

Basic Gate Repertoire

Are we sure we have all the gates we need?
Just how many two-input gates are there?

AND		OR		NAND		NOR	
AB	Y	AB	Y	AB	Y	AB	Y
00	0	00	0	00	1	00	1
01	0	01	1	01	1	01	0
10	0	10	1	10	1	10	0
11	1	11	1	11	0	11	0



Hmmmm... all of these have 2-inputs (no surprise)
... each with 4 combinations, giving 2^2 output cases

How many ways are there of assigning 4 outputs? $2^{2^2} = 2^4 = 16$

There are only so many gates

There are only 16 possible 2-input gates

... some we know already, others are just silly

I N P U T AB																
	Z	A	A		B		X		N	X	N	A	N	B	N	O
	E	R	N	>	>		O	O	O	O	O	<=	T	<=	A	N
	O	D	B	A	A	B	R	R	R	R	'B'	B	'A'	A	D	E
00	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
01	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
10	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
11	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

How many of these gates can be implemented using a single CMOS gate?



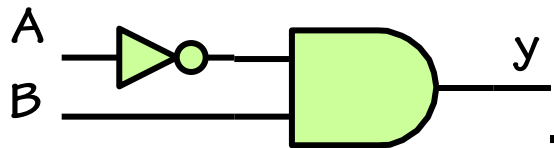
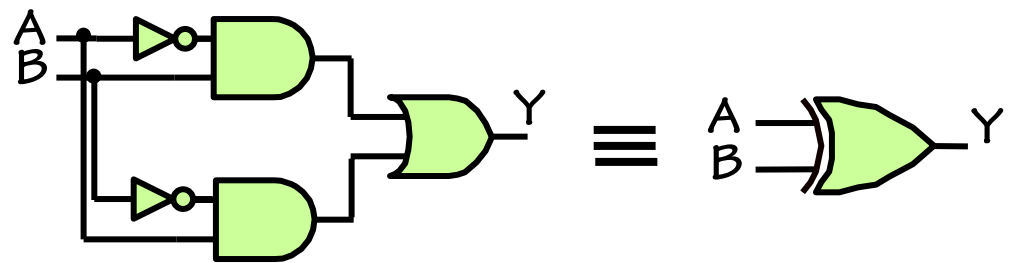
CMOS gates are inverting; we can always respond positively to positive transitions by cascaded gates. But suppose our logic yielded cheap *positive* functions, while inverters were expensive...

Fortunately, we can get by with a few basic gates...

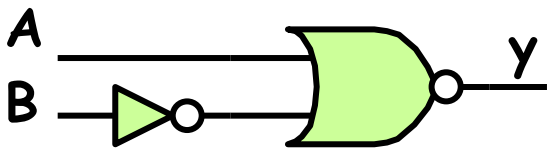
AND, OR, and NOT are sufficient... (cf Boolean Expressions):

B > A		
AB		Y
00		0
01		1
10		0
11		0

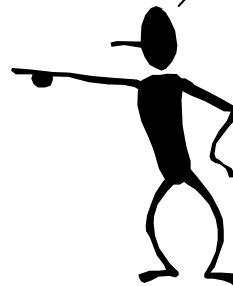
XOR		
AB		Y
00		0
01		1
10		1
11		0



$$\overline{A}B = \overline{A+B}$$



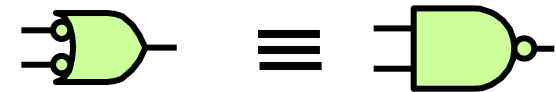
That is just
DeMorgan's
Theorem!



$$\overline{\overline{A}\overline{B}} = \overline{\overline{A+B}}$$



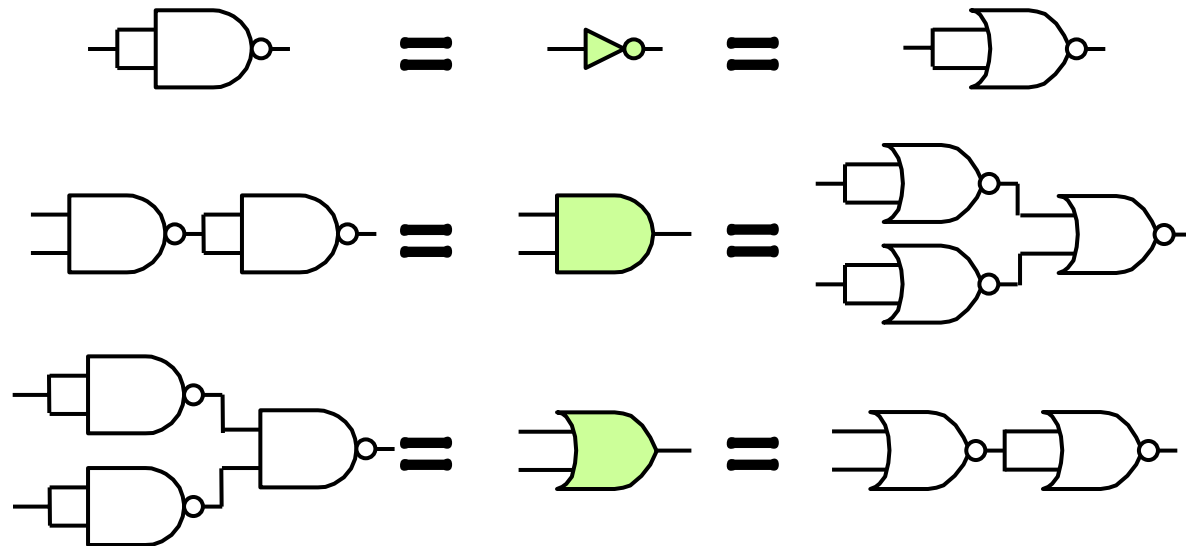
$$\overline{A} + \overline{B} = \overline{AB}$$



How many different gates do we *really* need?

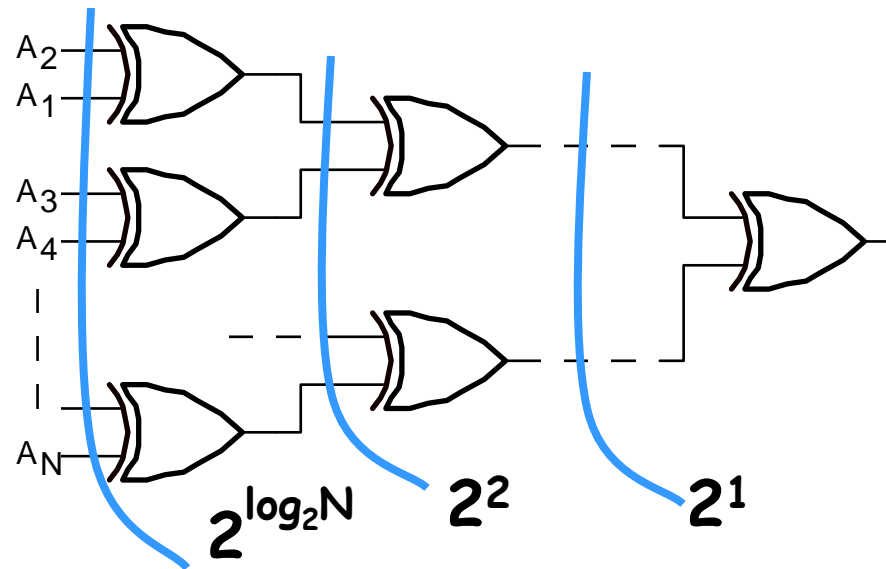
One will do!

NANDs and NORs are universal:



Ah!, but what if we want more than 2 inputs?

I think that I shall never see
a circuit lovely as...



N-input TREE has $O(\log N)$ levels...

Signal propagation takes $O(\log N)$ gate delays.

Question: Can EVERY N-Input Boolean function be implemented as a tree of 2-input gates?

Here's a Design Approach

Truth Table

C	B	A	Y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

1) Write out our functional spec as a truth table

2) Write down a Boolean expression for every '1' in the output

$$Y = \overline{C} \overline{B} A + \overline{C} B A + C B \overline{A} + C B A$$

3) Wire up the gates, call it a day, and declare success!

This approach will always give us Boolean expressions in a particular form:

SUM-OF-PRODUCTS

-it's systematic!
-it works!
-it's easy!
-are we done yet???

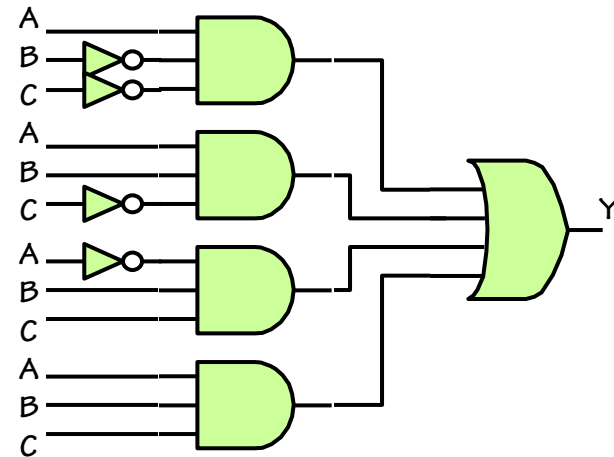


Straightforward Synthesis

We can implement
SUM-OF-PRODUCTS
with just three levels of
logic.

INVERTERS/AND/OR

Propagation delay --
No more than "3" gate delays
(well, it's actually $O(\log N)$ gate delays)



Logic Simplification

Can we implement the same function with fewer gates? Before trying we'll add a few more tricks in our bag.

BOOLEAN ALGEBRA:

OR rules: $a + 1 = 1, a + 0 = a, a + a = a$

AND rules: $a1 = a, a0 = 0, aa = a$

Commutative: $a + b = b + a, ab = ba$

Associative: $(a + b) + c = a + (b + c), (ab)c = a(bc)$

Distributive: $a(b+c) = ab + ac, a + bc = (a+b)(a+c)$

Complements: $a + \bar{a} = 1, a\bar{a} = 0$

Absorption: $a + ab = a, a + \bar{a}b = a + b$

$$a(a + b) = a, a(\bar{a} + b) = ab$$

Reduction: $ab + \bar{a}b = b, (a + b)(\bar{a} + b) = b$

DeMorgan's Law: $\bar{a} + \bar{b} = \overline{ab}, \overline{\bar{a}b} = \overline{a + b}$

Boolean Minimization:

An Algebraic Approach

Lets (again!) simplify

$$Y = \overline{C}\overline{B}A + C\overline{B}\overline{A} + CBA + \overline{C}BA$$

Using the identity

$$\alpha A + \alpha \overline{A} = \alpha$$

For any expression α and variable A :

$$Y = \overline{C}\overline{B}A + C\overline{B}\overline{A} + CBA + \overline{C}BA$$

$$Y = \overline{C}\overline{B}A + C\overline{B}\overline{A} + CBA + \overline{C}BA$$

$$Y = \overline{C}A + CB$$

Can't he come up
with a new example???



Hey, I could write
A program to do
That!



Karnaugh Maps: A Geometric Approach

K-Map: a truth table arranged so that terms which differ by exactly one variable are adjacent to one another so we can see potential reductions easily.

Here's the layout of a 3-variable K-map filled in with the values from our truth table:

Truth Table

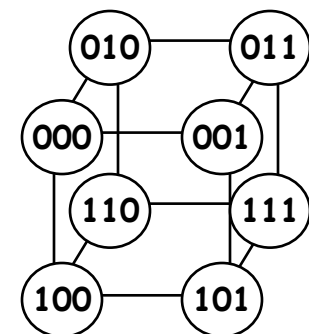
C	A	B	Y
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

C\AB	00	01	11	10
0	0	0	1	1
1	0	1	1	0

Why did he shade that row Gray?

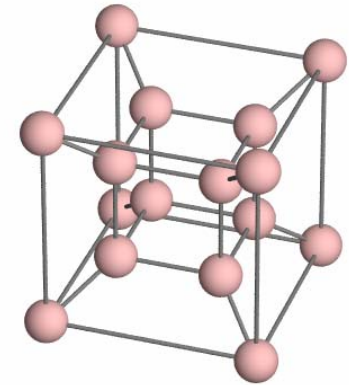


It's cyclic. The left edge is adjacent to the right edge. (It's really just a flattened out cube).



On to Hyperspace

4-variable K-map for a multipurpose logic gate:



$$Y = \begin{cases} A \times B & \text{if } CD = 00 \\ A + B & \text{if } CD = 01 \\ \overline{B} & \text{if } CD = 10 \\ A \oplus B & \text{if } CD = 11 \end{cases}$$

$\begin{smallmatrix} \backslash AB \\ CD \backslash \end{smallmatrix}$	00	01	11	10
00	0	0	1	0
01	0	1	1	1
11	0	1	0	1
10	1	0	0	1

Again it's cyclic. The left edge is adjacent to the right edge, and the top is adjacent to the bottom.

Finding Subcubes

We can identify clusters of “irrelevant” variables by circling adjacent subcubes of 1s. A subcube is just a lower dimensional cube.

C\AB	00	01	11	10
0	0	0	1	1
1	0	1	1	0

\AB CD\	00	01	11	10
00	0	0	1	0
01	0	1	1	1
11	0	1	0	1
10	1	0	0	1

The best strategy is generally a greedy one.

- Circle the largest N-dimensional subcube (2^N adjacent 1's)
4x4, 4x2, 4x1, 2x2, 2x1, 1x1
- Continue circling the largest remaining subcubes
(even if they overlap previous ones)
- Circle smaller and smaller subcubes until no 1s are left.

Write Down Equations

Write down a product term for the portion of each cluster/subcube that is invariant. You only need to include enough terms so that all the 1's are covered. Result: a **minimal sum of products** expression for the truth table.

C\AB	00	01	11	10
0	0	0	1	1
1	0	1	1	0

$$Y = \bar{C}A + CB$$

We're done!



\AB CD\	00	01	11	10
00	0	0	1	0
01	0	1	1	1
11	0	1	0	1
10	1	0	0	1

$$Y = ABC\bar{C} + \bar{A}BD + A\bar{B}D + \bar{B}C\bar{D}$$

Recap: K-map Minimization

- 1) Copy truth table into K-Map
- 2) Identify subcubes,
selecting the largest available subcube at each step, even if it involves some overlap with previous cubes, until all ones are covered. (Try: 4x4, 2x4 and 4x2, 1x4 and 4x1, 2x2, 2x1 and 1x2, finally 1x1)
- 3) Write down the minimal SOP realization

Truth Table

C	B	A	Y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

JARGON: The circled terms are called *implicants*. An implicant not completely contained in another implicant is called a *prime implicant*.

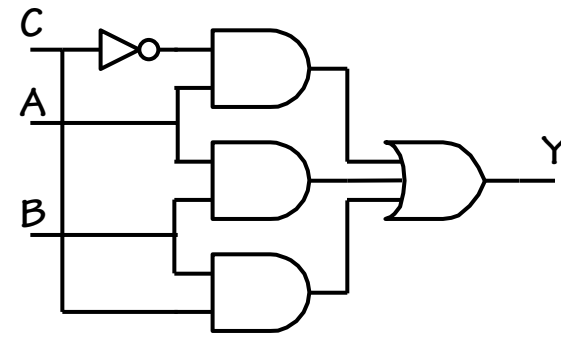
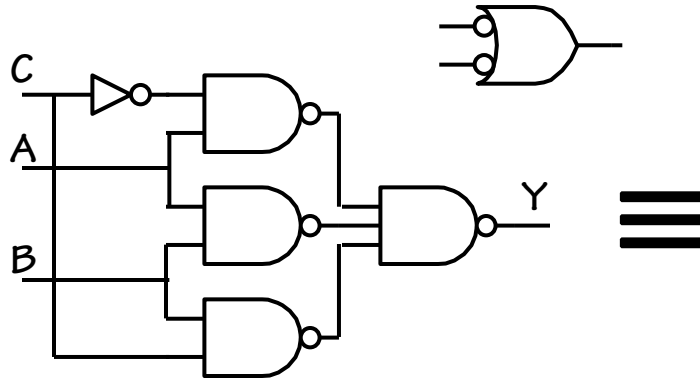
C\BA	00	01	11	10
0	0	1	1	0
1	0	0	1	1

$$Y = \overline{C}A + CB$$

Practical SOP Implementation

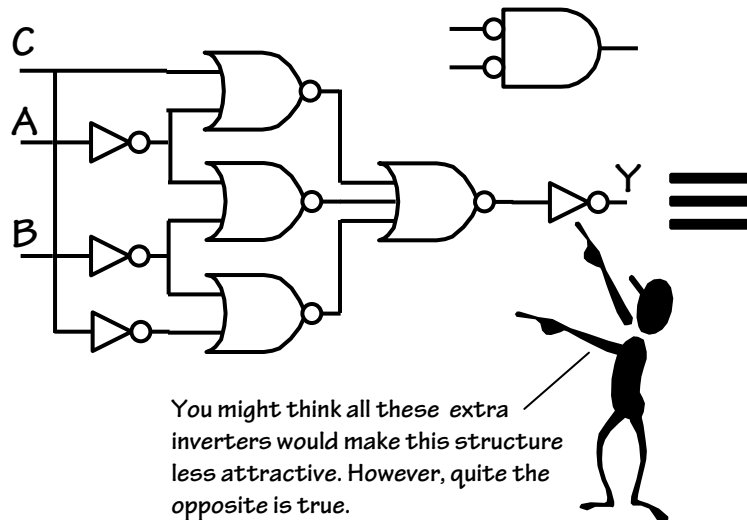
- NAND-NAND $\overline{AB} = \overline{A} + \overline{B}$

“Pushing Bubbles”

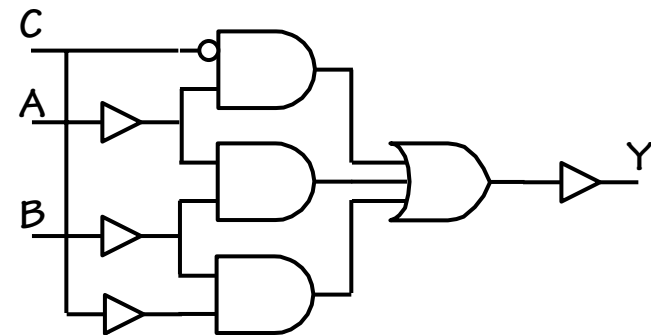


$$\overline{xyz} = \overline{x} + \overline{y} + \overline{z}$$

- NOR-NOR $\overline{\overline{A}\overline{B}} = \overline{\overline{A+B}}$



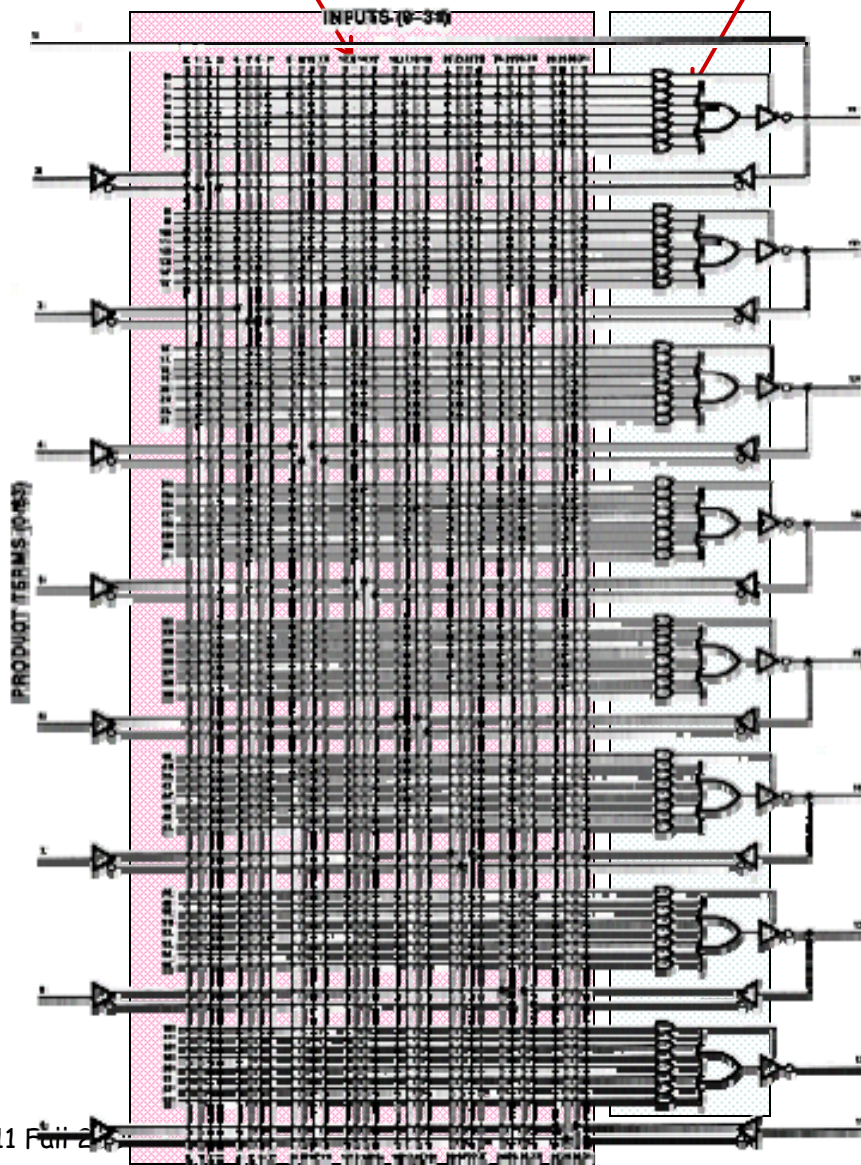
You might think all these extra inverters would make this structure less attractive. However, quite the opposite is true.



$$\overline{x + y} = \overline{x} \overline{y}$$

PALs: Programmable Array Logic

User-programmable ANDs Fixed ORs

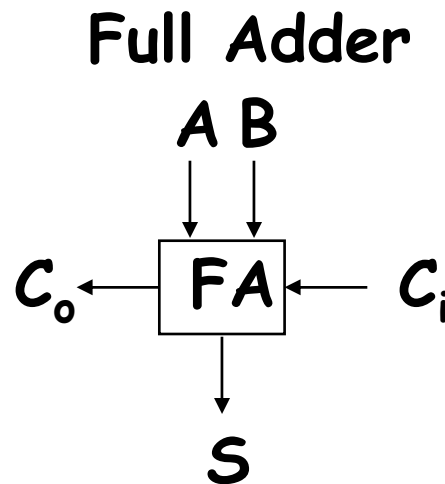


Another approach to structured logic design is Programmable Array Logic (PAL). These were once popular off-the-shelf devices. They basically replaced TTL gates in the '80s and fueled the minicomputer revolution.

PALs have a programmable decoder (AND plane) with fixed selector logic (OR plane). These devices were useful for implementing large fan-in gates and SOP logic expressions. They are purchased as unprogrammed chips and configured in the field using an inexpensive programmer.

Logic that defies SOP simplification

C_i	A	B	S	C_o
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1



S

C/AB	00	01	11	10
0	0	1	0	1
1	1	0	1	0

C_o

C/AB	00	01	11	10
0	0	0	1	0
1	0	1	1	1

$$S = A\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}\bar{B}C + ABC$$

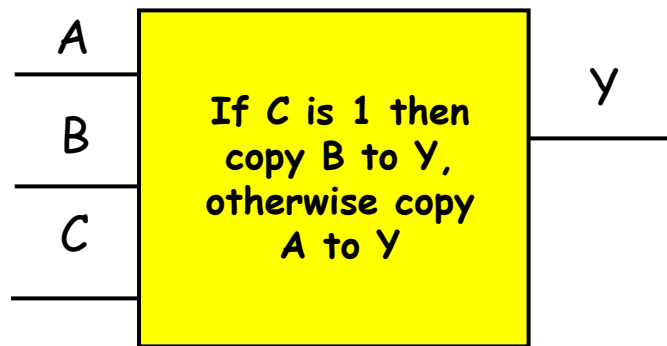
$$C_o = AB\bar{C} + \bar{A}BC + \bar{A}BC + ABC$$

Can simplify the carry out easily enough, eg...

$$C_o = BC + AB + AC$$

But, the sum, S , doesn't have a simple sum-of-products implementation even though it can be implemented using only two 2-input XOR gates.

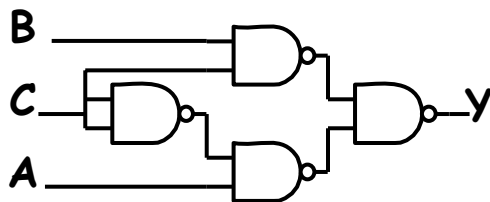
Logic Synthesis Using MUXes



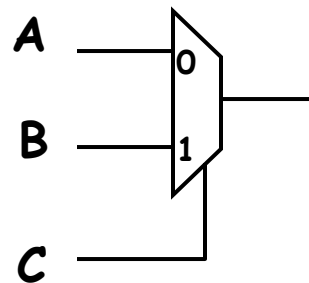
2-input Multiplexer

Truth Table

C	B	A	Y
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

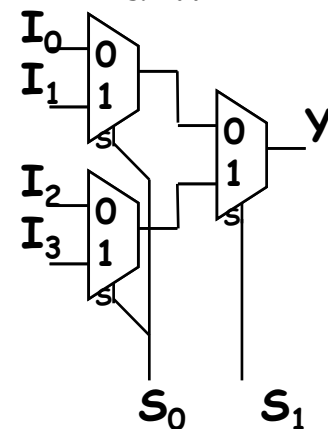


schematic



Gate
symbol

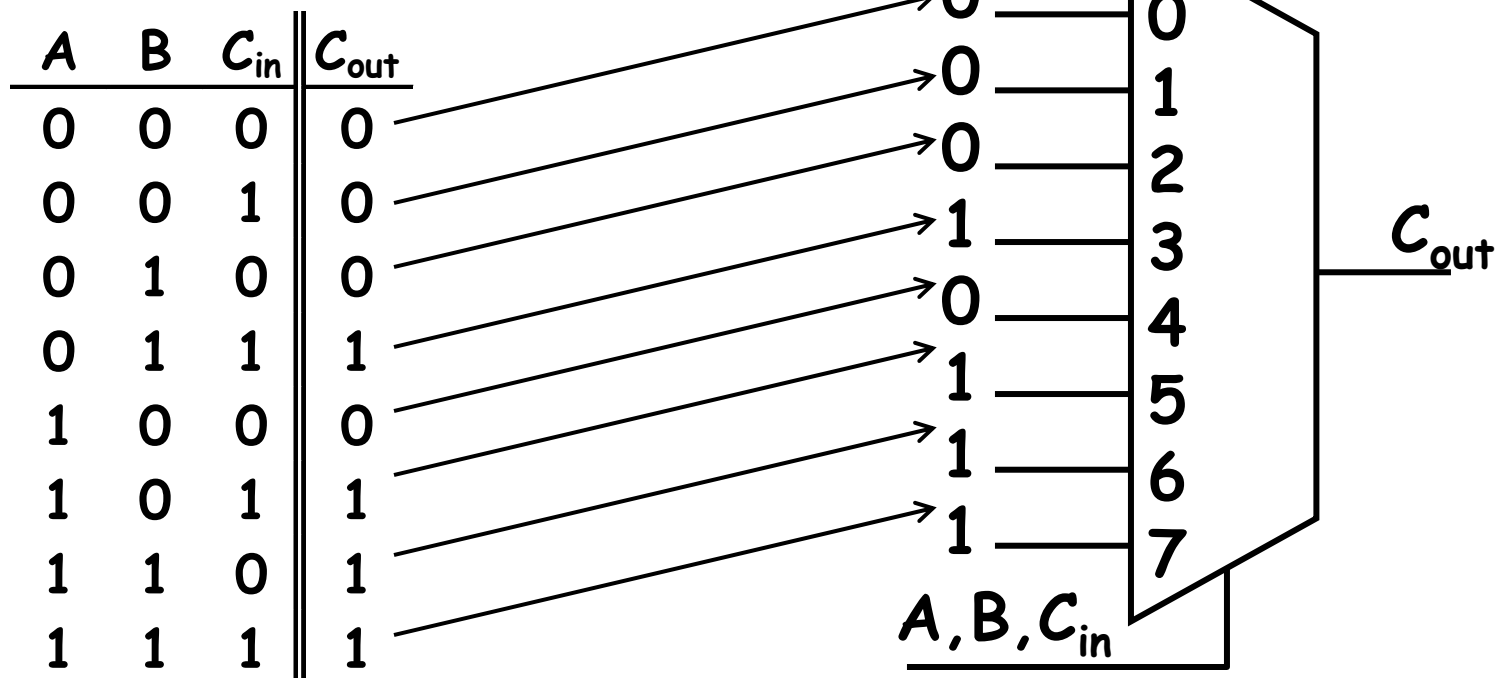
A 4-input Mux
implemented as
a tree



Systematic Implementation of Combinational Logic

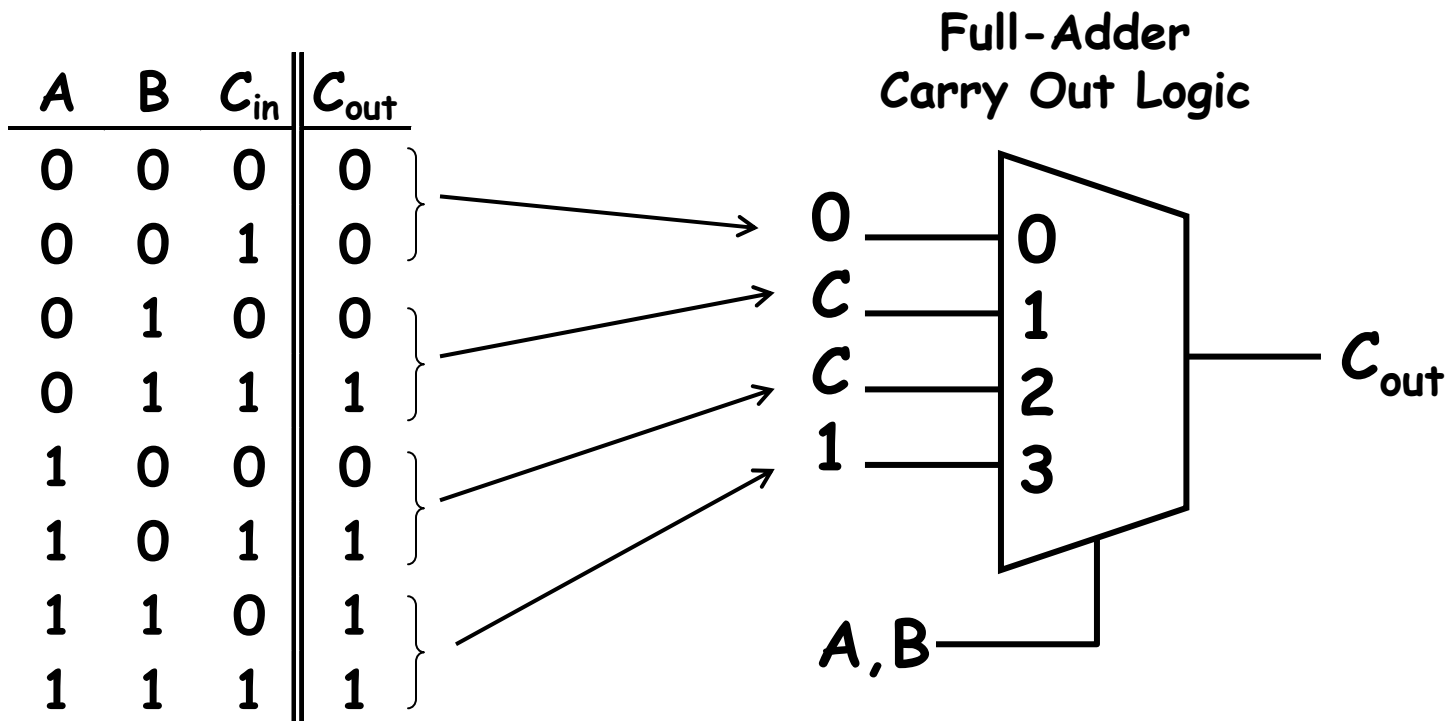
Consider implementation of some arbitrary Boolean function, $F(A,B)$

... using a MULTIPLEXER as the only circuit element:

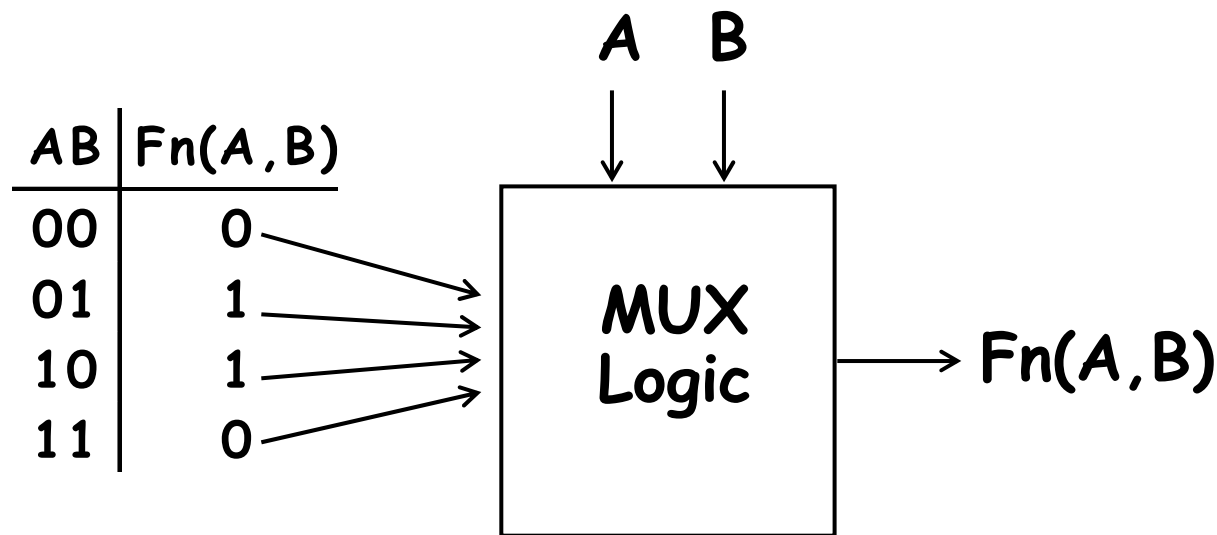


Systematic Implementation of Combinational Logic

Same function as on previous slide, but this time let's use a 4-input mux



General Table Lookup Synthesis



Generalizing:

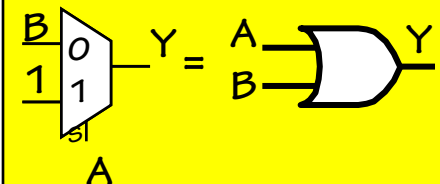
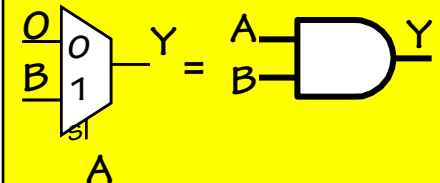
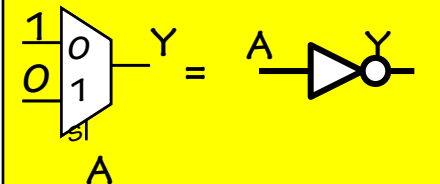
In theory, we can build any 1-output combinational logic block with multiplexers.

For an N-input function we need a 2^N input mux.

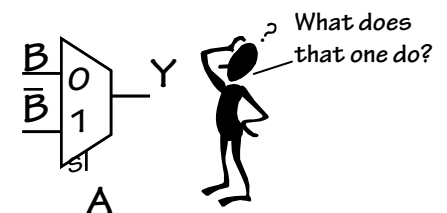
BIG Multiplexers?

How about 10-input function? 20-input?

Muxes are UNIVERSAL!

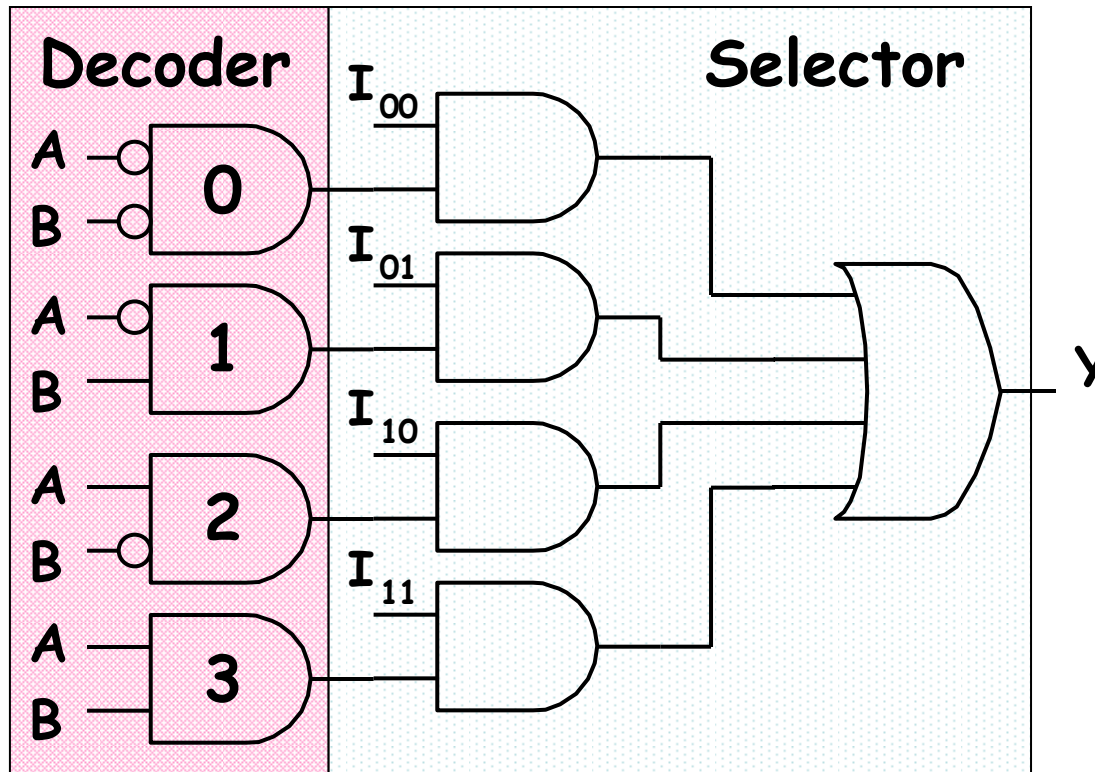


In future technologies
muxes might be the
“natural gate”.



A Mux's Guts

A decoder generates all possible product terms for a set of inputs



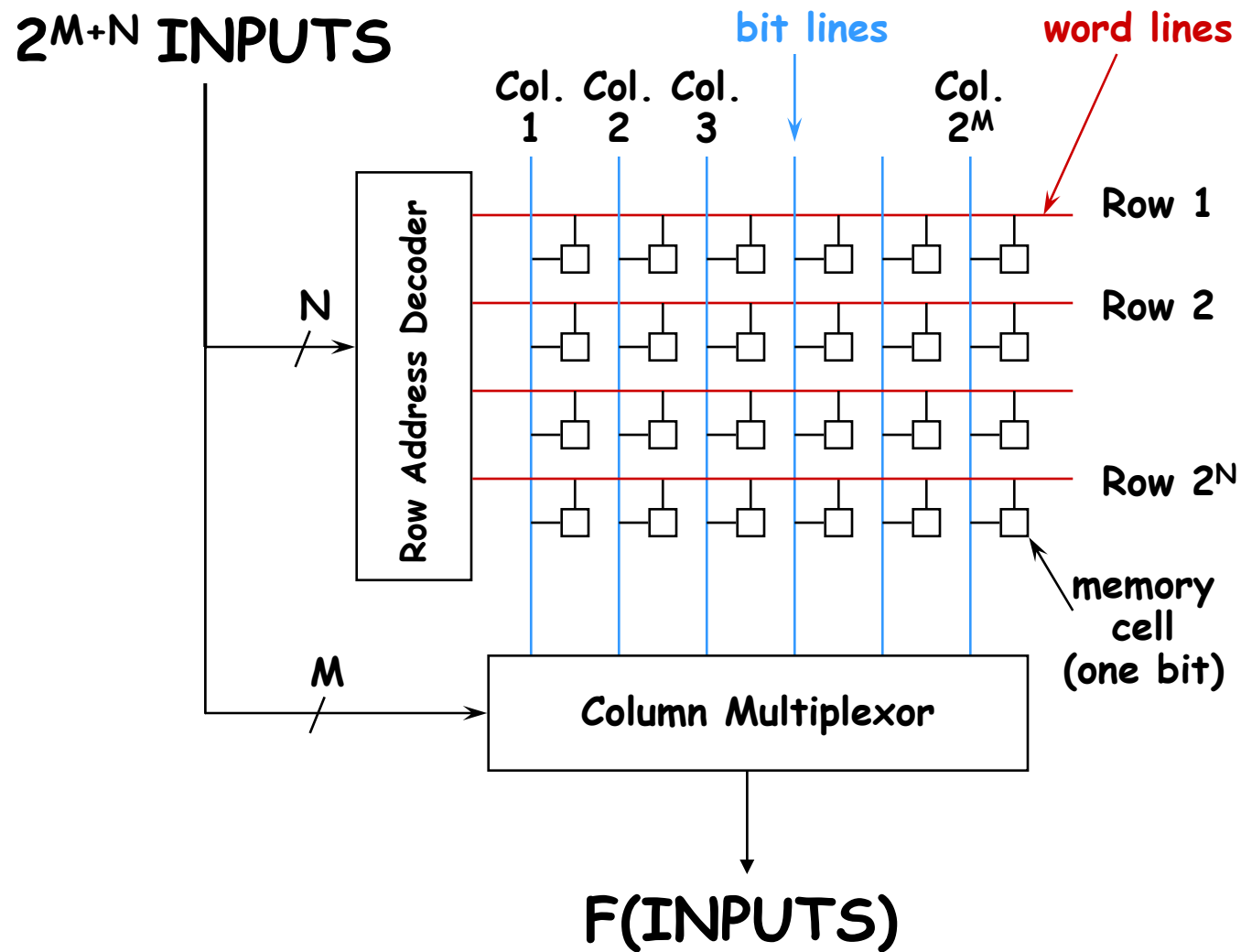
Multiplexers can be constructed into two sections:

A **DECODER** that identifies the desired input, and

a **SELECTOR** that enables that input onto the output.

Hmmm, by sharing the decoder part of the logic MUXs could be adapted to make lookup tables with any number of outputs

Using Memory as a Programmable Logic Device



FPGA Logic Block

