how the top of the hump moves, first left, then right. Can you explain this by looking at the characteristics?

Chapter 3

Diffusion

In this chapter we investigate equations which model diffusion processes, such as heat flow in a solid, or the spread of dye in water. There is an important difference between a diffusion process and the wave motion studied in Chapter 2. In diffusion, initial data is smeared and smoothed out; there are no sharp fronts. We shall see that this is a result of the constitutive relation (or diffusion law). In the latter part of this chapter we will combine nonlinear wave motion and diffusion.

3.1 The diffusion equation

We shall derive a basic diffusion equation in the context of heat flow. Assume that we have a bar of some material of constant cross section that is surrounded by insulation so that heat can only flow along the bar and not out of the cylindrical surface. We shall eventually assign conditions to the ends of the bar to control the flow of heat in or out of the ends, but for the moment we assume that the bar is so long that we can neglect what happens there, that is, we assume that the bar is infinitely long. Putting the x axis along the bar, we say that there should be a well-defined temperature $u(x,t)$ at each position x in the bar, and at each time $t > 0$. We are assuming that the temperature is uniform across each cross section. Let c be the specific heat of the material: the amount of heat energy, usually in calories, needed to raise the temperature of a unit mass of the material one degree centigrade. We shall assume that c does not depend on the temperature u of the material in the range of temperatures we consider but that c may depend on x. Let ρ be the linear density of the material. ρ may also depend on x. Then

$$\int_a^b u(x,t)c(x)\rho(x)dx$$

is an expression for the amount of heat energy in that portion of the bar in $a \le x \le b$. The rate of change of the heat energy in this portion of the bar is given by

$$\frac{d}{dt}\int_a^b u(x,t)c(x)\rho(x)dx.$$

The rate of change of heat energy in $[a, b]$ is the rate at which heat enters and leaves this portion of the bar through its ends, plus the rate at which heat energy is created or absorbed by an internal source (for instance an electric heating element). Assuming for the moment no internal sources, we can write

$$\frac{d}{dt}\int_a^b c(x)\rho(x)u(x,t)dx = F(a,t) - F(b,t), \tag{3.1}$$

where the flux $F(x, t)$ is the rate at which heat energy passes the point x, with the convention that $F(x, t) \ge 0$ if the heat energy is flowing from left to right. Thus if u and F are C^1, then

$$\int_a^b [c(x)\rho(x)u_t(x,t) + \partial_x F(x,t)]dx = 0$$

for all intervals $[a, b]$. We conclude that

$$c(x)\rho(x)\partial_t u(x,t) + \partial_x F(x,t) = 0. \tag{3.2}$$

This is virtually the same as equation (2.10) in Chapter 2 that arose in wave motion. The flux in Chapter 2 depended only on the density u. A moment's reflection, however, shows that the heat flux could not depend only on the temperature, for if the temperature is constant, there should be no heat flow. Rather the heat flux should depend on the spatial rate of change of the temperature. This is expressed in Fourier's law of cooling:

$$F(x,t) = -\kappa u_x(x,t).$$

$\kappa > 0$ is a constant of proportionality determined by the material. It is called the heat conductivity. Thus if the temperature is decreasing from left to right at x, heat flows from left to right, that is, $F(x, t) > 0$. If the material is homogeneous, κ is independent of x. However in a material which varies with x, perhaps because of impurities, κ may depend on x. Allowing κ to depend on x, we substitute $F(x,t) = -\kappa(x)u_x(x,t)$ in (3.2) under the additional assumption that u is C^2 in x. Then

$$c(x)\rho(x)u_t(x,t) = \partial_x(\kappa(x)u_x(x,t)). \tag{3.3}$$

If we add a source term to (3.1), it will be of the form

$$\int_a^b c(x)\rho(x)q(x,t)dx$$

where q has units degrees/time. This yields the inhomogeneous equation

$$u_t(x,t) = \frac{1}{c(x)\rho(x)}\partial_x(\kappa(x)u_x(x,t)) + q(x,t).$$

If c, ρ, κ are constant (a uniform material), and $q = 0$, we arrive at

$$u_t(x,t) = ku_{xx}(x,t), \tag{3.4}$$

where $k = \kappa/c\rho$ is the diffusion constant which has units $(\text{length})^2$/time. Equation (3.4) is the well known heat equation. When the source term is present, we have the inhomogeneous equation

$$u_t = ku_{xx} + q. \tag{3.5}$$

We can use essentially the same derivation to describe the diffusion of a chemical, say dye, in a liquid. In this case $u(x, t)$ is the concentration of the dye in gm/cm. The analogue of Fourier's law of cooling is known as Fick's law of diffusion which states that the flux is proportional to the spatial rate of change of the concentration, and that dye moves from regions of higher concentration to regions of lower concentration:

$$F(x,t) = -ku_x(x,t).$$

We are led to the same equation (3.5).

Equation (3.5) also arises in the study of Brownian motion, and we can give a probabilistic interpretation to the solutions of (3.5). See the book [Fe]. For diffusion in the biological context, including the spread of genes in a population, see [E].

Jump conditions

Suppose the bar consists of two different materials, with an interface at $x = a$, so that

$$c(x) = \begin{cases} c_r & \text{for } x > a \\ c_l & \text{for } x < a \end{cases},$$

$$\rho(x) = \begin{cases} \rho_r & \text{for } x > a \\ \rho_l & \text{for } x < a \end{cases},$$

and

$$\kappa(x) = \begin{cases} \kappa_r & \text{for } x > a \\ \kappa_l & \text{for } x < a \end{cases}.$$

Let $k_r = \kappa_r/c_r\rho_r$ and $k_l = \kappa_l/c_l\rho_l$ be the two diffusion constants. Heat flow in this bar is governed by two heat equations and conditions at the interface linking the two solutions. Let u_l solve the first equation and u_r the second:

$$u_t = k_l u_{xx} \qquad \text{for } x < a\ ,$$

$$u_t = k_r u_{xx} \qquad \text{for } x > a\ .$$

They are linked across the interface by the conditions

$$u_l(a) = u_r(a),$$

continuity of the temperature, and

$$\kappa_l \partial_x u_l(a) = \kappa_r \partial_x u_r(a), \tag{3.6}$$

which is continuity of the flux. Note that the second of these conditions forces the first derivative u_x to have a jump across the interface if $\kappa_r \neq \kappa_l$.

Equation (3.6), the continuity of the flux, is derived from the integral law of diffusion (3.1) as follows. Apply (3.1) to the short interval $[a-\varepsilon, a+\varepsilon]$. Now, from the left side of (3.1), we see that

$$\int_{a-\varepsilon}^{a+\varepsilon} c(x)\rho(x)u_t(x)dx \to 0, \quad \text{as } \varepsilon \to 0,$$

so from the other side of (3.1), we deduce that

$$F(a-\varepsilon, t) - F(a+\varepsilon, t) \to 0, \quad \text{as } \varepsilon \to 0\ .$$

This is exactly the statement that $x \to F(x,t)$ is continuous at $x = a$.

3.2 The maximum principle

Before finding any particular solutions of (3.4), let us look at some qualitative properties of solutions of (3.4). Let Q be the semi-infinite open set

$$Q = \{(x,t) \in R^2 : a < x < b, t > 0\}$$

for some $a < b$, and suppose that $u(x,t)$ is a solution of the heat equation (3.4) in Q. Further suppose that, at some point $(x_0, t_0) \in Q$, the function $u(x,t)$ has a local maximum at (x_0, t_0) with $u_{xx}(x_0, t_0) < 0$. Then at the point (x_0, t_0), (3.4) says that

$$u_t(x_0, t_0) = ku_{xx}(x_0, t_0) < 0,$$

so that the temperature at x_0 must be strictly decreasing. Consequently, for $t = t_0 - \delta < t_0$, it follows that $u(x_0, t) > u(x_0, t_0)$. This is impossible because (x_0, t_0) is supposed to be a local maximum point of the temperature. Thus, as a consequence of Fourier's law of cooling, we see that heat cannot concentrate to produce a local space time maximum at a point $(x_0, t_0) \in Q$. Heat must flow away from hot spots. A precise statement of this idea is the (weak form) of the maximum principle.

Theorem 3.1
Let u be a strict solution of (3.4) in the set Q, which is continuous on $\bar{Q}$. For any $T > 0$, let

$$Q_T = \{(x,t) \in Q : 0 < t < T\}.$$

Let Γ_T be that part of ∂Q_T described by

$$\Gamma_T = \{(x,t) : x = a, b,\ 0 \le t \le T\} \cup \{(x,t) : t = 0,\ a \le x \le b\}.$$

Then for each $T > 0$,

$$\max_{\bar{Q}_T} u = \max_{\Gamma_T} u.$$

In words, the temperature in the piece of the bar $a \le x \le b$ can never exceed the larger of the maximum of the initial temperature or the maximum temperature *(over time) at the ends of the interval $[a, b]$ (see Figure 3.1). A similar statement holds for the minimum.*

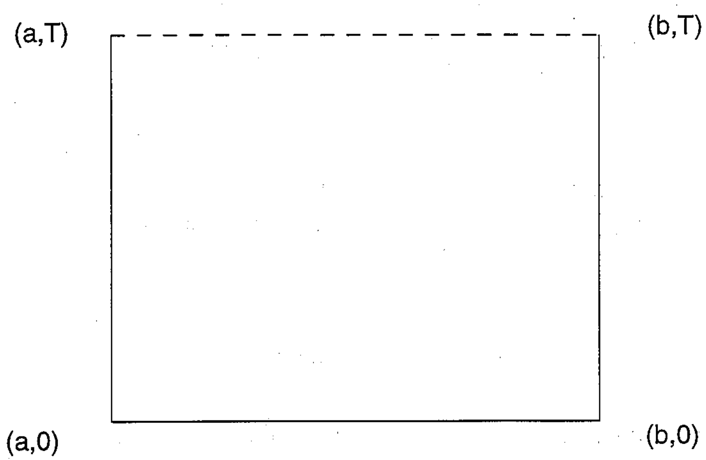

FIGURE 3.1
The part of the boundary of Q_T which is Γ_T indicated by the solid line.

The intuitive argument preceding the statement of the maximum principle is not quite correct because u can have a maximum in the interior of Q without having $u_{xx} < 0$. To make a tighter argument, we modify u slightly. Let $\varepsilon > 0$ be an arbitrarily small constant. Define

$$v_\varepsilon(x,t) = u(x,t) - \varepsilon t.$$

Both u and v_ε are continuous on $\bar{Q}_T$. Hence both u and v_ε have well-defined maximums on the sets $\bar{Q}_T$ and Γ_T. We see that v_ε satisfies

$$(v_\varepsilon)_t - k(v_\varepsilon)_{xx} = (u - \varepsilon t)_t - k u_{xx} = -\varepsilon.$$

If v_ε has a maximum at (x_0, t_0) with $a < x_0 < b$ and $0 < t_0 \le T$, then $(v_\varepsilon)_{xx}(x_0,t_0) \le 0$, which implies that $(v_\varepsilon)_t(x_0,t_0) = (v_\varepsilon)_{xx}(x_0,t_0) - \varepsilon \le -\varepsilon$. This is impossible since if the maximum occurs at a point with $t_0 < T$, then $(v_\varepsilon)_t(x_0,t_0) = 0$, and if the maximum occurs on the line $t = T$, then $(v_\varepsilon)_t(x_0,t_0) \ge 0$. Thus the maximum of v_ε on $\bar{Q}_T$ cannot occur in Q_T or on the line $t = T$. It must therefore occur on the part of the boundary Γ_T. In symbols,

$$\max_{\bar{Q}_T} v_\varepsilon = \max_{\Gamma_T} v_\varepsilon.$$

Now

$$\max_{\bar{Q}_T} u = \lim_{\varepsilon\to 0} \max_{\bar{Q}_T} v_\varepsilon = \lim_{\varepsilon\to 0} \max_{\Gamma_T} v_\varepsilon = \max_{\Gamma_T} u.$$

The argument is finished. □

Here is another version of the maximum principle, derived from Theorem 3.1. In this version we deal with a solution defined for all $x \in R$ and $t \ge 0$. Since there are no spatial boundaries, we must add another hypothesis to the type of solution we will consider.

Theorem 3.2
Let u be a strict solution of (3.4) in the open set

$$B_T = \{(x,t) : x \in R, \quad 0 < t < T\},$$

and suppose u is continuous on $\bar{B}_T$. Suppose further that there is a constant $M \ge 0$, such that $u(x,t) \le M$ for all $(x,t) \in \bar{B}_T$. Then

$$u(x,t) \le M_0 \quad \textit{for all} \quad (x,t) \in \bar{B}_T,$$

where M_0 is any number such that $u(x,0) \le M_0$ for all $x \in R$.

Example

Suppose that $u(x,t)$ satisfies the hypotheses of Theorem 3.2 with $u(x,0) = \arctan(x)$. Then we may take $M_0 = \pi/2$ and deduce that $u(x,0) \le \pi/2$ on B_T for any $T > 0$.

The proof goes as follows. Let $u(x,t)$ be the given solution of (3.4) and for $\varepsilon > 0$ an arbitrarily small real quantity, set $v_\varepsilon(x,t) = u(x,t) - \varepsilon(kt + x^2/2)$. Then v_ε also solves (3.4). Now we want to apply Theorem 3.1 to v_ε. We define the set

$$Q_{T,a} = B_T \cap \{-a < x < a\}$$

which has Γ boundary

$$\Gamma_{T,a} = \{(x,t) : |x| = a, \ 0 \le t \le T\} \cup \{(x,t) : |x| \le a, \ t = 0\}.$$

Then by Theorem 3.1, for each $\varepsilon > 0$,

$$\max_{\bar{Q}_{T,a}} v_\varepsilon = \max_{\Gamma_{T,a}} v_\varepsilon.$$

However this last quantity can be estimated using the function u. In fact,

$$\max_{\Gamma_{T,a}} v_\varepsilon = \max_{\Gamma_{T,a}}[u(x,t) - \varepsilon(kt + x^2/2)] \le \max\{M - \frac{\varepsilon}{2}a^2, M_0\}.$$

So far we have not made any choice of the constant a. Now we choose $a = a_0$, depending on ε, so that $M - \varepsilon a_0^2/2 = M_0$. Then for all $a \ge a_0$,

$$\max_{\bar{Q}_{T,a}} v_\varepsilon \le M_0.$$

This implies that

$$v_\varepsilon(x,t) \le M_0 \quad \text{for all} \quad (x,t) \in \bar{B}_T.$$

Thus we have established the conclusion of Theorem 3.1 for the auxilliary function v_ε for each $\varepsilon > 0$. Now take the limit as $\varepsilon \downarrow 0$. We see that

$$u(x,t) = \lim_{\varepsilon \downarrow 0} v_\varepsilon(x,t) \le M_0$$

for all $(x,t) \in B_T$. This concludes the proof of Theorem 3.2. □

The maximum principle is a very important tool in the study of solutions of the heat equation. We shall use it to establish the uniqueness of solutions of initial-value problems and initial-boundary-value problems. The book [PW] provides a thorough treatment of maximum principles in many contexts.

Exercises 3.2

1. Verify that each of the following functions satisfies the heat equation.
   (a) $u(x,t) = kt + \frac{1}{2}x^2 + C$.
   (b) $v(x,t) = \exp(-\gamma^2 kt)\sin(\gamma x)$, for any real γ.
   (c) $w(x,t) = \exp(-\gamma^2 kt)\cos(\gamma x)$, for any real γ.
   (d) $z(x,t) = \exp(kt \pm x)$.

2. (a) For each of the functions in exercise 1, find the maximum and minimum over the rectangle $[-a,a] \times [0,T]$. Verify that the maximum principle of Theorem 3.1 is satisfied in each case. To which of the cases can you apply the maximum principle of theorem 3.2?
   (b) Make 3-D plots of each of these functions using MATLAB, thereby visually verifying your results of part (a). For example, write a function mfile, say `u.m`, as follows:

```
function y = u(x,t,k)
y = k*t + .5*x.^2;
```

   Then use the MATLAB commands `meshgrid` and `surf` to make a 3-D plot.

3. Let $u(x,t)$ and $v(x,t)$ both be solutions of the equation

$$u_t - ku_{xx} = q.$$

   Suppose that $u(x,t) \le v(x,t)$ for $|x| \le L, t = 0$, and for $x = \pm L, 0 \le t \le T$. Show that $u(x,t) \le v(x,t)$ for $|x| \le L, 0 \le t \le T$. Hint: Consider the equation satisfied by $u - v$.

3.3 The heat equation without boundaries

3.3.1 The fundamental solution

It will be useful in our construction of a solution to determine if there is a conserved quantity asssociated with solutions of the heat equation in the absence of any sources. Recall that equation (3.3) is

$$c\rho u_t = \partial_x(\kappa u_x),$$

where c, ρ, κ are allowed to depend on x. Under suitable assumptions on c, ρ, κ and u, we can integrate the equation in x over the whole real line to deduce

$$\frac{d}{dt}\int c\rho u dx = \int c\rho u_t dx = \int \partial_x(\kappa u_x)dx$$
$$= \lim_{a\to\infty}(\kappa u_x)|_{-a}^{a} = \lim_{a\to\infty}(F(-a,t) - F(a,t)).$$

Here, and in the remainder of this subsection, we shall omit the upper and lower limits of integration when the integral is taken from $-\infty$ to ∞. If, for each t, the heat flux $F(\pm a, t)$ converges to the same number as, $a \to \infty$ (for example 0), we can conclude that the total heat energy

$$\int c(x)\rho(x)u(x,t)dx$$

is constant. In other words, if u solves the heat equation written above, $u(x,0) = f(x)$, and the heat flux $F(-a,t) - F(a,t) \to 0$, as $a \to \infty$, then

$$\int c(x)\rho(x)u(x,t)dx = \int c(x)\rho(x)f(x)dx$$

for all t. In particular, this would be the case if c, ρ, κ are constant and $u_x(x,t) \to 0$, as $x \to \pm\infty$. Then

$$\int u(x,t)dx = \int f(x)dx \quad \text{for all } t \geq 0\,.$$

From now on we assume that c, ρ, κ are constant.

We want to investigate the situation where a fixed amount of heat energy is concentrated initially in a very short portion of the bar. We can model this situation with a sequence of initial conditions $f_n(x)$ defined by

$$f_n(x) = \begin{cases} n & \text{for } |x| \leq \frac{1}{2n} \\ 0 & \text{for } |x| > \frac{1}{2n} \end{cases}.$$

As in Chapter 1, it is easy to verify that $f_n(x) \to 0$, as $n \to \infty$, for each $x \neq 0$, and that the amount of heat energy $\int_R f_n(x)dx = 1$ for all n. Let $u_n(x,t)$ be the solution of the IVP

$$(u_n)_t = k(u_n)_{xx}, \qquad u_n(x,0) = f_n(x). \tag{3.7}$$

We denote the limiting solution by $S(x,t)$, if it exists. Formally it would satisfy

$$S_t = kS_{xx}. \tag{3.8}$$

What would be the initial condition of S? We saw in Section 1.1.3 that the sequence f_n does not converge to a function in the usual sense. Rather it converges to a generalized function called the δ function, denoted $\delta(x)$. We write the initial condition for S as

$$S(x,0) = \delta(x). \tag{3.9}$$

Because $\int u_n(x,t)dx = \int f_n(x)dx = 1$ for all $t > 0$ and all n, we expect that the limiting solution S has the same property:

$$\int S(x,t)dx = 1 \quad \text{for all } t > 0\,. \tag{3.10}$$

For convenience, we set $k = 1$. When we are done, we can replace t by kt.

Next we note that if $v(x,t)$ is any solution of $v_t = v_{xx}$, then, for each $\lambda > 0$, the same is true of a new function v_λ defined by

$$v_\lambda(x,t) = v(\lambda x, \lambda^2 t),$$

because $(v_\lambda)_t - (v_\lambda)_{xx} = \lambda^2(v_t - v_{xx}) = 0$. If $v(x,t)$ has initial condition $v(x,0) = f(x)$, then $v_\lambda(x,0) = f(\lambda x)$. Now applying this invariance property to $S(x,t)$, we see that $S_\lambda(x,t) = S(\lambda x, \lambda^2 t)$ should also solve the heat equation, and since $S(x,0) = 0$ for $x \neq 0$, the same will be true for S_λ. We conjecture that

$$S(\lambda x, \lambda^2 t) = C(\lambda)S(x,t),$$

where $C(\lambda)$ is a constant depending on λ. To find $C(\lambda)$, apply the integral constraint. Combining (3.10) with the previous equation, we see that for any $t > 0$,

$$1 = \int S(x,t)dx = \frac{1}{C(\lambda)}\int S(\lambda x, \lambda^2 t)dx.$$

Now in the last integral, make the change of variable $y = \lambda x$ with $dx = dy/\lambda$. We deduce that

$$1 = \frac{1}{\lambda C(\lambda)}\int S(y, \lambda^2 t)dy = \frac{1}{\lambda C(\lambda)}$$

because by (3.10), $\int S(y,s)dy = 1$ for any value of $s > 0$. Thus $C(\lambda) = 1/\lambda$, and we find that

$$S(\lambda x, \lambda^2 t) = \frac{1}{\lambda}S(x,t). \tag{3.11}$$

We can use this scaling equation to reduce the problem of finding S to that of solving an ODE for a function of one variable. Since we can make any choice of λ in (3.11), let us choose λ so that the quantity $\lambda^2 t$ is constant. Specifically, let $\lambda = t^{-1/2}$. Then the scaling equation becomes

$$S(x,t) = \frac{1}{\sqrt{t}}S(\frac{x}{\sqrt{t}}, 1).$$

Let $g(s) = S(s, 1)$ so that

$$S(x,t) = \frac{1}{\sqrt{t}} g(\frac{x}{\sqrt{t}}). \tag{3.12}$$

Substituting this candidate into (3.8), with $k = 1$, we find that g must satisfy

$$-(\frac{1}{2})t^{-3/2}g(xt^{-1/2}) - \frac{1}{2}xt^{-2}g'(xt^{-1/2}) = t^{-3/2}g''(xt^{-1/2}).$$

Multiplying by $t^{3/2}$ and setting $s = xt^{-1/2}$ yields the following ODE for g:

$$g''(s) + \frac{1}{2}sg'(s) + \frac{1}{2}g(s) = 0. \tag{3.13}$$

Standard ODE methods (see [BD]) produce a family of solutions

$$g(s) = A\exp(-s^2/4),$$

where A must be determined by the condition (3.10). Thus

$$S(x,t) = \frac{A}{\sqrt{t}}\exp(-\frac{x^2}{4t}),$$

and, making the change of variable $y = \frac{x}{2\sqrt{t}}$,

$$1 = \int S(x,t)dx = \tfrac{A}{\sqrt{t}} \int \exp(-\tfrac{x^2}{4t})dx$$

$$= 2A \int \exp(-y^2)dy = 2A\sqrt{\pi}.$$

Thus we finally obtain (after replacing t by kt)

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}. \tag{3.14}$$

Referring to Section 1.1.3, we see that for each $t > 0$, $x \to S(x,t)$ is a Gaussian, that is, $S(x,t) = \gamma_r(x) = \sqrt{r/\pi}\exp(-rx^2)$ if we take $r = 1/4kt$. When $t \to 0$, $r \to \infty$. Thus by Theorem 1.7, $S(x,t) \to \delta(x)$, as $t \to 0$. In summary, $S(x,t)$ satisfies (3.8) - (3.10). $S(x,t)$ is called the *fundamental solution* of the heat equation, or the one-dimensional heat kernel.

3.3.2 Solution of the initial-value problem

Now we want to construct solutions of the initial-value problem

$$u_t = ku_{xx} \quad \text{for } t > 0, x \in R, \qquad u(x,0) = f(x) \quad \text{for } x \in R. \tag{3.15}$$

Assume for the moment that the initial data $f = 0$ for x outside some finite interval $I = [a,b]$. We subdivide I into n equal subintervals I_j of length $\Delta y = (b-a)/n$. Finally we sample the initial data f at the midpoints y_j of the I_j. Then the heat flow from the interval I_j is approximated by

$$(x,t) \to S(x - y_j, t)f(y_j)\Delta y,$$

which is the singular heat flow that results from concentrating the heat energy $f(y_j)\Delta y$ at the single point y_j. Because the PDE (3.15) is linear, the sum

$$v_n(x,t) = \sum_{j=1}^{n} S(x - y_j, t)f(y_j)\Delta y$$

is again a solution of the PDE (3.15), and it is an approximation to the solution of the IVP (3.15) which gets better as $\Delta y \to 0$ (i.e., as $n \to \infty$). On the other hand, for each $t > 0$, $v_n(x,t)$ is a Riemann sum approximation of the integral

$$\int S(x-y,t)f(y)dy.$$

Hence our candidate for the solution of the IVP (3.15) is given by

$$u(x,t) = \lim_{n\to\infty} v_n(x,t) = \int S(x-y,t)f(y)dy.$$

Combining with (3.14), this formula becomes explictly

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_R e^{-\frac{(x-y)^2}{4kt}} f(y)dy. \tag{3.16}$$

Now that we have derived the formula (3.16), we shall drop the assumption that $f = 0$ outside some finite interval.

Theorem 3.3
Assume that f is continuous on R and that there is an $M > 0$ such that $|f(x)| \le M$ for all $x \in R$. Then there is a unique bounded solution $u(x,t)$ of (3.15) which is continuous on $x \in R, t \ge 0$. u is given by (3.16) and satisfies

$$|u(x,t)| \le M \quad \text{for all } x \in R,\ t \ge 0.$$

In addition, the solution $u \in C^\infty$ for $x \in R,\ t > 0$.

Remark We conclude from this theorem that the IVP (3.15) is a well-posed problem in the class of continuous, bounded functions in the sense of Chapter 2. In fact, if $u(x,t)$ is the solution of (3.15) with initial data f, and $v(x,t)$ is the solution of (3.15) with initial data g, then $w = u - v$ solves (3.15) with initial data $f - g$. If $|f(x) - g(x)| \le \delta$ for all $x \in R$, then, by the estimate of the theorem,

$$|u(x,t) - v(x,t)| = |w(x,t)| \le \delta \quad \text{for all } x \in R, t \ge 0.$$

This shows that u and v never differ by more than the maximum difference in the initial data f and g.

Here is a proof of Theorem 3.3. From (3.14) we see that $y \to S(x-y,t)$ decays very rapidly as $y \to \pm\infty$ for each $x \in R$ and $t > 0$. This means that we will have no problems with the convergence of the integral in (3.16). Now differentiating (3.16) formally, we obtain

$$u_t = -\frac{1}{2\sqrt{4\pi k}t^{3/2}} \int \exp(-\frac{(x-y)^2}{4kt}) f(y)dy$$

$$+\frac{1}{\sqrt{4\pi kt}} \int \frac{(x-y)^2}{4kt^2} \exp(-\frac{(x-y)^2}{4kt}) f(y)dy.$$

The differentiated kernel still decays very rapidly so that these integrals converge. In fact one can differentiate to any order and still have convergent integrals:

$$\frac{\partial^k}{\partial t^k} u(x,t) = \int \frac{\partial^k}{\partial t^k} S(x-y,t) f(y)dy$$

and

$$\frac{\partial^k}{\partial x^k} u(x,t) = \int \frac{\partial^k}{\partial x^k} S(x-y,t) f(y)dy.$$

The rapid decay of the derivatives of $S(x-y,t)$, as $y \to \pm\infty$, ensures that the integrals will converge for $t > 0$. In particular,

$$u_t - ku_{xx} = \int [S_t(x-y,t) - kS_{xx}(x-y,t)] f(y)dy = 0.$$

Thus u given by (3.16) is a solution of the differential equation of (3.15).

Next we verify that u takes on the proper initial values. We cannot plug $t = 0$ into the formula (3.16) because $S(0,t)$ becomes infinite as $t \to 0$. Instead we will show that

$$\lim_{t \to 0} u(x,t) = f(x)$$

for each x. As we saw before, we can write

$$S(x-y,t) = \gamma_r(x-y) = \sqrt{r/\pi}\exp[-r(x-y)^2],$$

if $r = 1/4kt$. Hence

$$u(x,t) = \int S(x-y,t) f(y)dy = \int \gamma_r(y-x) f(y)dy$$

because $\gamma_r(x-y) = \gamma_r(y-x)$. Since $r \to \infty$ when $t \to 0$, we can apply part (b) of Theorem 1.7 to deduce that

$$\lim_{t \to 0} u(x,t) = \lim_{r \to \infty} \int \gamma_r(y-x) f(y)dy = f(x).$$

We have shown that the limit of $u(x,t)$, as $t \to 0$, exists and equals $f(x)$. However, this does not completely prove that u is continuous at $(x,0)$. A more complete argument shows that we can approach $(x,0)$ from any direction and u has the same limit.

To show that $u(x,t)$ is bounded for all $x \in R, t \ge 0$, we use the fact that $S(x,t) \ge 0$ and (3.10):

$$|u(x,t)| = |\int S(x-y,t) f(y)dy| \le M \int S(x-y,t)dy = M.$$

Thus the same constant M that bounds the initial data f is also a bound for $|u(x,t)|$.

The uniqueness of the bounded solution follows from the version of the maximum principle of Theorem 3.2. Suppose that u and v are two bounded solutions of (3.15) with the same initial data f. Then $w = u - v$ is also a bounded solution of (3.15), but with zero initial data. By the maximum principle, $u - v \leq max_{x \in R}(u(x,0) - v(x,0)) = 0$. Reversing the roles of u and v we see that $v - u \leq 0$ as well. We conclude that $u = v$. The proof of the theorem is finished. □

We began our search for the fundamental solution $S(x,t)$ by thinking of S as the limit, in some sense, of solutions $u_n(x,t)$ of (3.7). In the exercises we will see that indeed the solutions u_n converge to S, as $n \to \infty$.

The error function

For most choices of initial data, the formula (3.16) is hard to evaluate analytically. However, when $f(x)$ is piecewise constant, we can express the solution in terms of the error function.

The error function, $erf(x)$, is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds.$$

In most applications, $erf(x)$ is used with $x \geq 0$. However, from the expression for erf, we see that it can be considered an odd function of x:

$$erf(-x) = -erf(x).$$

We shall refer to this defintion of the error function as the *extended* error function. You can compute the value of $erf(x)$, for a real number x, or for a vector of real numbers **x**, with the MATLAB command `erf(x)` .

Now let the initial data $f(x)$ for problem (3.15) be the function

$$f(x) = \begin{cases} \alpha, & x < 0 \\ 0, & x > 0 \end{cases}.$$

Then the solution formula (3.16) becomes

$$u(x,t) = \int_R S(x-y,t) f(y) dy = \frac{\alpha}{\sqrt{4\pi kt}} \int_{-\infty}^0 e^{\frac{-(x-y)^2}{4kt}} dy.$$

To express this integral in terms of the error function we make the change of variable $z = \frac{(y-x)}{\sqrt{4kt}}$ so that $dy = \sqrt{4kt}\, dz$. Then we find that

$$u(x,t) = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{-x/\sqrt{4kt}} e^{-z^2} dz$$

$$= \frac{\alpha}{\sqrt{\pi}} \int_{x/\sqrt{4kt}}^{\infty} e^{-z^2} dz$$

$$= \frac{\alpha}{\sqrt{\pi}} \left(\int_0^{\infty} e^{-z^2} dz - \int_0^{x/\sqrt{4kt}} e^{-z^2} dz \right)$$

$$= \frac{\alpha}{2}(1 - erf(x/\sqrt{4kt})).$$

3.3.3 Sources and the principle of Duhamel

Now let us turn our attention to finding solutions of the heat equation with a source,

$$u_t - ku_{xx} = q, \qquad u(x,0) = f(x). \tag{3.17}$$

First, using the linearity of the equation, we can split the solution $u = v + w$ where

$$v_t - kv_{xx} = 0, \qquad v(x,0) = f(x), \tag{3.18}$$

and

$$w_t - kw_{xx} = q, \qquad w(x,0) = 0. \tag{3.19}$$

Since we already know how to solve (3.18), we consider (3.19). We will construct the solution of (3.19) using the *principle of Duhamel*. By way of motivation, recall from Chapter 1 that the solution of the ODE problem (λ constant)

$$\varphi' + \lambda\varphi = q, \qquad \varphi(0) = 0, \tag{3.20}$$

is

$$\varphi(t) = \int_0^t \exp[-\lambda(t-s)] q(s) ds.$$

Note that for each fixed s, $z(t,s) = \exp[-\lambda(t-s)]q(s)$ is in fact the solution of the IVP

$$z'(t,s) + \lambda z(t,s) = 0, \qquad z(s,s) = q(s),$$

where $'$ is differentiation with respect to t. Thus

$$\varphi(t) = \int_0^t z(t,s) ds.$$

To verify that φ solves (3.20), we calculate

$$\varphi'(t) = z(t,t) + \int_0^t z'(t,s)ds$$
$$= q(t) - \lambda \int_0^t z(t,s)ds = q(t) - \lambda\varphi(t),$$

where we have used Theorem 1.3.

Now we wish to use the same construction to solve (3.19). Let $z(x,t,s)$ be the solution for $t > s$ of

$$z_t = kz_{xx}, \qquad z(x,s,s) = q(x,s).$$

The solution of this problem is given by (3.16), replacing t by $t-s$ because we are solving for $t \geq s$:

$$z(x,t,s) = \int S(x-y,t-s)q(y,s)dy.$$

Then the solution $w(x,t)$ of (3.19) is

$$w(x,t) = \int_0^t z(x,t,s)ds = \int_0^t \int S(x-y,t-s)q(y,s)dyds.$$

Finally the solution of (3.17) is

$$u(x,t) = \int S(x-y,t)f(y)dy + \int_0^t \int S(x-y,t-s)q(y,s)dyds. \quad (3.21)$$

Exercises 3.3

1. Make an mfile, say `S.m`, to plot profiles of $S(x,t,k)$ for various values of t and k. Try the following

```
function u = S(x,t,k)
  u = exp(-x.^2./(4*k*t))./sqrt(4*pi*k*t);
```

After choosing an x vector, say `x = -4:.05:4` you can plot the profile for $t = 1$, and $k = .5$ with the command


```
>> plot(x,S(x,1,.5) )
```

You can plot several profiles on the same graph (in different colors), say for $k = .5$ and $t = .1, .2, .5$, with the command

```
>> plot(x,S(x,.1,.5),x,S(x,.2,.5), x,S(x,.5,.5) )
```

(a) Plot several profiles with $k = 1$ and several values of t. What happens to the profiles as t increases? What happens as $t \downarrow 0$?

(b) Plot several profiles with $t = 1$ and various values of k. Verify that, if k_1, t_1 and k_2, t_2 are values of k and t such that $k_1t_1 = k_2t_2$, then $S(x,t_1,k_1) = S(x,t_2,k_2)$.

(c) Fix $t = .5$ and $k = 1$. Plot $x \to S(x-y,.5,1)$ for several values of y. For example, if $y = 2$, you can use the command

```
>> plot(x,S(x-2,.5,1))
```

Now plot the result of adding together heat sources at several points. Plot the profile $x \to .5S(x-2,t) + .4S(x-1,t) - .2S(x+1),t)$ with the command

```
>> u=.5*S(x-2,.5,1)+.4*S(x-1,.5,1)-.2*S(x+1, .5,1);
>> plot(x,u)
```

2. In this exercise we graphically investigate the way in which the solution of the initial-value problem (3.15) is built up as a superposition of point heat sources. This will be a further development of the technique of part (c) of exercise 1. In our derivation we said that the solution should be given approximately by the discrete sum

$$\sum_{j=1}^{n} S(x-y_j)f(y_j)\Delta y$$

and that the formula for $u(x,t)$ is obtained by taking the limit as $\Delta y \to 0$. The program `heat1` computes this sum for initial data given on the interval $[-1,1]$. You must enter the choice of k, n (n must be even), and t. The following initial data is "built in " to program `heat 1`.

$$f(x) = \begin{cases} -1 & -1 \leq x < 0 \\ 2 & 0 < x \leq 1 \\ 0 & \text{elsewhere} \end{cases}.$$

We subdivide the interval $[-1, 1]$ into n subintervals of length $\Delta y = 2/n$, and we choose y_j as the midpoint of each subinterval. For small $t > 0$, the solution $x \to u(x, t)$ should resemble f, but with the corners rounded off.

You can also run `heat1` with different initial data which must be provided in a supporting mfile, `ff.m`. For more information enter `help heat1`. Run `heat1` with time $t = .01$, $k = 1$, and with $n = 2, 4, 6, 8, 12$. To get a better view, restrict the plots to the interval $[-5, 5]$ with the command `axis([-5 5 -1.5 3])`. What happens as n increases?

3. Now run the program `heat1` with the same f, $n = 20$, $k = 1$, and $t = .1, .5, 1, 2, 5$. To get the profiles on the same graph, run the program with $t = .1$, then type the command `hold on`. To get a better view, change the axes with the command `axis([-5 5 -1 2])`. After you have seen the graphs you want, type the command `hold off`. Otherwise, all succeeding graphs will be plotted together.

   (a) Notice which parts warm up and which parts cool off. How can this behavior be predicted from the differential equation?

   (b) Try to estimate the rate at which the solution decays. Look at the value of the profile at $x = 1/2$ at several times (say $t = 10, 11$, or $t = 20, 21$), and see if you can find constants C and γ such that the heights at $x = 1/2$ fit on a curve $Ct^{-\gamma}$. To get the values of the approximate solution (with $n = 20$) at $x = 1/2$, you must first figure out what index of the plotting vector x corresponds to $x = 1/2$. In the program `heat1`, `x = -10:.05:10`. Then each time you run the program `heat1`, save the component of the vector snap1 that has this index.

   (c) Explain the rate of decay you found in part b) by analyzing the solution formula (3.16), using, of course, (3.14).

4. Let $f(x) = \alpha$ for $x < 0$, and $f(x) = \beta$ for $x > 0$ and let $u(x, t)$ be the solution of (3.15).

   (a) Show that for this piecewise constant data, the formula (3.16) can be rewritten in terms of the (extended) error function. You will need to make the change of variable $z = \frac{(y-x)}{\sqrt{4kt}}$.

   (b) Show that for each x, $\lim_{t\to\infty} u(x, t) = (\alpha + \beta)/2$.

   (c) Using the error function of MATLAB `erf(x)`, plot the solution on $[-5, 5]$ for $k = 1$ and $t = .01, .1, .5, 2$.

5. A bit harder. Let $f(x) = \alpha$ for $x \leq -B$, and $f(x) = \beta$ for $x \geq B$ where B is some positive number. Assume that f is continuous on $[-B, B]$, and hence bounded on all of R. Show that conclusion 4 (b) still holds.

6. Assume that $u(x, t)$ is a solution of $u_t = ku_{xx}$, such that u and u_x tend to zero rapidly, as $x \to \pm\infty$. Let $Q = \int_R u(x, t)dx$. We have already seen that Q is a conserved quantity. Here are two more quantities associated with solutions of the heat equation that have a special behavior. In the following, assume that $Q \neq 0$.

   (a) Show that $m = \frac{1}{Q}\int_R xu(x, t)dx$ is independent of t. (Hint: Differentiate with respect to t under the integral and use the fact that u solves the heat equation.)

   (b) Let
   $$p(t) = \frac{1}{Q}\int_R (x - m)^2 u(x, t)dx.$$
   Show that $p(t) = p(0) + 2kt$. Use the same hint as in part (a).

   (c) Show that
   $$p(t) = \frac{1}{Q}\int_R x^2 u(x, t)dx - m^2.$$

   (d) Find m and $p(t)$ for the fundamental solution $S(x, t)$.

   (e) Evaluate the integral
   $$\int_{-\sqrt{p(t)}}^{\sqrt{p(t)}} S(x, t)dx$$
   using the error function. The amount of heat contained in the interval $[-\sqrt{p(t)}, \sqrt{p(t)}]$ is a constant fraction of the total amount of heat.

Remark When $u(x, t) \geq 0$, $x \to u(x, t)/Q$ can be thought of as the probability distribution of a random variable $U(t)$. The probability that $U(t)$ lies in the interval $[a, b]$ is given by
$$\frac{1}{Q}\int_a^b u(x, t)dx.$$
m is the mean of this random variable, which remains constant, and $p(t) = \sigma^2(t)$ where $\sigma(t)$ is the standard deviation which increases as $t^{1/2}$.

7. We want to construct an approximation $v(x, t)$ to the solution $u(x, t)$ of the heat equation which is easy to evaluate and such that $|u(x, t) - v(x, t)| \to 0$ in a appropriate sense as, $t \to \infty$. This result will extend the analysis of exercise 6.

(a) Let $Q = \int_R f(y)dy$ (assume again that $Q \neq 0$), $m = \frac{1}{Q}\int_R yf(y)dy$ and $p_0 = \frac{1}{Q}\int (y-m)^2 f(y)dy$ be the quantities associated with the initial data f.
Expand the function $y \to \exp[-(x-y)^2/4kt]$ in a Taylor series about the point $y = m$.

(b) Show that

$$\int_R e^{-\frac{(x-y)^2}{4kt}} f(y)dy = Qe^{-\frac{(x-m)^2}{4kt}}(1 - \frac{p_0}{4kt}) + R$$

where, for each x, $R = O(1/t^2)$, as $t \to \infty$.
Set

$$v(x,t) = Q\frac{(1 - p_0/4kt)}{\sqrt{4\pi kt}} e^{-\frac{(x-m)^2}{4kt}}.$$

We shall refer to v as the *Gaussian approximation.* Note that v is not a solution of the heat equation, but that

$$\int_R v(x,t)dx = Q(1 - \frac{p_0}{4kt}), \qquad \int_R xv(x,t)dx = mQ(1 - \frac{p_0}{4kt}).$$

(c) Show that $|u(x,t) - v(x,t)| \leq C/t^{5/2}$ for x in a bounded interval $[a,b]$. The constant C will depend on the interval $[a,b]$.

(d) Compute Q, m, and p_0 for the initial data of exercise 2. Note that $p_0 < 0$, so that in this example $x \to u(x,t)/Q$ is not the probability distribution of a random variable.

(e) The program `heat1` also computes and plots the Gaussian approximation for the initial data of `heat1`. To see the graph of the Gaussian approximation, run `heat1`, and then give the command `plot(x,gauss)`. Run `heat1` this time with $n = 20$, $k = 1$, and $t = .5, 1, 2, 5, 10$. Compare the graphs of the solution u and the Gaussian approximation v for each of these values of t by using the command `plot(x,snap1, x,gauss)`. How large must t be so that the relative error $|u - v|/\max|u|$ is less than 5%?

8. Show that the solutions $u_n(x,t)$ defined in (3.7) converge to the fundamental solution $S(x,t)$, as $n \to \infty$.

9. Solve the initial-value problem

$$u_t - ku_{xx} + \gamma u = 0, \qquad u(x,0) = f(x).$$

Hint: Set $v(x,t) = \exp(\gamma t) \cdot u(x,t)$. Find the equation satisfied by v and solve it.

10. Solve the initial value problem

$$u_t + cu_x - \varepsilon u_{xx} = 0,$$

$$u(x,0) = f(x) = \begin{cases} \alpha & x < 0 \\ \beta & x > 0 \end{cases}.$$

This equation combines linear diffusion and a linear convection term cu_x. Call the solution $u_\varepsilon(x,t)$. Show that, for each x,

$$\lim_{\varepsilon \to 0} u_\varepsilon(x,t) = f(x - ct).$$

Hint: Change to a coordinate system that moves with speed c. Let $v_\varepsilon(s,t) = u_\varepsilon(s+ct,t)$ so that $u_\varepsilon(x,t) = v_\varepsilon(x-ct,t)$. Use the chain rule to calculate the derivatives of u_ε in terms of those of v_ε, and substitute in the equation for u_ε to find the equation satisfied by v_ε. Solve this equation using the results of exercise 4. Then transform back and take the limit.

3.4 Boundary value problems on the half-line

Up to now, we have assumed that our bar in which heat is flowing is infinitely long, whereas in any real physical situation, we shall have to deal with bars of finite length. Various assumptions can be made about the behavior of the heat flow at the ends of the bar. These are called *boundary conditions*. We start with a bar which occupies the infinite half-line $x \geq 0$. We want to solve a problem

$$u_t - ku_{xx} = 0 \quad \text{in } x > 0, t > 0, \qquad u(x,0) = f(x) \quad \text{for } x > 0.$$

In addition we must specify boundary conditions at $x = 0$. There are three principal types.

First kind or Dirichlet condition: $u(0,t) = 0$
The temperature is held fixed at 0, perhaps by putting ice cubes on the end.

Second kind or Neumann condition: $u_x(0,t) = 0$
The end of the bar is insulated so that no heat can flow (Flux = 0).

Third kind or Robin condition: $u_x(0,t) - hu(0,t) = 0$
Here we assume $h > 0$. In this case the heat flux $F(0,t) = -\kappa u_x(0,t) = -\kappa h u(0,t)$ is negative when $u(0,t) > 0$. Heat flows out of the end of the bar when the temperature is positive and flows into the bar when the temperature is negative. In both cases the temperature will tend toward zero. If this condition is imposed with $h < 0$, the temperature may grow when $u(0,t) > 0$ which is an unstable situation.

In all three conditions we can introduce a nonzero function of t on the right-hand side, making the boundary condition inhomogeneous.

We will solve the homogeneous boundary conditions by extending the initial data in an appropriate manner to all of R and then solving the equation on the whole line using (3.16). First we recall the definitions of two classes of functions on R.

$f(x)$ defined on R is an *even* function if $f(-x) = f(x)$ for all $x \in R$.
$f(x)$ is an *odd* function if $f(-x) = -f(x)$ for all $x \in R$.

For example $\cos(x)$ is an even function and $\sin(x)$ is odd. If f is odd, then we must have $f(0) = 0$. If f is even and differentiable at $x = 0$, then $f'(0) = 0$.

Now suppose that f is odd and that u is the solution of the IVP $u_t = ku_{xx}$, $u(x,0) = f(x)$. Then we claim that for each $t > 0$, the function $x \to u(x,t)$ is odd. In fact the solution is given by (3.16):

$$u(x,t) = \int S(x-y,t) f(y) dy,$$

so that

$$u(-x,t) = \int S(-x-y,t) f(y) dy.$$

Setting $z = -y$ and using the fact that $S(-x+z,t) = S(x-z,t)$ we find that

$$u(-x,t) = \int_{\infty}^{-\infty} S(-x+z,t) f(-z)(-dz)$$

$$= -\int_{-\infty}^{\infty} S(x-z,t) f(z) dz = -u(x,t).$$

It follows that $u(0,t) = 0$ for all $t \geq 0$. A similar calculation shows that when f is even, $x \to u(x,t)$ is even, so that $u_x(0,t) = 0$ for all $t \geq 0$.

Now we solve the initial-boundary-value problem (IBVP)

$$u_t = ku_{xx} \text{ for } x > 0, t > 0, \qquad u(x,0) = f(x) \text{ for } x > 0 \tag{3.22}$$

$$u(0,t) = 0 \text{ for } t > 0.$$

We think of the given function $f(x)$ as defined to be zero for $x < 0$. Then we define the *odd extension* of f as

$$\tilde{f}(x) = f(x) - f(-x).$$

$\tilde{f}$ is an odd function on R, and because $f(x) = 0$ for $x < 0$, we see that

$$\tilde{f} = \begin{cases} f(x) & \text{for } x > 0 \\ -f(-x) & \text{for } x < 0 \end{cases}.$$

Then let $\tilde{u}(x,t)$ be the solution given by (3.16) with initial data $\tilde{f}$. Thus

$$\tilde{u}(x,t) = \int S(x-y,t)\tilde{f}(y)dy = \int S(x-y,t)[f(y) - f(-y)]dy$$

$$= \int_0^{\infty} S(x-y,t) f(y) dy + \int_{-\infty}^{0} S(x-y,t)[-f(-y)]dy$$

$$= \int_0^{\infty} [S(x-y,t) - S(x+y,t)] f(y) dy.$$

The restriction of $\tilde{u}(x,t)$ to $x \geq 0$ is the solution of (3.22).

Next we turn to the solution of the second boundary value problem

$$u_t = ku_{xx} \text{ for } x > 0, t > 0 \qquad u(x,0) = f(x) \text{ for } x > 0, \tag{3.23}$$

$$u_x(0,t) = 0 \text{ for } t > 0.$$

To solve this problem, we define the *even extension* of f by

$$\tilde{f}(x) = f(x) + f(-x).$$

Clearly $\tilde{f}$ is an even function. We substitute the even extension of f in (3.16) to produce an even solution $\tilde{u}(x,t)$ of (3.15). Its restriction to $x > 0$ is given by

$$u(x,t) = \int_0^\infty [S(x-y,t) + S(x+y,t)]f(y)dy.$$

This is the solution of the IBVP (3.23).

Uniqueness of solutions to these IBVPs follows from the maximum principle, and continuous dependence of the solution on the initial data follows as in Theorem 3.3.

Exercises 3.4

1.
   (a) Reformulate the maximum principle as stated in Theorem 3.2 for the quadrant $\{x, t \geq 0\}$, and modify the argument of Theorem 3.2 to prove it.

   (b) Use the maximum principle as stated in part (a) to prove uniqueness of bounded solutions of (3.22).

2. Let u be a solution of either (3.22) or (3.23). Assume that $u(x,t) \geq 0$ and that $u, u_x, u_{xx} \to 0$ rapidly as, $x \to \infty$.

   (a) Compute the rate of change of the total heat energy $\int_0^\infty u(x,t)dx$.

   (b) For which boundary conditions (Dirichlet or Neumann) is this quantity conserved? Why?

   (c) Assume the Dirichlet boundary condition. If the temperature in the bar $u(x,t) > 0$ for x near zero, does the total amount of heat increase or decrease? If the temperature outside the bar is greater than the temperature inside, does the total amount of heat increase or decrease?

3. The program `heat2` solves the initial-boundary-value problems (IBVP) (3.22) (Dirichlet) and (3.23) (Neumann) with initial data

$$f(x) = \begin{cases} 0 & 0 < x < 1 \\ 1 & 1 < x < 2. \\ 0 & x > 2 \end{cases}$$

The solution to each of these problems can be expressed in terms of the error function $erf(x)$. Program `heat2` evaluates these expressions and plots them together. To get information on program heat 2, invoke MATLAB and type `help heat2` . To see the expressions, look at the file `heat2.m`.

Which solution decays faster? Set $k = 1$, and compare the values at $x = 1$. In the code `heat2.m`, the vector `x = 0:.05:10`. Note that `x(21)` corresponds to $x = 1$. The solutions at time t are stored in vectors `dirch` and `neumann`. The values of the solutions at $x = 1$ also have index `21`. Find constants C_D and γ_D, and C_N and γ_N, such that the solution of (3.22) decays like $C_D t^{-\gamma_D}$ at $x = 1$, and the solution of (3.23) decays like $C_N t^{-\gamma_N}$ at $x = 1$. Use pairs of values of t, like $t = 10, 11$ or $t = 20, 21$.

4. Now we find an analytic reason for the different rates of decay observed in exercise 3.

   (a) Use the power series for $\exp(-x^2)$ to expand both terms in the solution formula for (3.22). What is the leading power of t in the expansion after subtraction?

   (b) Do the same for the solution formula of (3.23). Now what is the leading power of t? Does this explain your observations in exercise 3?

5. (a) Starting with the solution formula for IBVP (3.22), show that the solution of (3.22) with initial data $f(x) \equiv U$ is given by

$$u(x,t) = U - 2U \int_x^\infty S(y,t)dy.$$

   (b) Verify directly that this function u solves the IBVP (3.22).

   (c) Make the change of variable $z = y/\sqrt{4kt}$, and show that $u(x,t) \to 0$ for each x, as $t \to \infty$.

6. (a) Using the solution of exercise 5, show that the solution of the inhomogeneous IBVP

$$v_t - kv_{xx} = 0 \quad x,\ t > 0, \qquad v(x,0) = 0, \quad x > 0,$$

$$v(0,t) = U, \quad t > 0,$$

is given by

$$v(x,t) = 2U \int_x^\infty S(y,t)dy.$$

   (b) Show that $v(x,t) \to U$ for each x, as $t \to \infty$.

   (c) Write the solution v in terms of the error function erf(x). For the definition of the error function, see the exercises of Section 3.3.

7. A heat source of temperature 100^o C is placed at the end of a long metal rod whose cylindrical surface is insulated. If the rod is initially at temperature 0^o C, how long does it take for the temperature to reach 50^o C at a distance of 10 cm. from the end of the rod? Use the solution of exercise 6, the error function from a table or from a software package, and the following values of the diffusion constants k for iron, aluminum, and copper.

iron	$k = .230$ cm^2/sec.
aluminum	$k = .975$ cm ^{2}/sec.
copper	$k = 1.156$ cm^2/sec.

8. Consider the half-line problem with a source:

$$u_t - ku_{xx} = q, \qquad x,\ t > 0$$

$$u(0,t) = 0, \quad t \geq 0, \qquad u(x,0) = f(x), \quad x > 0.$$

Let $H(x, y, t) = S(x - y, t) - S(x + y, t)$.

(a) Using the principle of Duhamel, find the solution of this IBVP (see formula (3.21)).

(b) Assume that $q = 0$ for $x > a$ and for $t > T$. Make a Gaussian approximation to the solution that is valid for large T.

9. Suppose that u solves the IBVP with the Robin condition:

$$u_t - ku_{xx} = 0 \quad x,\ t \geq 0, \qquad u(x,0) = f(x), \quad x \geq 0,$$

$$u_x(0,t) = hu(0,t), \qquad t \geq 0,$$

and that $u, u_x, u_{xx} \to 0$ rapidly, as $x \to \infty$. Assume that $u(x,t) \geq 0$. How does the sign of h affect the rate of change of the total heat energy $\int_0^\infty u(x,t)dx$? For which sign of h is the Robin condition a radiating (absorbing) boundary condition?

10. We want to solve the IBVP of exercise 9.

(a) Assume u solves this problem and let $v(x,t) = u_x(x,t) - hu(x,t)$. v is again a solution of the heat equation. What boundary and initial conditions does v satisfy?

(b) Solve the IBVP for v in terms of integrals of the fundamental solution S.

(c) Now solve the first-order ODE for u in terms of v. Assume $h > 0$. The solution which is bounded, as $x \to \infty$, is

$$u(x,t) = -\int_x^\infty e^{h(x-\xi)} v(\xi, t) d\xi.$$

Substitute the formula for v obtained in part (b). This will express u in terms of its initial data.

11. (a) Let $u(x,t)$ be the solution of the IBVP of exercise 9 (as solved in exercise 10) with initial data $u(x,0) \equiv U$. Now the intermediate solution $v(x,t)$ has initial value $v(x,0) = u_x(x,0) - hu(x,0) = -hU$, and we can use exercise 5 to find v. Finally show that

$$u(x,t) = U + 2U \int_x^\infty S(y,t)[e^{h(x-y)} - 1]dy.$$

Note that the formula works equally well for $h > 0$ and $h < 0$.

(b) What happens to the temperature $u(0,t)$ as t increases (depending on the sign of h)?

3.5 Diffusion and nonlinear wave motion

Now is a good time to compare the properties of wave motion, seen in Chapter 2, and the properties of diffusion. First note that for any $t > 0$, no matter how small, the fundamental solution of the heat equation, $S(x,t)$, is strictly positive for all $x \in R$. Thus the information about a heat source at $x = 0$ spreads instantaneously to all of R, which contrasts with the finite speed of propagation of waves. Next note that the formula (3.16) for the solution of the IVP for the heat equation shows that the value of u at (x,t), $t > 0$, depends on the values of $f(y)$ for all $y \in R$ in the form of an integral. This is quite different from the wave motion we studied where $u(x,t)$ depends only on the initial data at a single point $x - ct$. A third difference which follows from the dependence on initial data is that, in the case of wave motion, $u(x,t) = f(x - ct)$ has the same regularity as f, but in the case of diffusion, $u(x,t)$ given by (3.16) is C^∞, even though the data may be only continuous.

To see what happens when we combine diffusion and nonlinear wave motion we consider

$$u_t + uu_x - \varepsilon u_{xx} = 0, \qquad u(x,0) = f(x) \text{ for } x \in R. \tag{3.24}$$

Equation (3.24) is Burger's equation (with viscosity). Recall from Chapter 2 that when $f'(x_0) < 0$ for some x_0, solutions of

$$u_t + uu_x = 0, \qquad u(x,0) = f(x) \tag{3.25}$$

breakdown at a time t_* by developing a vertical piece in the wave profile which corresponds to a discontinuity in the solution. For $t > t_*$, the solution must be continued as a weak solution. An example of a weak solution of (3.25) is the shock wave

$$u(x,t) = f(x - ct), \tag{3.26}$$

where

$$f(x) = \begin{cases} u_l & \text{for } x < 0 \\ u_r & \text{for } x > 0 \end{cases},$$

with $u_l > u_r$. The translation speed (shock speed) c is determined from the Rankine-Hugoniot condition

$$c = \frac{u_l + u_r}{2}. \tag{3.27}$$

The addition of a diffusion term in (3.24) models a viscous effect in gas dynamics which tends to smooth solutions. The small parameter ε is the viscosity. We imagine that the addition of the diffusion term would prevent the breakdown that can happen to the solutions of (3.25). In particular we conjecture that there are solutions of (3.24) which look like (3.26) except that they are smoothed off, without the discontinuity. We call such solutions, when they exist, *travelling waves*.

Let us look for a travelling wave solution of (3.24)

$$u(x,t) = \varphi(x - ct),$$

such that

$$\varphi(s) \to u_l, \quad \text{as } s \to -\infty \tag{3.28}$$

and

$$\varphi(s) \to u_r, \quad \text{as } s \to +\infty.$$

We assume that $u_l > u_r$, and we expect the speed c to be determined by the values u_l and u_r.

We substitute this form in (3.24) and see that φ must satisfy the nonlinear second-order ODE

$$\varepsilon\varphi''(s) = (\varphi(s) - c)\varphi'(s). \tag{3.29}$$

We seek a solution of (3.29) and a value $c > 0$ such that (3.28) is satisfied. Rewrite (3.29) as

$$\varepsilon\varphi'' = [\frac{1}{2}\varphi^2 - c\varphi]',$$

which we can immediately integrate once to get

$$\varepsilon\varphi' = \frac{1}{2}\varphi^2 - c\varphi + A, \tag{3.30}$$

where A is the constant of integration. Now assuming $\varphi'(s) \to 0$, as $s \to \pm\infty$, we take limits in (3.30) and use (3.28) to deduce that

$$\frac{1}{2}u_l^2 - cu_l + A = 0 = \frac{1}{2}u_r^2 - cu_r + A. \tag{3.31}$$

The A drops out and we can solve for c:

$$c = \frac{u_l + u_r}{2}.$$

Returning to (3.31), we see that

$$A = \frac{u_r u_l}{2}.$$

With A and c determined this way, substitute in (3.30) to arrive at

$$\varepsilon\varphi' = \frac{1}{2}(\varphi - u_r)(\varphi - u_l).$$

Up to this point we have not used the assumption $u_l > u_r$. If we expect the solution φ to lie between the values u_l and u_r, then the right-hand side of this first-order ODE is negative, which says that φ is decreasing. Thus we must require that $u_l > u_r$ to be consistent with (3.28).

This equation is a separable ODE which can be written

$$(\frac{1}{u_r - u_l})[\frac{1}{\varphi - u_r} + \frac{1}{u_l - \varphi}]d\varphi = \frac{ds}{2\varepsilon}.$$

Now integrating so that $\varphi(0) = c$, we obtain

$$(\frac{1}{u_r - u_l})\log(\frac{\varphi - u_r}{u_l - \varphi}) = \frac{s}{2\varepsilon},$$

whence

$$\varphi(s) = u_l + \frac{u_r - u_l}{1 + \exp[-\frac{s}{2\varepsilon}(u_l - u_r)]}.$$

In the exercises we shall compute an approximate width of the region where the graph of φ makes its drop from u_l to u_r, and shall see that it tends to zero as $\varepsilon \to 0$.

If we think of both (3.24) and (3.25) as models for a fluid or gas, (3.24) is more sophisticated because it takes viscosity into account. The travelling wave solution $u_\varepsilon(x,t) = \varphi_\varepsilon(x-ct)$ of the viscous equation converges to the shock wave (3.26) as the viscosity coefficient $\varepsilon \to 0$. To get a travelling wave solution we had to assume $u_l > u_r$, and the limiting discontinuous solution of the inviscid Burgers' equation is an admissible shock wave. The procedure of approaching the shock wave by solutions of the viscous equation picks out the physically meaningful shock waves which satisfy the entropy condition. The simpler model (3.25) provides a good approximation to the viscous solution for small ε. In particular the shock wave and the viscous travelling wave have the same speed of propagation.

It is a happy accident that the solution of (3.24) can in fact be found in closed form using the Cole-Hopf transformation. We have concentrated on the travelling wave solutions to illustrate a technique that we shall use later with another equation where such a general transformation is not available. For a discussion of the Cole-Hopf transformation, see [Lo].

Exercises 3.5

1. Show that the only steady-state solutions of (3.24) which exist for all x are constant functions.

2. Make an mfile for the function φ, including the dependence on the parameters ε, u_l and u_r. Try

```
function u = phi(s,myeps, uleft, uright)
  denom = 1+exp( -.5*s*(uleft - uright)/myeps );
  u = uleft + (uright -uleft)./denom ;
```

Note that we cannot use `eps` as a variable, because this is already a system constant. Plot φ for $u_l = 2$, $u_r = 1$, and several values of ε, say $\varepsilon = .02, .05, .1, .2$. Plot the graphs on $[-2, 2]$. In each case estimate the width of the transition region where φ makes 90% of its drop from 2 to 1, that is, from 1.95 to 1.05.

3. Make an analytic estimate of the width of the transition region for general u_l and u_r by solving the defining equation for φ in terms of s. Let s_l be the value of s such that $\varphi(s_l) = u_l - .05(u_l - u_r)$ and s_r such that $\varphi(s_r) = u_r + .05(u_l - u_r)$. Show that the width of the transition region $\Delta s = s_r - s_l$ tends to zero as the first power of the viscosity coefficient ε. Does this estimate agree with your graphical estimates in exercise 2?

4. Consider the nonlinear equation

$$u_t + uu_x + \gamma u = \varepsilon u_{xx}.$$

What is the ODE that φ must satisfy, so that $u(x,t) = \varphi(x - ct)$ is a travelling wave solution of (3.24)? Can you solve it the same way as we solved (3.29)?

5. The nonlinear equation

$$u_t - ku_{xx} = f(u)$$

is called a reaction-diffusion equation. The nonlinear term $f(u)$ represents the reaction of chemicals while the term ku_{xx} as usual represents diffusion.

(a) Let $f(u) = u(1-u)$. What constant solutions are there?

(b) Look for a travelling wave solution $u(x,t) = \varphi(x - ct)$. What second-order nonlinear ODE must φ satisfy?

(c) Write this second-order ODE as a first-order system by introducing the new dependent variable $\psi = \varphi'$. Find the critical points of this system. Linearize the system about each of these critical points, and discuss the stability of each (this will depend on the relationship between c and k).

(d) It can be proved that for each $c > 0$, there is a unique trajectory $s \to (\varphi_c(s), \psi_c(s))$ which approaches $(1, 0)$ as $s \to \infty$, and approaches $(0, 0)$ as $s \to -\infty$. Sketch the graph of $\varphi_c(s)$ for different values of c.

3.6 Numerical methods for the heat equation

In this section we shall examine finite difference methods for the heat equation

$$u_t - ku_{xx} = 0, \qquad u(x,0) = f(x). \tag{3.32}$$

We shall use these methods in the next chapter to solve boundary value problems for the heat equation. These methods are easily generalized to treat situations where the coefficients c, ρ, κ depend on both x and t.

Make a grid with mesh points $x_j = j\Delta x, t_n = n\Delta t$. Let $u_{j,n}$ denote the approximate value at (x_j, t_n). Replace the derivatives by their finite difference approximations. As a first attempt, replace u_t by the forward difference

$$u_t = \frac{u_{j,n+1} - u_{j,n}}{\Delta t} + O(\Delta t)$$

and the second derivative u_{xx} by the centered difference

$$u_{xx} = \frac{u_{j+1,n} - 2u_{j,n} + u_{j-1,n}}{\Delta x^2} + O(\Delta x^2).$$

If we substitute these expressions for the derivatives and solve for $u_{j,n+1}$, we arrive at the scheme

$$u_{j,n+1} = (1-2s)u_{j,n} + s(u_{j+1,n} + u_{j-1,n}) \tag{3.33}$$

where $s = k\Delta t/\Delta x^2$. The truncation error for this scheme is $\sigma = O(\Delta t) + O(\Delta x^2)$. The scheme is *explicit*. If we have already computed $u_{j,n}$ at level n, then we can compute $u_{j,n+1}$ from (3.33). The computational diagram is shown in Figure 3.2.

Unfortunately this scheme is unstable (and does not converge) unless the coefficient of $u_{j,n}$ is nonnegative. Suppose we take initial data $f(x) = \varepsilon\cos(\pi x/\Delta x)$. Then

$$f(x_j) = f(j\Delta x) = \varepsilon(-1)^j.$$

This data oscillates exactly with the spatial frequency of the grid. Then it is easy to show that, for this initial data,

$$u_{j,n} = \varepsilon(-1)^j(1-4s)^n. \tag{3.34}$$

Now the exact solution of (3.32) with this initial data has $|u(x,t)| \le \varepsilon$ for all x, t. However the computed discrete approximation will be unbounded, as $n \to \infty$, unless

$$|1-4s| \le 1,$$

which is true if and only if $0 \le s \le 1/2$, or

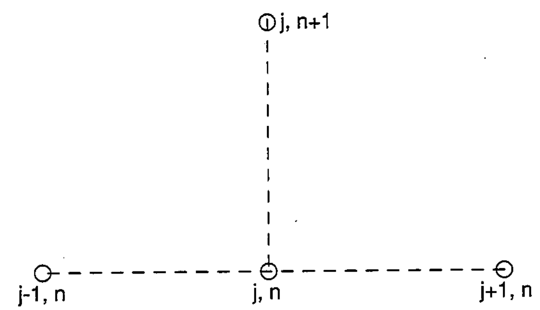

FIGURE 3.2
Computational diagram for (3.33).

$$k\Delta t \le \frac{(\Delta x)^2}{2}.$$

Thus for $\Delta x = .1$, say, this numerical scheme will become unstable if the initial data contains oscillations on the order of Δx, unless $k\Delta t \le (.1)^2/2 = .005$. This is a very small step size, and so, to make this scheme stable, is not practical.

Now let us try an *implicit* scheme. Instead of centering the difference quotient for the second derivative at (x_j, t_n), let us center it at (x_j, t_{n+1}), that is,

$$u_{xx}(x_j, t_{n+1}) = \frac{u_{j+1,n+1} - 2u_{j,n+1} + u_{j-1,n+1}}{(\Delta x)^2} + O(\Delta x^2).$$

This is equivalent to replacing the difference quotient for u_t by a backward difference. We arrive at the scheme

$$(1+2s)u_{j,n+1} - s(u_{j+1,n+1} + u_{j-1,n+1}) = u_{j,n}.$$

The computational diagram is shown in Figure 3.3. The truncation error is again $O(\Delta t) + O(\Delta x^2)$. We must solve this system of equations for each time step. To see better what is involved we write out the system for a small number of spatial grid points, say x_j, $j = 0, 1, 2, 3, 4$. From the initial condition $u(x,0) = f(x)$, we have $u_{j,0} = f(j\Delta x)$ given. Then to advance to the next level ($n = 1$), we must solve three equations in the five unknowns $u_{j,1}$, $j = 0, \cdots 4$:

$$\begin{array}{lllll} -su_{0,1} & +(1+2s)u_{1,1} & -su_{2,1} & & = u_{1,0} \\ & -su_{1,1} & +(1+2s)u_{2,1} & -su_{3,1} & = u_{2,0} \\ & & -su_{2,1} & +(1+2s)u_{3,1} \; -su_{4,1} & = u_{3,0} \end{array}$$

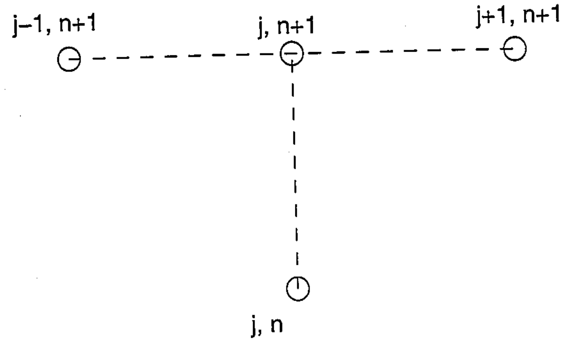

FIGURE 3.3
Computational diagram for the implicit scheme.

Usually such an overdetermined system does not have a solution. We should have a system of three equations in three unkowns. One way to reduce the number of unknowns is to impose the boundary conditions discussed in Section 3.4 at both ends of the bar. We shall discuss this kind of boundary value problem in Chapter 4. Here we use these concepts as a means of getting a solvable system of linear equations. We suppose that the bar has length L and that, for purposes of illustration, $\Delta x = L/4$ with $x_0 = 0$, and $x_4 = L$. Recall that the Dirichlet boundary condition specifies the temperature of the ends of bar in advance for all time. This means that $u_{0,n}$ and $u_{4,n}$ are given for all n. These values are then moved to the right-hand side of the first and third equations, resulting in a system of three equations in three unknowns.

$$\begin{bmatrix} 1+2s & -s & 0 \\ -s & (1+2s) & -s \\ 0 & -s & (1+2s) \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} u_{1,0} + su_{0,1} \\ u_{2,0} \\ u_{3,0} + su_{4,1} \end{bmatrix}.$$

For instance, if the ends of the bar are kept at temperature zero, then $u_{0,n} = u_{4,n} = 0$ for all n. Now to find $u_{1,1}, u_{2,1}, u_{3,1}$, we must solve the system above. This may be done for any $s \geq 0$. In the general case, we use mesh points $x_0, x_1, \ldots, x_J$ and the matrix is $J-1 \times J-1$ corresponding to the $J-1$ interior points $x_1, \ldots x_{J-1}$.

A second important boundary condition is the (homogeneous) Neumann condition in which we assume that the ends of the bar are insulated, so that the heat flux through the ends of the bar is zero, that is, $u_x(0,t) = u_x(L,t) = 0$ for all t. For the moment, we add "ghost points" x_{-1} and x_5 to our mesh. These points lie outside the bar. We can approximate $u_x(0,t)$ and $u_x(L,t)$ with centered differences

$$0 = u_x(0,t) \approx \frac{u_{-1,n} - u_{1,n}}{2\Delta x} \quad \text{or} \quad u_{-1,n} = u_{1,n},$$

and

$$0 = u_x(L,t) \approx \frac{u_{5,n} - u_{3,n}}{2\Delta x} \quad \text{or} \quad u_{5,n} = u_{3,n}.$$

Using the ghost points, augment the system of equations centered at the interior points by an equation centered at x_0 and another equation centered at x_4.

$$-su_{-1,1} + (1+2s)u_{0,1} - su_{1,1} = u_{0,0}$$

$$-su_{3,1} + (1+2s)u_{4,1} - su_{5,1} = u_{4,0}.$$

Now we have a system of five equations in the seven unknowns $u_{-1,1}, u_{0,1}, u_{1,1}, u_{2,1}, u_{3,1}, u_{4,1}, u_{5,1}$. However, when we use the Neumann condition, which translates into the discrete framework as $u_{-1,1} = u_{1,1}$ and $u_{5,1} = u_{3,1}$, we can reduce the number of unknowns in these last two equations to arrive at the systems of five equations in five unknowns:

$$\begin{bmatrix} 1+2s & -2s & 0 & 0 & 0 \\ -s & 1+2s & -s & 0 & 0 \\ 0 & -s & 1+2s & -s & 0 \\ 0 & 0 & -s & 1+2s & -s \\ 0 & 0 & 0 & -2s & 1+2s \end{bmatrix} \begin{bmatrix} u_{0,1} \\ u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{4,1} \end{bmatrix} = \begin{bmatrix} u_{0,0} \\ u_{1,0} \\ u_{2,0} \\ u_{3,0} \\ u_{4,0} \end{bmatrix}.$$

Again, we get a tridiagonal matrix. See also exercise 3 of Section 10.1.

In both cases the system is solved by making an LU factorization of the matrix and then using forward and backward substitution. This method is stable for any $s \geq 0$ and converges.

One can get improved accuracy in the approximation of the PDE by using an average of the explicit and the implicit scheme. This is called the Crank-Nicolson method. It has a truncation error $\sigma = (\Delta t^2) + O(\Delta x^2)$. The method is stable for all $s \geq 0$ and converges. The system of equations becomes

$$-su_{j-1,n+1} + (1+2s)u_{j,n+1} - su_{j+1,n+1}$$

$$= su_{j-1,n} + (1-2s)u_{j,n} + su_{j+1,n},$$

$j = 0, \cdots, J$, where now $s = (1/2)k\Delta t/(\Delta x)^2$. Again, these equations with proper boundary conditions can be solved efficiently with LU factorization. This is the method which is implemented in the programs `heat3`, `heat4`, and `heat5` which you will use in Chapter 4.

General references for finite difference methods are the books [RM] and [St].

3.7 Projects

1. Write a MATLAB program to implement the explicit finite difference scheme (3.33). Set $k = 1$ and use the boundary conditions $u = 0$ at $x = 0$ and at $x = 10$. Write an mfile `f.m` for the initial data. Try out your program with initial data $f(x) = \sin(\pi x/10)$.

   Fix $\Delta x = .5$ and then experiment with various values of Δt to see when the scheme becomes stable. Your observations should agree with the results of Section 3.6.

   Compare your computed results with the exact solution

$$u(x,t) = \sin(\pi x/10)\exp[-(\pi/10)^2 t]$$

   at various times, with $\Delta t = .125$. Find the maximum error at each time. Use `help max` to see how to have MATLAB do this. Now reduce the spatial step size to $\Delta x = .25$, and make $\Delta t = (\Delta x)^2/2$. Again compare the computed solution and the exact solution at the same times you did before when $\Delta x = .5$. Is the error smaller? How much?

2. Write a MATLAB program which uses Simpson's rule to evaluate the convolution integral

$$u(x,t) = \int S(x-y,t)f(y)dy.$$

   Assume that the initial data $f(y) = 0$ for $|y| > 1$. Compare the plots of the solution at some $t > 0$, as $h \to 0$. Do you see convergence?

3. Write a MATLAB program which calculates the solution of exercise 11 of Section 3.4 and plots $x \to u(x,t)$ for each t. Let h be an input parameter, and compare the plots for different values of h.

4. Use MATLAB to make a phase-plane portrait of the system of ODE's that arises in exercise 5 of Section 3.5. Make c and k input parameters and comment on how the phase-plane portrait changes as c and k change.

Chapter 4

Boundary Value Problems for the Heat Equation

Up to now we have studied fairly broad qualitative properties of solutions of the heat equation. We have not attempted to solve problems with boundary conditions assigned at both ends of a bar of finite length. In this chapter we study these problems and exploit the theory of linear operators to give a unified treatment of the many possible combinations of boundary conditions.

4.1 Separation of variables

In this section we develop new techniques of solution based on the venerable method of separation of variables. The formulas we obtain for the solutions will contain much valuable information about their structure. However, for computational purposes, we will use the finite difference schemes discussed in Chapter 3. They are easily extended to the case of variable coefficients.

As a model problem, we consider a bar of length L with zero boundary conditions at both ends:

$$u_t = ku_{xx}, \quad \text{for} \quad 0 < x < L,\ t > 0, \qquad u(0,t) = u(L,t) = 0 \quad \text{for} \quad t \geq 0, \tag{4.1}$$

$$u(x,0) = f(x) \quad \text{for} \quad 0 < x < L.$$

This is an initial-boundary-value problem. We shall use the abbreviation IBVP from now on.