)dz.
$$

- It is simple to check D'Alembert's formula does, in fact, solve problem (34). Indeed, if f is *any* twice differentiable function, then $f(x \pm ct)$ solves the wave equation: $[\frac{\partial}{\partial t^2} - \frac{\partial}{\partial x^2}]f(x \pm ct) = c^2 f'' - c^2 f'' = 0$. It is also easy to check the initial data are satisfied. Hence, assuming u_0 is twice differentiable, and v_0 is once differentiable, we have solved problem (34).

- **Finite speed of propagation:** The solution at space-time point (x,t) depends only on the initial data (u_0 and v_0) in the interval $[x-ct, x+ct]$ (draw a graph!) – that is, signals propagate with speed at most c.

- *Sidenote:* D'Alembert's formula makes sense even if u_0 and v_0 are *not* differentiable (just continuous) – though then the PDE doesn't hold in the classical sense, only in the sense of distributions.

6. The wave equation in higher dimensions.

2D free-space Green's function for the wave equation:
We look for a Green's function depending on the spatial variable y only through $r := |y - x|$, leading to the problem

$$\begin{cases} G_{\sigma\sigma} - c^2 \left[G_{rr} + \frac{1}{r} G_{rr}\right] = \delta(\sigma)\delta(y - x), & r > 0, \sigma > 0 \\ G \equiv 0, & \sigma < 0 \end{cases}.$$

Taking again the Laplace transform in the variable σ we are lead to

$$s^2 \tilde{G} - c^2 \left[\tilde{G}_{rr} + \frac{1}{r}\tilde{G}_r\right] = \delta(y - x).$$

For $r > 0$, the ODE

$$\tilde{G}_{rr} + \frac{1}{r}\tilde{G}_r - \frac{s^2}{c^2}\tilde{G} = 0$$

is solved by $\tilde{G} = AK_0((s/c)r)$, where $K_0(z)$ is the *modified Bessel function of order* 0 which solves

$$\begin{cases} z^2 K_0''(z) + zK_0'(z) - z^2 K_0(z) = 0 \\ K_0(z) \sim \log z \text{ as } z \to 0 \\ K_0(z) \sim \text{ (const) } \frac{e^{-z}}{\sqrt{z}} \text{ as } z \to \infty \end{cases}.$$

The constant A can be determined using the divergence theorem: denoting the disk of radius ϵ about x by B_ϵ,

$$\begin{aligned} 1 &= \lim_{\epsilon \to 0+} \int_{B_\epsilon} \delta(y - x) dy = -c^2 \lim_{\epsilon \to 0+} \int_{B_\epsilon} \Delta \tilde{G} dy = -c^2 \lim_{\epsilon \to 0+} \int_{r=\epsilon} \frac{\partial \tilde{G}}{\partial n} dS \\ &= -c^2 A \frac{s}{c} \lim_{\epsilon \to 0+} \int_{r=\epsilon} \frac{c}{sr} dS = -2\pi c^2 A \end{aligned}$$

so $A = -(2\pi c^2)^{-1}$, and

$$\tilde{G} = -\frac{1}{2\pi c^2} K_0\left(\frac{s}{c} r\right).$$

It turns out we can invert the Lalpace transform on the Bessel function explicity. In fact, for $b > 0$, we have

$$\mathcal{L}\left(\frac{1}{\sqrt{t^2 - b^2}} H(t - b)\right) = -K_0(bs),$$

and so

$$G = \frac{1}{2\pi c^2} \frac{1}{\sqrt{\sigma^2 - r^2/c^2}} H(\sigma - r/c)$$

and replacing $\tau = t - \sigma$, $r = |y - x|$, our two-dimensional free-space Green's function for the wave equation is

$$G(x,t;y,\tau) = \frac{1}{2\pi c^2 \sqrt{(t-\tau)^2 - \frac{1}{c^2}|y-x|^2}} H\left(t - \tau - \frac{1}{c}|y-x|\right).$$

Remark: As with the 1D wave equation, the fact that the Green's function is supported in $\{\ |y-x| \leq c(t-\tau)\ \}$ shows that signals propagate with speed no greater than c. Notice also, as in the 1D case, signals do not propagate "sharply" in 2D, since for fixed x, y, τ, the Green's function is non-zero for times t beyond $\tau + \frac{1}{c}|y-x|$ (though it does decay like $1/t$).

It is left as an excercise to use this expression for the free-space Green's function to show that the solution of the Cauchy problem

$$\begin{cases} u_{tt} = c^2 \Delta u & x \in \mathbb{R}^2,\ t > 0 \\ u(x,0) = u_0(x), & u_t(x,0) = v_0(x) \end{cases},$$

for the 2D wave equation, is

$$u(x,t) = \frac{1}{2\pi c^2}\left[\int_{|y-x|\leq ct} \frac{v_0(y)}{\sqrt{t^2 - \frac{1}{c^2}|y-x|^2}} dy + \frac{\partial}{\partial t}\int_{|y-x|\leq ct} \frac{u_0(y)}{\sqrt{t^2 - \frac{1}{c^2}|y-x|^2}} dy\right],$$

sometimes known as *Poisson's formula.*

3D free-space Green's function for the wave equation:
We again look for a Green's function depending on the spatial variable y only through $r := |y-x|$, leading to the problem

$$\begin{cases} G_{\sigma\sigma} - c^2\left[G_{rr} + \frac{2}{r}G_{rr}\right] = \delta(\sigma)\delta(y-x), & r > 0, \sigma > 0 \\ G \equiv 0, & \sigma < 0 \end{cases}.$$

Taking again the Laplace transform in the variable σ we are lead to

$$s^2\tilde{G} - c^2\left[\tilde{G}_{rr} + \frac{2}{r}\tilde{G}_r\right] = \delta(y-x).$$

For $r > 0$, the ODE

$$\tilde{G}_{rr} + \frac{2}{r}\tilde{G}_r - \frac{s^2}{c^2}\tilde{G} = 0$$

is solved by (we've seen this before in a homework problem)

$$\tilde{G} = A\frac{e^{-\frac{s}{c}r}}{r}.$$

Again, we find the constant A using the diveregence theorem. Let B_ϵ be the ball of radius ϵ about x. Then

$$
\begin{aligned}
1 &= \lim_{\epsilon\to 0+}\int_{B_\epsilon}\delta(y-x)dy = -c^2\lim_{\epsilon\to 0+}\int_{B_\epsilon}\Delta\tilde{G}dy = -c^2\lim_{\epsilon\to 0+}\int_{r=\epsilon}\frac{\partial\tilde{G}}{\partial n}dS \\
&= -c^2\lim_{\epsilon\to 0+}\int_{r=\epsilon}A\left[-\frac{s}{c}\frac{e^{-\frac{s}{c}r}}{r}-\frac{e^{-\frac{s}{c}r}}{r^2}\right]dS = 4\pi c^2 A
\end{aligned}
$$

so $A=(4\pi c^2)^{-1}$, and

$$\tilde{G}=\frac{1}{4\pi c^2}\frac{e^{-\frac{s}{c}r}}{r}.$$

Now notice that for $b>0$,

$$\mathcal{L}(\delta(t-b))=\int_0^\infty e^{-st}\delta(t-b)dt=e^{-bs}$$

and so by comparison,

$$G=\frac{1}{4\pi c^2 r}\delta(\sigma-r/c).$$

Reinstating $\tau=t-\sigma$, we have the 3D free-space Green's function for the wave equation,

$$G(x,t;y,\tau)=\frac{1}{4\pi c^2|y-x|}\delta\left(t-\tau-\frac{1}{c}|y-x|\right).$$

Remark: Notice that the Green's function is supported *exactly* on the cone $\{\ |y-x|=c(t-\tau)\ \}$ (sketch a graph). So not only do signals propagate with speed c, they are also *crisp* in 3D – standing at y, you receive a signal emitted at time $\tau=0$ and point x, exactly at time $t=|y-x|/c$ and then it is gone! It's nice to live in 3D.

Again, it is left as an excercise to use this expression for the free-space Green's function to show that the solution of the Cauchy problem

$$\left\{\begin{array}{ll} u_{tt}=c^2\Delta u & x\in\mathbb{R}^3,\ t>0 \\ u(x,0)=u_0(x), & u_t(x,0)=v_0(x) \end{array}\right.,$$

for the 3D wave equation, is

$$u(x,t)=\frac{1}{4\pi c^2}\left[\frac{1}{t}\int_{|y-x|=ct}v_0(y)dS(y)+\frac{\partial}{\partial t}\frac{1}{t}\int_{|y-x|=ct}u_0(y)dS(y)\right],$$

sometimes known as *Kirchoff's formula*. Note again the crisp signal propagation, reflected in the fact that the solution at (x,t) is given in terms of integrals of the data over the sphere of radius ct about x.